Model reduction of infinite dimensional systems: An application to TDS and linear PDE cases

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Introduction

Optimal model approximation

Projection-based approximation framework

Rational interpolation Loewner framework

Stability regions estimation

Conclusions
Introduction

Context

Considered benchmarks

Optimal model approximation

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Conclusions
LARGE-SCALE DYNAMICAL MODELS

... some motivating examples in the simulation & control domains

Large-scale systems are present in many engineering fields: aerospace, computational biology, building structure, VLI circuits, automotive, weather forecasting, fluid flow…

- difficulties with simulation & memory management (e.g. ODE solvers)
- difficulties with analysis (e.g. frequency response, $\mu_{ssv}$ and $\mathcal{H}_\infty$ computation . . .)
- difficulties with controller design (e.g. robust, optimal, predictive, . . .)
INTRODUCTION

Topics addressed in this presentation about model approximation:

- Some projection based methods in the finite dimensional case where a realization is available: IRKA/ITIA\(^1\), IETIA\(^2\) ...
- Interpolation method using Loewner framework\(^3\) \(^4\)
- Approximation of stability regions for large-scale time-delay systems\(^5\) \(^6\)

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4 C. Beattie and S. Gugercin, "Realization-independent \(H_2\)-approximation", IEEE Conference on Decision and Control, 2012, pp. 4953-4958.
Considered benchmarks

**Benchmark NSS**: Navier-Stokes equation in a open cavity flow: discretization and linearisation for different Reynolds Numbers

\[ \begin{align*}
E \dot{x}(t) &= A(Re)x(t) + Bu(t) \\
y(t) &=Cx(t)
\end{align*} \]  

(1)

- **Two Reynolds cases** (\(Re = 7000\) and \(Re = 7500\))
- **SISO DAE**, 8 unstable modes, order \(\approx 700,000\) states

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Considered benchmarks

**Benchmark TDS-#1: Feedback delay and controller gain**

Let us consider

\[
\dot{x}(t) = Ax(t) + Bu(t); \quad y(t) = Cx(t),
\]

where

\[
A = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-10 & 10 & 0 & 0 \\
5 & -15 & 0 & -0.25 \\
\end{pmatrix}, \quad B = \begin{pmatrix}
0 \\
0 \\
1 \\
0 \\
\end{pmatrix}, \quad C = \begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
\end{pmatrix}^T.
\]

We add to this model the delayed static output feedback \( u(t) = -ky(t) + ky(t - \tau) \)

The resulted model \( H_{TDS1} \) is governed by

\[
\dot{x}(t) = A_0x(t) + A_1x(t - \tau)
\]

where \( A - BCk \) and \( A_1 = BCk \).

**Question:** Given \((k, \tau)\), what is the stability of (4)?

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8 A. Seuret and F. Gouaisbaut, "Hierarchy of LMI conditions for the stability analysis of time delay systems", Repport LAAS 14429.
Considered benchmarks

Benchmark TDS-#2: Multiple delays (in feedback) large-scale system

\[
y(t - \tau_i) = \begin{pmatrix} a_{\text{cont}}(t - \tau_1) \\ q_{\text{cont}}(t - \tau_2) \\ u_{\text{pit}}(t - \tau_3) \end{pmatrix}
\]

- Vibration control of aircraft model.
- \( \text{order}(H) \approx 600 \) states.
- Controller \( H \) designed without taking into account time-delays.
- Three output delays \( \{\tau_1, \tau_2, \tau_3\} \).

**Question:** Stability function of \( \{\tau_1, \tau_2, \tau_3\} \)? How to measure loss of performance?

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INTRODUCTION

Considered benchmarks

Benchmark PDE: Example string vibration with dissipation
Vibrating string of length $L = 1$ whose ends are fixed with control and observation are both distributed along the string.

$$\frac{\partial^2 z(x, t)}{\partial t^2} + \epsilon \langle \frac{\partial z(x, t)}{\partial t}, 1_{[0, \frac{1}{2}]} \rangle 1_{[0, \frac{1}{2}]}(x) = \frac{\partial^2 z(x, t)}{\partial x^2} + 1_{[0, \frac{1}{2}]}(x) u(t), \quad 0 < x < 1, \quad t \geq 0$$

(5)

where, $1_{[0, \frac{1}{2}]}(x) = \left\{ \begin{array}{ll} 1 & \text{if } 0 \leq x \leq 1/2 \\ 0 & \text{if } 1/2 < x \leq 1 \end{array} \right.$, with

$z(0, t) = 0, z(1, t) = 0$

, and

$$y(t) = \int_0^1 \frac{\partial z(x, t)}{\partial t} 1_{[0, \frac{1}{2}]}(x) dx.$$

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Considered benchmarks

**Benchmark PDE: Example string vibration with dissipation**

The transfer function of this model is given by

\[
H(s) = \frac{\frac{s}{2} \sinh(s) + 2 \cosh\left(\frac{s}{2}\right) - 3 \cosh^2\left(\frac{s}{2}\right) + 1}{s\left(s + \frac{1}{2}\right) \sinh(s) + 2 \cosh\left(\frac{s}{2}\right) - 3 \cosh^2\left(\frac{s}{2}\right) + 1}
\]  

(6)
Introduction

Optimal model approximation
  Approximation in the $\mathcal{H}_2$, $\mathcal{H}_2,\Omega$ and $\mathcal{L}_2$-norm
  $\mathcal{H}_2$ and $\mathcal{L}_2$ optimality conditions

Projection-based approximation framework

Rational interpolation Loewner framework

Stability regions estimation

Conclusions
Optimal model approximation

Approximation in the $\mathcal{H}_2$, $\mathcal{H}_2\Omega$ and $\mathcal{L}_2$-norm

