Towards more general stability analysis of systems with delay-dependent coefficients

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In this work we study a special type of delay systems, namely systems with the delay parameter also appearing in the coefficients.

We consider systems whose linearized dynamics has a characteristic equation of the following form

\[ P(\lambda, \tau) + Q(\lambda, \tau)e^{-\tau \lambda} = 0 \]

where \( \tau \) is restricted to Interval \( I \), \( I = [\tau^l, \tau^u] \).

The objective is to find all the sub-intervals of \( I \) that guarantees the asymptotical stability of the system.
Motivating Examples

The stella dynamo model:

\[
\dot{B}_\phi(t) = c_1 e^{-c_2 T_0} A(t - T_0) - c_2 B_\phi(t)
\]

\[
\dot{A}(t) = c_3 e^{-c_2 T_1} B_\phi(t - T_1) - c_2 A(t)
\]

where \( B_\phi \) is the strength of toroidal field, and \( A \) is the strength of poloidal field, and \( c_1, c_2, c_3, T_0, T_1 \) are positive constants.

The characteristic equation of the above system can be easily obtained as

\[
\lambda^2 + 2c_2 \lambda + c_2^2 - c_1 c_3 e^{-c_2 \tau} e^{-\tau \lambda} = 0,
\]

where \( \tau = T_0 + T_1 \).
A model of hematopoietic stem cell:

\[
\dot{S}(t) = -\delta S(t) + e^{-\delta \tau} \beta(S(t - \tau)) N(t - \tau)
\]

\[
\dot{N}(t) = -\delta N(t) - \beta(S(t)) N(t) + 2e^{-\delta \tau} \beta(S(t - \tau)) N(t - \tau)
\]

The model is nonlinear, and possess two equilibria. The linearized equation in the neighborhood of the nonzero equilibrium has the following characteristic equation

\[
\lambda + A(\tau) - B(\tau)e^{-\lambda \tau} = 0
\]

where \(A, B\) are nonlinear functions of \(\tau\). Therefore delay-dependent coefficients may arise from the linearized dynamics of a nonlinear system.
The idea of the $\tau$—decomposition approach is to regard the time delay as a variable while the system coefficients are fixed.

The general idea:

- Identify the critical delay values corresponding to the imaginary roots
- The delay domain $\mathcal{I}$ are divided into subintervals.
- On the boundary of each subinterval, determine the migration direction of the imaginary roots.
Assumption I

Assumption I. For all \( \tau \in \mathcal{I} \), \( P_\tau \) satisfies

\[
\text{ord}(P_\tau) = n,
\]

where \( \text{ord}(\cdot) \) is the order of the polynomial, and

\[
\lim_{\omega \to \infty} \left| \frac{Q_\tau(j\omega)}{P_\tau(j\omega)} \right| < 1.
\]

Assumption II

No \( (\omega, \tau) \in \mathbb{R}_+ \times \mathcal{I} \) satisfies simultaneously

\[
P(j\omega, \tau) = 0, \quad Q(j\omega, \tau) = 0
\]
Define $F(\omega, \tau) = P(j\omega, \tau)P(-j\omega, \tau) - Q(j\omega, \tau)Q(-j\omega, \tau)$

**Assumption III**

For any $(\omega^*, \tau^*) \in \mathbb{R}_+ \times \mathcal{I}$ that satisfies $D(j\omega^*, \tau^*) = 0$ then

$$\left. \frac{\partial}{\partial \omega} F(\omega, \tau) \right|_{\omega=\omega^*} = 0.$$  

Furthermore, $D(j\omega, \tau) = 0$ admits a finite number of solutions for $(\omega, \tau) \in \mathbb{R} \times \mathcal{I}$

**Assumption IV**

There are only a finite number of $(\omega, \tau)$ in $\mathbb{R}_+ \times \mathcal{I}$ that simultaneously satisfy $F(\omega, \tau) = 0$ and

$$\frac{\partial}{\partial \omega} F(\omega, \tau) = 0.$$
Let \( \tau^{(1)}, \ldots, \tau^{(L)} \) be all the delay values contained in \( \mathcal{I} \) that satisfy

\[
F(\omega, \tau) = 0, \partial_\omega F(\omega, \tau) = 0
\]

for some real \( \omega \).

**Proposition**

The number of real roots of \( F(\omega, \tau) = 0 \) in \( \omega \) are the same for all \( \tau \in \mathcal{I}_o^{(i)} = (\tau^{(i)}, \tau^{(i+1)}) \), and they are all simple. These real simple roots are continuously differentiable functions of \( \tau \), and may be expressed as \( \pm \omega_k^{(i)}(\tau) \), \( k = 1, 2, \ldots, m^{(i)} \), where \( m^{(i)} \leq n \), and \( \omega_k^{(i)}(\tau) > 0 \) for all \( \tau \in \mathcal{I}_o^{(i)} \).

- The signature of \( \partial_\omega F(\omega_k^{(i)}(\tau), \tau) \) is constant in each subinterval \( \mathcal{I}^{(i)} \).
Recall the characteristic equation is $P + Q e^{-\lambda \tau} = 0$

Now for each subinterval $I^{(i)}$ and some critical frequency function $\omega^{(i)}_k$ we define the corresponding phase function:

$$\theta^{(i)}_k(\omega^{(i)}_k(\tau), \tau) = \angle P(j\omega^{(i)}_k \tau, \tau) - \angle Q(j\omega^{(i)}_k \tau, \tau) + \omega \tau + \pi$$

Then it is easy to see that the sufficient and necessary condition for $j\omega^*$ to be an imaginary root when $\tau = \tau^* \in I^{(i)}$ is that for some integer $k$ and $l$

$$\omega^{(i)}_k(\tau^*) = \omega^*, \theta^{(i)}_k(\tau^*) = 2l\pi$$
For each critical delay $\tau^*$, and imaginary root $\lambda = j\omega^*$