$L_2$ model approximation

\[
\hat{H} := \arg \min_{G \in \mathcal{L}_2^{n_y \times n_u}, \dim(G) = r} ||H - G||_{\mathcal{L}_2}
\]

\[
||H||_{\mathcal{L}_2}^2 := \text{trace} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} (H(i\nu)H(i\nu)^T)d\nu \right)
\]

- $\mathcal{L}_2(i\mathbb{R})$ the Hilbert space of matrix-valued functions $\mathbb{C} \rightarrow \mathbb{C}^{n_y \times n_u}$ satisfying
  \[
  \int_{\mathbb{R}} \text{trace}[F(i\omega)F(i\omega)^T]d\omega < \infty.
  \]
- $\mathcal{L}_2(i\mathbb{R}) = \mathcal{H}_2(\mathbb{C}^-) \oplus \mathcal{H}_2(\mathbb{C}^+)$

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I. Pontes Duff et al. [Onera] Model reduction of infinite dimensional systems: An application to TDS and linear PDE cases.
**Optimal Model Approximation**

**Approximation in the $\mathcal{H}_2$, $\mathcal{H}_2,\Omega$ and $\mathcal{L}_2$-norm**

**$\mathcal{H}_2$ model approximation**

$$\hat{H} := \arg \min_{G \in \mathcal{H}_2^{ny \times nu}} \|H - G\|_{\mathcal{H}_2}$$

with

$$\dim(G) = r$$

$$\|H\|_{\mathcal{H}_2}^2 := \operatorname{trace} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} (H(i\nu)H(i\nu)) \, d\nu \right)$$

$$= \operatorname{trace} \left( C P C^T \right) = \operatorname{trace} \left( B^T Q B \right)$$

$$= \sum_{i=1}^{n} \operatorname{trace} \left( \phi_i H(-\lambda_i)^T \right)$$

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Assume that \( H \) and \( \hat{H} \) have semi-simple poles and suppose that \( \hat{H} \) is a \( r^{th} \)-order finite-dimensional model with transfer function

\[
\hat{H}(s) = \sum_{k=1}^{r} \frac{\hat{c}_k \hat{b}_k^T}{s - \hat{\lambda}_k}. \tag{11}
\]

**\( H_2 \)-optimality conditions**

If \( H, \hat{H} \in H_2 \) and \( \hat{H} \) is a local minimum of the \( H_2 \) approximation problem, then the following interpolations equations hold

\[
H(-\hat{\lambda}_k) \hat{b}_k = \hat{H}(-\hat{\lambda}_k) \hat{b}_k, \quad \hat{c}_k^T H(-\hat{\lambda}_k) = \hat{c}_k^T \hat{H}(-\hat{\lambda}_k) \tag{12}
\]

\[
\hat{c}_k^T \left. \frac{dH}{ds} \right|_{s=-\hat{\lambda}_k} \hat{b}_k = \hat{c}_k^T \left. \frac{d\hat{H}}{ds} \right|_{s=-\hat{\lambda}_k} \hat{b}_k, \tag{13}
\]

for all \( k = 1, \ldots, r \) where \( \hat{\lambda}_k \) are the poles of \( \hat{H} \) and \( \hat{b}_k \) and \( \hat{c}_k \) are its tangential directions, respectively.

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In the case where $H \in L_2(i\mathbb{R})$ is a SISO LTI system and $H = H^+ + H^-$ where $H^+ \in \mathcal{H}(\mathbb{C}^+)$ and $H^- \in \mathcal{H}(\mathbb{C}^-)$, it is possible to state the following result:

**$L_2$ optimality conditions**

Given $H \in L_2(i\mathbb{R})$ and its decomposition $H = H^+ + H^-$ where $H^+ \in \mathcal{H}(\mathbb{C}^+)$ and $H^- \in \mathcal{H}(\mathbb{C}^-)$. Let $\hat{H}$ be the local minimizer of order $r$ whose poles are all simple $\{\hat{\lambda}_1, \ldots, \hat{\lambda}_k\} \in \mathbb{C}^-$ and $\{\hat{\lambda}_{k+1}, \ldots, \hat{\lambda}_r\} \in \mathbb{C}^+$. If $\hat{H}(s)$ is given as (11) and if it is a local minimal of the $L_2$ approximation problem, then following hold for $i = 1, \ldots, k$

$$H^+(-\hat{\lambda}_i) = \hat{H}^+(-\hat{\lambda}_i), \quad \frac{dH^+}{ds} \bigg|_{s=-\hat{\lambda}_i} = \frac{d\hat{H}^+}{ds} \bigg|_{s=-\hat{\lambda}_i} \quad (14)$$

and for $i = k + 1, \ldots, r$,

$$H^-(\hat{\lambda}_i) = \hat{H}^-(\hat{\lambda}_i), \quad \frac{dH^-}{ds} \bigg|_{s=-\hat{\lambda}_i} = \frac{d\hat{H}^-}{ds} \bigg|_{s=-\hat{\lambda}_i}. \quad (15)$$

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I. Pontes Duff et al. [Onera]

Model reduction of infinite dimensional systems: An application to TDS and linear PDE cases.
**Optimal Model Approximation**

**Approximation in the $H_2, \Omega$-norm**

$$\hat{H} := \arg\min_{G \in H^{n_y \times n_u}_\infty} \|H - G\|_{H_2, \Omega} \quad \text{dim}(G) = r$$

$$\|H\|_{H_2, \Omega}^2 := \text{trace}\left(\frac{1}{\pi} \int_\Omega (\overline{H(i\nu)H(i\nu)}) d\nu\right)$$

$$= \text{trace}(CP_\Omega C^T) = \text{trace}(B^T Q_\Omega B)$$

$$= \sum_{i=1}^n \text{trace}\left(\phi_i H(-\lambda_i)^T \left[-\frac{2}{\pi} \text{atan}\left(\frac{\omega}{\lambda_i}\right)\right]\right)$$


### OPTIMAL MODEL APPROXIMATION

#### Approximation in the $\mathcal{H}_2,\Omega$-norm

\[
\begin{align*}
\hat{H} &:= \arg \min_{G \in \mathcal{H}_{n_y \times n_u}^\infty, \dim(G) = r} ||H - G||_{\mathcal{H}_2,\Omega} \\
||H||_{\mathcal{H}_2,\Omega}^2 &:= \text{trace} \left( \frac{1}{\pi} \int_{\Omega} (\overline{H(i\nu)} H(i\nu)) d\nu \right) \\
&= \text{trace} \left( C \mathcal{P}_\Omega C^T \right) = \text{trace} \left( B^T Q \Omega B \right) \\
&= \sum_{i=1}^{n} \text{trace} \left( \phi_i H(-\lambda_i)^T \right) \left[ -\frac{2}{\pi} \tan \left( \frac{\omega}{\lambda_i} \right) \right]
\end{align*}
\]

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I. Pontes Duff et al. [Onera]

Model reduction of infinite dimensional systems: , An application to TDS and linear PDE cases
\textbf{Mismatch objective and eigenvector preservation}

\[
\hat{H} := \arg \min_{G \in \mathcal{L}_{2}^{n_y \times n_u}} \|H - G\|_{\mathcal{H}_2}
\]

\text{dim}(G) = r \ll n \quad \lambda_k(G) \subseteq \lambda(H) \quad k = 1, \ldots, q < r

\begin{itemize}
  \item More than a $\mathcal{H}_2$ (sub-optimal) criteria
  \item Keep some user defined eigenvalues... \textit{e.g.} the unstable ones
\end{itemize}

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\begin{footnotesize}
\textsuperscript{18} C. Poussot-Vassal and P. Vuillemin, "\textit{An Iterative Eigenvector Tangential Interpolation Algorithm for Large-Scale LTI and a Class of LPV Model Approximation}”, European Control Conference, 2013, pp. 4490-4495.
\end{footnotesize}
Introduction

Optimal model approximation

Projection-based approximation framework
  Projectors
  MIMO IRKA (or ITIA)
  IETIA
  Fluid flow dynamical model approximation

Rational interpolation Loewner framework

Stability regions estimation

Conclusions
Let $H : \mathbb{C} \rightarrow \mathbb{C}^{n_y \times n_u}$ be a $n_u$ inputs $n_y$ outputs, full order $\mathcal{H}^{n_y \times n_u}_2$ (or $\mathcal{L}^{n_y \times n_u}_2$) complex-valued function describing a LTI dynamical system as a DAE of order $n$, with realization $H$:

\[
H : \begin{cases} 
E \dot{x}(t) = A x(t) + B u(t) \\
y(t) = C x(t) 
\end{cases}
\] (19)
**Projection-based approximation framework**

Let \( H : \mathbb{C} \to \mathbb{C}^{n_y \times n_u} \) be a \( n_u \) inputs \( n_y \) outputs, full order \( \mathcal{H}_{2}^{n_y \times n_u} \) (or \( \mathcal{L}_{2}^{n_y \times n_u} \)) complex-valued function describing a **LTI** dynamical system as a DAE of order \( n \), with realization \( \mathbf{H} \):

\[
\begin{align*}
E \dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{align*}
\]  

(19)

the approximation problem consists in finding \( V, W \in \mathbb{R}^{n \times r} \) (with \( r \ll n \)) spanning \( V \) and \( W \) subspaces and forming a projector \( \Pi_{V,W} = VW^T \), such that

\[
\hat{H} : \begin{cases} 
W^T EV \dot{x}(t) &= W^T AV \hat{x}(t) + W^T Bu(t) \\
\hat{y}(t) &= CV \hat{x}(t)
\end{cases}
\]  

(20)

well approximates \( H \).
**PROJECTION-BASED APPROXIMATION FRAMEWORK**

### Projectors

Let $H : \mathbb{C} \rightarrow \mathbb{C}^{n_y \times n_u}$ be a $n_u$ inputs $n_y$ outputs, full order $H_{2}^{n_y \times n_u}$ (or $L_{2}^{n_y \times n_u}$) complex-valued function describing a LTI dynamical system as a DAE of order $n$, with realization $H$:

$$H : \begin{cases}
E \dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{cases} \quad (19)$$

the approximation problem consists in finding $V, W \in \mathbb{R}^{n \times r}$ (with $r \ll n$) spanning $\mathcal{V}$ and $\mathcal{W}$ subspaces and forming a projector $\Pi_{V,W} = VW^T$, such that

$$\hat{H} : \begin{cases}
W^T EV \dot{x}(t) &= W^T AV \dot{x}(t) + W^T Bu(t) \\
\dot{y}(t) &= CV \dot{x}(t)
\end{cases} \quad (20)$$

well approximates $H$.

- Small approximation error and/or global error bound
- Stability / passivity preservation
- Numerically stable & efficient procedure
**PROJECTION-BASED APPROXIMATION FRAMEWORK**

**MIMO IRKA (or ITIA) - $\mathcal{H}_2$ optimality conditions (Tangential subspace approach)** \(^{19} \ 20\)

Given $H(s)$, let $V \in \mathbb{C}^{n \times r}$ and $W \in \mathbb{C}^{n \times r}$ be matrices of full column rank $r = q_2$ such that $W^* V = I_r$. If, for $j = 1, \ldots, q_2$,

$$
\begin{bmatrix}
(\sigma_j E - A)^{-1} B \hat{b}_j
\end{bmatrix} \in \text{span}(V)\quad \text{and} \quad
\begin{bmatrix}
(\sigma_j E - A^T)^{-1} C^T \hat{c}_j^*
\end{bmatrix} \in \text{span}(W)
$$

where $\sigma_j \in \mathbb{C}$, $\hat{b}_j \in \mathbb{C}^{n_u}$ and $\hat{c}_j \in \mathbb{C}^{n_y}$, be given sets of interpolation points and left and right tangential directions, respectively.