We define

$$\text{Inc}(\tau^*, \omega^*) = \frac{\text{sgn}(\Re(\lambda(\tau^*+))) - \text{sgn}(\Re(\lambda(\tau^-)))}{2}.$$ 

for all $\tau^* \neq \tau^l$. If $\tau_1 = \tau^l$ we define instead

$$\text{Inc}(\tau_1, \omega^*) = \text{sgn}(\Re(\lambda(\tau^*+)))$$

Further for each critical delay $\tau_l$, define

$$\text{Inc}(\tau_l) = 2 \sum_{h=1}^{H_l} \text{Inc}(\tau^*, \omega_{lh})$$

where $\omega_{lh}$, $h = 1, ..., H_l$ are all the non-negative frequencies of the imaginary roots corresponding to $\tau_l$. 
Now we can count the number of unstable roots for delay $\tau$ as

$$N^u(\tau) = N^u(\tau^l) + \sum_{l=1}^{L(\tau)} \text{Inc}(\tau_l)$$

where $L(\tau)$ is the largest number such that $\tau_{L(\tau)} < \tau$.

It is easy to see that if $\Re(\lambda'_{lh}(\tau))_{\tau=\tau_l} \neq 0$, then the following holds:

$$\text{Inc}(\tau_l, \omega_{lh}) = \begin{cases} 
\text{sgn} \left( \Re(\lambda'_{lh}(\tau))_{\tau=\tau_l} \right), & \text{if } \tau_l > \tau^l, \\
\max \left\{ 0, \text{sgn} \left( \Re(\lambda'_{lh}(\tau))_{\tau=\tau_l} \right) \right\}, & \text{if } \tau_l = \tau^l,
\end{cases}$$
Lemma

\[ \partial_\omega F(\omega_k^{(i)}(\tau), \tau) \neq 0 \text{ implies } \partial_\lambda D(j\omega_k^{(i)}(\tau), \tau) \neq 0 \]

It follows from this lemma that if \( \tau^* \) is a critical delay and \( j\omega^* \) is a corresponding imaginary root, then for \( \tau \) in a neighbourhood of \( \tau^* \), \( \lambda \) can be viewed as a function of \( \tau \) denoted as \( \lambda(\tau) \).

Theorem

Let \( (\omega^*, \tau^*) \), \( \omega^* \in \mathbb{R}, \tau^* \in I \), satisfies \( D(j\omega^*, \tau^*) = 0 \). Let \( i, k \) be the integers such that \( \tau^* \in I^{(i)}, \omega_k^{(i)}(\tau^*) = \omega^* \)

\[
\text{sgn} \left( \Re \left( \frac{d\lambda}{d\tau} \right)_{\tau=\tau^*} \right) = \text{sgn} \left( \frac{\partial}{\partial \omega} F(\omega, \tau) \right) \times \text{sgn} \left( \frac{d\theta_k^{(i)}(\tau)}{d\tau} \right)_{\tau=\tau^*}^{\omega=\omega^*}
\]
We summarize our stability analysis procedure.

- Solve $F(\omega, \tau) = 0$ and $\partial_\omega F(\omega, \tau) = 0$ to obtain $\tau^{(1)}, \ldots, \tau^{(L)}$ and thus $\mathcal{I}^{(i)}$, $i = 1, \ldots, L$.

- In each sub-interval $\mathcal{I}^{(i)}$, solve $\theta_k^{(i)}(\tau) = 2l\pi$, for some integer $l$ and thus obtain all the critical delay values $\tau_1, \tau_2, \ldots, \tau_H$.

- At each critical delay value $\tau_l$, apply our root crossing criterion to obtain $\text{Inc}(\tau_l)$.

- For any $\tau \in \mathcal{I}$, count the number of unstable roots using

$$N^u(\tau) = N^u(\tau^l) + \sum_{l=1}^{L(\tau)} \text{Inc}(\tau_l)$$
Consider the stability of the stellar dynamos model with the delay interval $\mathcal{I} = [0, 2]$. The characteristic equation can be written as

$$\lambda^2 + 2c_2\lambda + c_2^2 - c_1c_3e^{-c_2\tau}e^{-\lambda\tau} = 0. \quad (0)$$

We have:

$$F(\omega, \tau) = \omega^4 + 2c_2^2\omega^2 + c_2^4 - c_1^2c_3^2e^{-2c_2\tau},$$

$$\frac{\partial}{\partial\omega} F(\omega, \tau) = 4\omega(\omega^2 + c_2^2).$$

Solving these equations, we have $\tau^{(1)} \approx 1.006.. \mathcal{I}^{(1)} = [0, \tau^{(1)}]$, $\mathcal{I}^{(2)} = [\tau^{(1)}, \tau^{(2)}]$, $\tau^{(2)} = 2$. 
$ω_1^{(1)}$ is the only positive frequency curve defined in $I^{(1)}$. The phase angle curve $θ_1^{(1)}$ intersects the horizontal line $2\pi$ at $τ_1 \approx 0.2748$ and $τ_2 \approx 0.5314$. Therefore, $H_1 = 1$, $ω_{11} = ω_1^{(1)}(τ_1)$, and $H_2 = 1$, $ω_{21} = ω_1^{(1)}(τ_2)$. 
Future research directions

- Our results indicate a strong correlation between roots crossing the imaginary axis and the phase curve crossing $2l\pi$. We want to find the essential reason behind this interesting correlation.
- Extend our root migration direction criterion to the case where $\Re\left(\frac{d}{d\tau}\lambda(\tau^*)\right) = 0$. High-order analysis is necessary.
- Extend the analysis approach for systems of more general forms.