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MIMO IRKA (or ITIA) - $\mathcal{H}_2$ optimality conditions (Tangential subspace approach) \(^{19, 20}\)

Given $H(s)$, let $V \in \mathbb{C}^{n \times r}$ and $W \in \mathbb{C}^{n \times r}$ be matrices of full column rank $r = q_2$ such that $W^*V = I_r$. If, for $j = 1, \ldots, q_2$,

\[
\begin{bmatrix}
(\sigma_j E - A)^{-1} B \hat{b}_j \\
(\sigma_j E - A^T)^{-1} C^T \hat{c}_j^*
\end{bmatrix} \in \text{span}(V) \text{ and } \begin{bmatrix}
(\sigma_j E - A)^{-1} B \hat{b}_j \\
(\sigma_j E - A^T)^{-1} C^T \hat{c}_j^*
\end{bmatrix} \in \text{span}(W) \tag{21}
\]

where $\sigma_j \in \mathbb{C}$, $\hat{b}_j \in \mathbb{C}^{n_u}$ and $\hat{c}_j \in \mathbb{C}^{n_y}$, be given sets of interpolation points and left and right tangential directions, respectively. Then, the reduced order system $\hat{H}(s)$ satisfies the tangential interpolation conditions

\[
\begin{align*}
H(-\hat{\sigma}_j) \hat{b}_j &= \hat{H}(-\hat{\sigma}_j) \hat{b}_j \\
\hat{c}_j^* H(-\hat{\sigma}_j) &= \hat{c}_j^* \hat{H}(-\hat{\sigma}_j) \\
\hat{c}_j^* \frac{d}{ds} H(s) \bigg|_{s=-\hat{\sigma}_j} \hat{b}_j &= \hat{c}_j^* \frac{d}{ds} \hat{H}(s) \bigg|_{s=-\hat{\sigma}_j} \hat{b}_j
\end{align*} \tag{22}
\]

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PROJECTION-BASED APPROXIMATION FRAMEWORK

Require: \( H = (E, A, B, C) \), \( \{\sigma_j^{(0)}, \ldots, \sigma_{q_2}^{(0)}\} \in \mathbb{C}^{q_2} \), \( \{\hat{b}_1, \ldots, \hat{b}_{q_2}\} \in \mathbb{C}^{n_u \times q_2} \), 
\( \{\hat{c}_1, \ldots, \hat{c}_{q_2}\} \in \mathbb{C}^{n_y \times q_2} \) and \( r = q_2 \in \mathbb{N} \)

1: Construct,
\[
\text{span}(V(\sigma_j^{(0)}, \hat{b}_j)) \quad \text{and} \quad \text{span}(W(\sigma_j^{(0)}, \hat{c}_j^*))
\] (23)

2: Compute \( W \leftarrow W(V^TW)^{-1} \)
3: while Stopping criteria do
4: \( k \leftarrow k + 1 \)
5: \( \hat{E} = W^TEV, \hat{A} = W^TAV, \hat{B} = W^TB, \hat{C} = CV \)
6: Compute \( \hat{A}R = \Lambda(\hat{A}, \hat{E})R \) and \( L\hat{A} = \Lambda(\hat{A}, \hat{E})L \)
7: Compute \( \{\hat{b}_1, \ldots, \hat{b}_{q_2}\} = \hat{B}^TL \) and \( \{\hat{c}_1^*, \ldots, \hat{c}_{q_2}^*\} = \hat{C}R \)
8: Set \( \sigma^{(i)} = -\Lambda(\hat{A}, \hat{E}) \)
9: Construct,
\[
\text{span}(V(\sigma_j^{(k)}, \hat{b}_j)) \quad \text{and} \quad \text{span}(W(\sigma_j^{(k)}, \hat{c}_j^*))
\] (24)
10: Compute \( W \leftarrow W(V^TW)^{-1} \)
11: end while
12: Construct \( \hat{H} := (W^TEV, W^TAV, W^TB, CV) \)
Ensure: \( V, W \in \mathbb{R}^{n \times r}, W^TV = I_r \)
PROJECTION-BASED APPROXIMATION FRAMEWORK

**Require:** \( H = (E, A, B, C), \{\sigma_1^{(0)}, \ldots, \sigma_{q_2}^{(0)}\} \in \mathbb{C}^{q_2}, \{\hat{b}_1, \ldots, \hat{b}_{q_2}\} \in \mathbb{C}^{n_u \times q_2}, \{\hat{c}_1, \ldots, \hat{c}_{q_2}\} \in \mathbb{C}^{n_y \times q_2} \) and \( r = q_2 \in \mathbb{N} \)

1: Construct,

\[
\text{span}(V(\sigma_j^{(0)}, \hat{b}_j)) \text{ and span}(W(\sigma_j^{(0)}, \hat{c}_j^*))
\]  \hspace{1cm} (23)

2: Compute \( W \leftarrow W(V^TW)^{-1} \)

3: while Stopping criteria do
4: \( k \leftarrow k + 1 \)
5: \( \hat{E} = W^TEV, \hat{A} = W^TAV, \hat{B} = W^TB, \hat{C} = CV \)
6: Compute \( \hat{A}R = \Lambda(\hat{A}, \hat{E})R \text{ and } L\hat{A} = \Lambda(\hat{A}, \hat{E})L \)
7: Compute \( \{\hat{b}_1, \ldots, \hat{b}_{q_2}\} = \hat{B}^TL \text{ and } \{\hat{c}_1^*, \ldots, \hat{c}_{q_2}^*\} = \hat{C}R \)
8: Set \( \sigma^{(i)} = -\Lambda(\hat{A}, \hat{E}) \)
9: Construct,

\[
\text{span}(V(\sigma_j^{(k)}, \hat{b}_j)) \text{ and span}(W(\sigma_j^{(k)}, \hat{c}_j^*))
\]  \hspace{1cm} (24)

10: Compute \( W \leftarrow W(V^TW)^{-1} \)
11: end while
12: Construct \( \hat{H} := (W^TEV, W^TAV, W^TB, CV) \)

**Ensure:** \( V, W \in \mathbb{R}^{n \times r}, W^TV = I_r \)
IETIA - $\mathcal{H}_2$ & spectral optimality conditions (Tangential subspace approach) 21

Given $H(s)$, let $V \in \mathbb{C}^{n \times r}$ and $W \in \mathbb{C}^{n \times r}$ be matrices of full column rank $r = q_1 + q_2$ such that $W^*V = I_r$. If, for $i = 1, \ldots, q_1$ and $j = 1, \ldots, q_2$,

$$\begin{bmatrix} \ell_i^* (\sigma_j E - A)^{-1} B \hat{b}_j \\ r_i^* (\sigma_j E - A^T)^{-1} C^T \hat{c}_j \end{bmatrix} \in \text{span}(V) \quad \text{and} \quad \begin{bmatrix} \ell_i^* (\sigma_j E - A^T)^{-1} C^T \hat{c}_j \\ r_i^* (\sigma_j E - A)^{-1} B \hat{b}_j \end{bmatrix} \in \text{span}(W) \quad (25)$$

$l_i^* \in \mathbb{C}^n$ and $r_i^* \in \mathbb{C}^n$ are left and right eigenvectors associated to $\lambda_i^* \in \mathbb{C}$ eigenvalues associated to $A$, $E$ and $\sigma_j \in \mathbb{C}$, $\hat{b}_j \in \mathbb{C}^{n_u}$ and $\hat{c}_j \in \mathbb{C}^{n_y}$, be given sets of interpolation points and left and right tangential directions, respectively.

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IETIA - $\mathcal{H}_2$ & spectral optimality conditions (Tangential subspace approach) \(^\text{21}\)

Given $H(s)$, let $V \in \mathbb{C}^{n \times r}$ and $W \in \mathbb{C}^{n \times r}$ be matrices of full column rank $r = q_1 + q_2$ such that $W^*V = I_r$. If, for $i = 1, \ldots, q_1$ and $j = 1, \ldots, q_2$,

$$
\begin{bmatrix}
r_i^* & (\sigma_j E - A)^{-1} B \hat{b}_j \\
\end{bmatrix} \in \text{span}(V) \quad \text{and} \quad
\begin{bmatrix}
l_i^* & (\sigma_j E - A^T)^{-1} C^T \hat{c}_j^* \\
\end{bmatrix} \in \text{span}(W)
$$

PROJECTION-BASED APPROXIMATION FRAMEWORK

Require: $H = (E, A, B, C)$, $\{\lambda_1^*, \ldots, \lambda_{q_1}^*\} \in \mathbb{C}^{q_1}$, $\{\sigma_1^{(0)}, \ldots, \sigma_{q_2}^{(0)}\} \in \mathbb{C}^{q_2}$, $\{\hat{b}_1, \ldots, \hat{b}_{q_2}\} \in \mathbb{C}^{n_u \times q_2}$, $\{\hat{c}_1, \ldots, \hat{c}_{q_2}\} \in \mathbb{C}^{n_y \times q_2}$ and $r = q_1 + q_2 \in \mathbb{N}$

1. Compute $\{l_1^*, \ldots, l_{q_1}^*\}$ and $\{r_1^*, \ldots, r_{q_1}^*\}$, eigenvectors of $\{\lambda_1^*, \ldots, \lambda_{q_1}^*\}$
2. Construct,

$$\text{span}(V(l_i^*, \sigma_j^{(0)}, \hat{b}_j)) \text{ and } \text{span}(W(r_i^*, \sigma_j^{(0)}, \hat{c}_j^*))$$ (28)

3. Compute $W \leftarrow W(V^TW)^{-1}$
4. while Stopping criteria do
5. \hspace{1cm} $k \leftarrow k + 1$
6. \hspace{1cm} $\hat{E} = W^TEV$, $\hat{A} = W^TAV$, $\hat{B} = W^TB$, $\hat{C} = CV$
7. \hspace{1cm} Compute $\hat{A}R = \hat{E}\Lambda(\hat{A}, \hat{E})R$ and $L\hat{A} = \Lambda(\hat{A})L$
8. \hspace{1cm} Compute $\{\hat{b}_1, \ldots, \hat{b}_{q_2}\} = \hat{B}^T L$ and $\{\hat{c}_1, \ldots, \hat{c}_{q_2}\} = \hat{C}R$
9. \hspace{1cm} Set $\sigma^{(i)} = -\Lambda(\hat{A}, \hat{E})$
10. \hspace{1cm} Construct,

$$\text{span}(V(l_i^*, \sigma_j^{(k)}, \hat{b}_j)) \text{ and } \text{span}(W(r_i^*, \sigma_j^{(k)}, \hat{c}_j^*))$$ (29)

11. Compute $W \leftarrow W(V^TW)^{-1}$
12. end while
13. Construct $\hat{H} := (W^TEV, W^TAV, W^TB, CV)$

Ensure: $V, W \in \mathbb{R}^{n \times r}$, $W^TV = I_r$, $\{\lambda_1^*, \ldots, \lambda_{q_1}^*\} \subset \Lambda(\hat{A}, \hat{E})$
PROJECTION-BASED APPROXIMATION FRAMEWORK

\textbf{Require:} \( \mathbf{H} = (E, A, B, C), \{\lambda_1^*, \ldots, \lambda_{q_1}^*\} \in \mathbb{C}^{q_1}, \{\sigma_1^{(0)}, \ldots, \sigma_{q_2}^{(0)}\} \in \mathbb{C}^{q_2}, \)
\( \{\hat{b}_1, \ldots, \hat{b}_{q_2}\} \in \mathbb{C}^{n_u \times q_2}, \{\hat{c}_1, \ldots, \hat{c}_{q_2}\} \in \mathbb{C}^{n_y \times q_2} \) and \( r = q_1 + q_2 \in \mathbb{N} \)

1: Compute \( \{l_1^*, \ldots, l_{q_1}^*\} \) and \( \{r_1^*, \ldots, r_{q_1}^*\} \), eigenvectors of \( \{\lambda_1^*, \ldots, \lambda_{q_1}^*\} \)
2: Construct,
\[
\text{span}(V(l_i^*, \sigma_j^{(0)}, \hat{b}_j)) \text{ and span}(W(r_i^*, \sigma_j^{(0)}, \hat{c}_j^*)))
\]
(28)

3: Compute \( W \leftarrow W(V^T W)^{-1} \)
4: \textbf{while} Stopping criteria \textbf{do}
5: \quad \quad \quad k \leftarrow k + 1
6: \quad \quad \quad \hat{E} = W^T EV, \hat{A} = W^T AV, \hat{B} = W^T B, \hat{C} = CV
7: \quad \quad \quad \text{Compute } \hat{A} \Lambda = \hat{E} \Lambda(\hat{A}, \hat{E}) R \text{ and } L \hat{A} = \Lambda(\hat{A}) L
8: \quad \quad \quad \text{Compute } \{\hat{b}_1, \ldots, \hat{b}_{q_2}\} = \hat{B}^T L \text{ and } \{\hat{c}_1^*, \ldots, \hat{c}_{q_2}^*\} = \hat{C} R
9: \quad \quad \quad \text{Set } \sigma^{(i)} = -\Lambda(\hat{A}, \hat{E})
10: \quad \quad \quad \text{Construct,}
\[
\text{span}(V(l_i^*, \sigma_j^{(k)}, \hat{b}_j)) \text{ and span}(W(r_i^*, \sigma_j^{(k)}, \hat{c}_j^*)))
\]
(29)

11: \quad \quad \quad \text{Compute } W \leftarrow W(V^T W)^{-1}
12: \quad \quad \quad \textbf{end while}
13: \quad \quad \quad \text{Construct } \hat{\mathbf{H}} := (W^T EV, W^T AV, W^T B, CV)
\textbf{Ensure:} \ V, W \in \mathbb{R}^{n \times r}, W^T V = I_r, \{\lambda_1^*, \ldots, \lambda_{q_1}^*\} \subset \Lambda(\hat{A}, \hat{E})
**Fluid flow dynamical model approximation - Re=7000 and Re=7500**

![Graphs showing gain and phase plots for Open cavity flow reduced order models - ITIA at Re=7000 and Re=7500.](image)

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RATIONAL INTERPOLATION LOEOWNER FRAMEWORK

Rational interpolation

Given $H(s)$, complex points $\sigma_1, \ldots, \sigma_n$, and tangential directions $\hat{b}_1, \ldots, \hat{b}_n$, $\hat{c}_1, \ldots, \hat{c}_n$, one constructs $(\hat{E}, \hat{A}, \hat{B}, \hat{C})$ such that the transfer function $\hat{H}(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B}$ satisfies the tangential interpolation conditions:

$$
\begin{align*}
H(\sigma_j)\hat{b}_j &= \hat{H}(\sigma_j)\hat{b}_j \\
\hat{c}_j^T H(\sigma_j) &= \hat{c}_j^T \hat{H}(\sigma_j) \\
\hat{c}_j^T \left. \frac{d}{ds} H(s) \right|_{s=\sigma_j} \hat{b}_j &= \hat{c}_j^T \left. \frac{d}{ds} \hat{H}(s) \right|_{s=\sigma_j} \hat{b}_j
\end{align*}
$$

(30)

This is possible thanks to Loewner matrices.
RATIONAL INTERPOLATION LOEWNER FRAMEWORK

The rational function $\hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B}$ interpolates $H(s)$ at points $\sigma_i$ and directions $\hat{b}_i$ and $\hat{c}_i$ iff.

$$(\hat{E})_{ij} = \begin{cases} 
\hat{c}_i^T (H(\sigma_i) - H(\sigma_j)) \hat{b}_j & i \neq j \\
\frac{\sigma_i - \sigma_j}{\hat{c}_i^T \hat{H}'(\sigma_i) \hat{b}_i} & i = j
\end{cases}$$

$$(\hat{A})_{ij} = \begin{cases} 
-\hat{c}_i^T (\sigma_i H(\sigma_i) - \sigma_j H(\sigma_j)) \hat{b}_j & i \neq j \\
\frac{\sigma_i - \sigma_j}{\hat{c}_i^T \left( sH(s) \right)'}_{s=\sigma_i} \hat{b}_i & i = j
\end{cases}$$

$$\hat{C} = [H(\sigma_1) \hat{b}_1, \ldots, H(\sigma_r) \hat{b}_r] \quad \text{and} \quad \hat{B} = \begin{bmatrix} \hat{c}_1^T H(\sigma_1) \\ \vdots \\ \hat{c}_r^T H(\sigma_r) \end{bmatrix}.$$ 

▶ An analogous to IRKA iterative method was proposed.

---


RATIONAL INTERPOLATION LOEOWNER FRAMEWORK

TF-IRKA algorithm

1: **Initialization:** transfer function $H(s)$, dimension $r$, $\sigma^0 = \{\sigma_1^0, \ldots, \sigma_r^0\} \in \mathbb{C}$ initial interpolation points and tangential directions $\{b_1, \ldots, b_r\} \in \mathbb{C}^{n_u \times 1}$ and $\{c_1, \ldots, c_r\} \in \mathbb{C}^{n_y \times 1}$.

2: **while** not convergence **do**
   3: **Build** $\hat{E}$, $\hat{A}$, $\hat{B}$ and $\hat{C}$ using *Loewner Matrices*.
   4: Solve the generalized eigenvalue problem $\hat{A}(k)x_i^{(k)} = \lambda_i^{(k)} \hat{E}(k)x_i^{(k)}$ and $y_i^{(k)}$ such that $y_i^{(k)} \ast \hat{E}(k)x_j^{(k)} = \delta_{i,j}$.
   5: **Set** $\sigma_i^{(k+1)} \leftarrow -\lambda_i^{(k)}$, $b_i^{(k+1)T} \leftarrow y_i^{(k)} \hat{B}(k)$ and $c_i^{(k+1)} \leftarrow \hat{C}(k)x_i^{(k)}$, for $i = 1, \ldots, r$.

6: **end while**

7: **Ensure** conditions (31) are satisfied.

8: **Build** $\hat{E}$, $\hat{A}$, $\hat{B}$ and $\hat{C}$.

---

Example TDS-#1 for $k = 1$ and $\tau = 3$
RATIONAL INTERPOLATION LOEWNER FRAMEWORK

Example DPS

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Introduction

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Projection-based approximation framework

Rational interpolation Loewner framework

**Stability regions estimation**

  Proposed Strategy - Approximation & Eigenvalues
  Results about stability approximation in $\mathcal{L}_2$
  Application to TDS

Conclusions
STABILITY REGIONS ESTIMATION

Proposed Strategy - Approximation & Eigenvalues (accompanied with proofs)

- **Procedure**: Estimating stability regions using model approximation & eigenvalues
- **Arguments for proof**: Provide some arguments why this procedure is valid
Proposed Strategy - Approximation & Eigenvalues (accompanied with proofs)

- **Procedure**: Estimating stability regions using model approximation & eigenvalues
- **Arguments for proof**: Provide some arguments why this procedure is valid

**Example**: Let us consider the model described by the transfer function

\[ H(s) = \frac{1}{1 + e^{-\tau s} + 2 e^{-\gamma s}} \], with \( \tau, \gamma \in [0, 2] \). After discretizing \([0, 2]\) and finding LTI approximation via TF-IRKA, the stability of the reduced model is plotted.
Results about stability approximation in $L_2$

- $L_2(i\mathbb{R})$ the Hilbert space of matrix-valued functions $\mathbb{C} \rightarrow \mathbb{C}^{n_y \times n_u}$ satisfying
  \[ \int_{\mathbb{R}} \text{trace}[F(i\omega)F(i\omega)^T]d\omega < \infty. \]

- $\langle H, G \rangle_{L_2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}(H(i\omega)G(i\omega)^T)d\omega.$

- $H_2(\mathbb{C}^+) (H_2(\mathbb{C}^-))$ closed subspace of $L_2(i\mathbb{R})$ containing the matrix functions $F(s)$ analytic in the open right-half plane (open left-half plane).

- $L_2(i\mathbb{R}) = H_2(\mathbb{C}^-) \oplus H_2(\mathbb{C}^+)$

- $L_2(i\mathbb{R}) \setminus H_2(\mathbb{C}^+)$ set of unstable LTI systems

- **Remark:** TF-IRKA allows us to obtain a system of order $r$ which satisfies the interpolation conditions:

\[ H(-\hat{\lambda}_k)\hat{b}_k = \hat{H}(-\hat{\lambda}_k)\hat{b}_k, \quad \hat{c}_k^T H(-\hat{\lambda}_k) = \hat{c}_k^T \hat{H}(-\hat{\lambda}_k) \]  
\[ \hat{c}_k^T \frac{dH}{ds} \bigg|_{s=-\hat{\lambda}_k} \hat{b}_k = \hat{c}_k^T \frac{d\hat{H}}{ds} \bigg|_{s=-\hat{\lambda}_k} \hat{b}_k, \]
STABILITY REGIONS ESTIMATION

Results about stability approximation in $L_2$

Proposition 1

If $H \in H_2(\mathbb{C}^+)$ and there exists a global minimizer $\hat{H} \in L_2(i\mathbb{R})$ of the $L_2$ approximation problem, then $\hat{H} \in H_2(\mathbb{C}^+)$. Similarly, if $H \in H_2(\mathbb{C}^-)$ and there exists a global minimizer $\hat{H} \in L_2(i\mathbb{R})$ of the $L_2$ approximation problem, then $\hat{H} \in H_2(\mathbb{C}^-)$.

Proof.

Let $\hat{H} \in L_2(i\mathbb{R})$ be the global minimizer of $L_2$ approximation problem. Since $H \in H_2(\mathbb{C}^+)$, one has $H^- = 0$. Seeing that $L_2(i\mathbb{R}) = H_2(\mathbb{C}^-) \oplus H_2(\mathbb{C}^+)$ and this an orthogonal decomposition, thus

$$\|H - \hat{H}\|_{L_2}^2 = \|H^- + \hat{H}^+\|_{L_2}^2 + \|0 - \hat{H}^-\|_{L_2}^2$$

(33)

Thus, $\hat{H}^- = 0$, otherwise $\hat{H}$ is not a global minimizer. □
Proposition 1

If $H \in H_2(\mathbb{C}^+)$ and there exists a global minimizer $\hat{H} \in L_2(i\mathbb{R})$ of the $L_2$ approximation problem, then $\hat{H} \in H_2(\mathbb{C}^+)$. Similarly, if $H \in H_2(\mathbb{C}^-)$ and there exists a global minimizer $\hat{H} \in L_2(i\mathbb{R})$ of the $L_2$ approximation problem, then $\hat{H} \in H_2(\mathbb{C}^-)$.

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Let $\hat{H} \in L_2(i\mathbb{R})$ be the global minimizer of $L_2$ approximation problem. Since $H \in H_2(\mathbb{C}^+)$, one has $H^- = 0$. Seeing that $L_2(i\mathbb{R}) = H_2(\mathbb{C}^-) \oplus H_2(\mathbb{C}^+)$ and this an orthogonal decomposition, thus

$$\|H - \hat{H}\|_{L_2}^2 = \|H^+ - \hat{H}^+\|_{L_2}^2 + \|0 - \hat{H}^-\|_{L_2}^2$$  \hspace{1cm} (33)

Thus, $\hat{H}^- = 0$, otherwise $\hat{H}$ is not a global minimizer.
**Proposition 2**

For every unstable system $H$, there exists a neighborhood $V$ of $H$ such that if $G \in V$, $G$ is also unstable. In other words, the set of unstable systems $(\mathcal{L}_2(i\mathbb{R}) \setminus \mathcal{H}_2(\mathbb{C}^+))$ is open for the $\mathcal{L}_2$ norm.

**Proof.**

Since $\mathcal{H}_2(\mathbb{C}^+)$ is a closed set, its complement $(\mathcal{H}_2(\mathbb{C}^+))^c = \mathcal{L}_2(i\mathbb{R}) \setminus \mathcal{H}_2(\mathbb{C}^+)$ is open. \qed
STABILITY REGIONS ESTIMATION

Results about stability approximation in $\mathcal{L}_2$

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Theorem 1

Given a unstable system $H \in \mathcal{L}_2(i\mathbb{R})\setminus \mathcal{H}_2(\mathbb{C}^+)$, there exists $n \in \mathbb{N}^*$ for which the minimizer $G_k$ of order $k \in \mathbb{N}^*, k > n$, obtained from the $\mathcal{L}_2$-approximation problem is also unstable.

Proof.

Proposition 2 states that if a system is sufficiently close to a unstable system in the $\mathcal{L}_2(i\mathbb{R})$ norm, it is also unstable. Furthermore, the subspace of rational functions which represents the finite LTI systems is dense in $\mathcal{L}_2(i\mathbb{R})$. Hence, for a given LTI unstable system $H \in \mathcal{L}_2(i\mathbb{R})\setminus \mathcal{H}_2(\mathbb{C}^+)$, a sequence $G_k$ of systems of order $k$ which satisfies the tangential interpolation conditions, will converge to $H$. Thus, due to Proposition proposition 2, there exists an order $n \in \mathbb{N}^*$ such that if $k \geq n$, $G_k$ will be unstable as well.
STABILITY REGIONS ESTIMATION

Results about stability approximation in $\mathcal{L}_2$

Theorem 1

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$\square$
STABILITY REGIONS ESTIMATION

Results about stability approximation in $L_2$

**Proposition 3**

For every stable system $H \in \mathcal{H}_2(\mathbb{C}^+)$, there exists a sequence of unstable systems $G_k \in L_2(i\mathbb{R}) \setminus \mathcal{H}_2(\mathbb{C}^+)$, $k \in \mathbb{N}^*$, such that

$$
\|H - G_k\|_{L_2(i\mathbb{R})} \to 0, \quad \text{when } k \to \infty
$$

(34)

In other words, the set $\mathcal{H}_2(\mathbb{C}^+)$ is not an open set of $L_2(i\mathbb{R})$.

**Proof.**

Given $H \in \mathcal{H}_2(\mathbb{C}^+)$, let $h \in \mathcal{H}_2(\mathbb{C}^-)$ be an element such that $\|h\|_{L_2(i\mathbb{R})} = 1$. The system $G_k = H + \frac{1}{k} h \in L_2(i\mathbb{R}) \setminus \mathcal{H}_2(\mathbb{C}^+)$ and $\|H - G_k\|_{L_2(i\mathbb{R})} = \frac{1}{k} \|h\|_{L_2(i\mathbb{R})} \to 0$ when $k \to \infty$. $\square$
STABILITY REGIONS ESTIMATION

Results about stability approximation in $L_2$

**Proposition 3**

For every stable system $H \in \mathcal{H}_2(\mathbb{C}^+)$, there exists a sequence of unstable systems $G_k \in L_2(i\mathbb{R}) \setminus \mathcal{H}_2(\mathbb{C}^+)$, $k \in \mathbb{N}^*$, such that

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Given $H \in \mathcal{H}_2(\mathbb{C}^+)$, let $h \in \mathcal{H}_2(\mathbb{C}^-)$ be an element such that $\|h\|_{L_2(i\mathbb{R})} = 1$. The system $G_k = H + \frac{1}{k} h \in L_2(i\mathbb{R}) \setminus \mathcal{H}_2(\mathbb{C}^+)$ and $\|H - G_k\|_{L_2(i\mathbb{R})} = \frac{1}{k} \|h\|_{L_2(i\mathbb{R})} \to 0$ when $k \to \infty$. \qed

Benchmark TDS-#1

- $\approx 0.13s$ for each approximation
- Approximation of order $r = 6$. 

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Benchmark TDS-#2

- $\tau_1$ fixed as 17ms.
- $\approx 30s$ for approximation
- Approximation of order $r = 12$. 
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Conclusion

- Projection model approximation method using realization.
- Loewner interpolation method using transfer function.
- Method to estimate the stability of large-scale TDS and PDE is proposed.
- Some arguments are given to justify this method.
- No borne of estimation error.

Perspectives

- Algorithm 'branch and bound' to find borders.
- $\mathcal{H}_2$-LPV model reduction.
- LSS reduction to TDS-system.
Thanks for your Attention. Questions ?


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