Converse Lyapunov–Krasovskii theorems for uncertain retarded differential equations ¹

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Outline

- Retarded Functional Differential Equation (RFDE)
- Switching system approach
- Results
- Conclusion
Consider the following Retarded Functional Differential Equation (RFDE)

\[ (\Sigma) \quad \dot{x}(t) = L(t)x_t \quad t \geq 0, \]

where

- \( x(t) \in \mathbb{R}^n \) : the system state at time \( t \)
- \( x_t : \theta \mapsto x(t + \theta), \quad \theta \in [-r, 0] \) : the history function
- \( x_0 = \varphi \in X \) : an initial condition
- \( L : [0, +\infty) \to \mathcal{L}(X, \mathbb{R}^n) \) : a bounded linear operator
Typical examples

1. \[ \begin{align*} \dot{x}(t) &= A_0 x(t) + A_1 x(t - \tau(t)) \quad t \geq 0 \\ x(\theta) &= \varphi(\theta), \quad \theta \in [-r, 0] \end{align*} \]

for some $n \times n$ matrices $A_0$ and $A_1$ and $\tau : [0, +\infty) \to [-r, 0]$.

2. \[ \begin{align*} \dot{x}(t) &= \int_0^r A(t, \theta) x(t - \theta) d\theta, \quad t \geq 0 , \\ x(\theta) &= \varphi(\theta), \end{align*} \]

$A(t, \theta)$ is a $n \times n$ matrix uniformly bounded with respect to $t$ and $\theta \in [0, r]$ and measurable with respect to $\theta$. 
Existence and uniqueness of a solution

- \( X = C([-r, 0], \mathbb{R}^n) \) or \( X = H^1([-r, 0], \mathbb{R}^n) \)

- \( L(\cdot)\varphi : t \mapsto L(t)\varphi \) is a measurable function \( \forall \varphi \in C([-r, 0], \mathbb{R}^n) \)

- there exists a positive constant \( m \) such that

\[
(K) : \quad |L(t)\varphi| \leq m\|\varphi\|_C \quad \forall \varphi \in C([-r, 0], \mathbb{R}^n)
\]

Lemma

Consider the linear RFDE given by system \((\Sigma)\). Let \( X \) be the Banach spaces \( C([-r, 0], \mathbb{R}^n) \) or \( H^1([-r, 0], \mathbb{R}^n) \). Assume that condition \((K)\) holds. For every \( \varphi \in X \) there exists a unique solution of \((\Sigma)\) with initial condition \( \varphi \).
Three principal approaches

- Lyapunov–Krasovskii: consists of finding a positive functional that decays along the trajectories of the considered systems.


- Barnea: consists in reducing the stability problem to an optimization problem.
Lyapunov–Krasovskii Theorem

Let $u, v, w : [0, +\infty) \rightarrow [0, +\infty)$ are continuous nondecreasing functions, $u(s)$ and $v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$. If there exists a continuous function $V : C([-r, 0], \mathbb{R}^n) \rightarrow \mathbb{R}$ such that

$$u(|\varphi(0)|) \leq V(\varphi) \leq v(\|\varphi\|_C)$$

$$\overline{DV}(\varphi) \leq -w(|\varphi(0)|)$$

then the solution $x = 0$ of equation (2) is uniformly stable. If $w(s) > 0$ for $s > 0$, then the solution $x = 0$ is exponentially stable.

$$\overline{DV}(\varphi) = \limsup_{t \to 0} \frac{V(x_t(\varphi)) - V(\varphi)}{t}$$

$$\underline{DV}(\varphi) = \liminf_{t \to 0} \frac{V(x_t(\varphi)) - V(\varphi)}{t}$$
\[ x(k+1) = A_0 x(k) + A_1 x(k - \tau(k)), \quad 0 < \tau(k) \leq m \]

Let

\[ z(k) = [x^T(k), \ldots, x^T(k - m)]^T \quad \text{and} \quad \sigma : \mathbb{Z}^+ \to \mathbb{S} = \{1, \ldots, m\} \]

\[ z(k + 1) = \bar{A}_{\sigma(k)} z(k) \quad \text{with} \quad \sigma(k) = \tau(k) \]

where the matrix \( \bar{A}_{\sigma(k)} \) switches in the set of possible matrices \( \{\bar{A}_1, \cdots, \bar{A}_m\} \)

\[ \bar{A}_i = \begin{pmatrix}
A_0 & 0 & \cdots & 0 & A_1 & 0 & \cdots & 0 \\
I & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & I & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & I & 0 & \cdots
\end{pmatrix}. \]

We parametrize the operator $t \mapsto L(t)$

- Let $\mathbb{S}$ be an index set (which can be uncountable).
- Let $\sigma(\cdot) : [0, +\infty) \rightarrow \mathbb{S}$ be a measurable signal
- $\sigma(\cdot)$ parametrizes $(\Sigma)$
  $$\dot{x}(t) = L_{\sigma(t)}x_t,$$
- There exists a positive constant $m$ such that
  $$(K) : \quad |L_{\sigma}\varphi| \leq m\|\varphi\|_C \quad \forall \varphi \in C([-r, 0], \mathbb{R}^n), \sigma \in \mathbb{S}$$
With any $\sigma \in S$
\[
\dot{x}(t) = L_\sigma x_t,
\]

one can associate a $C_0$-semigroup
\[
T_\sigma(t) : X \to X \quad \text{defined by} \quad T_\sigma(t)(\phi) = x_t
\]

with infinitesimal generator $A_\sigma$ given by
\[
D(A_\sigma) = \left\{ \phi \in X : \frac{d\phi}{d\theta} \in X, \frac{d\phi}{d\theta}(0) = L_\sigma \phi \right\},
\]
\[
A_\sigma \phi = \frac{d\phi}{d\theta}.
\]
The evolution operator corresponding to a piecewise constant signal

\[ \sigma(t) = \sum_{k \geq 0} 1_{[t_k, t_{k+1})}(t) \sigma_k \]

with \( t_0 = 0 \), \( t_k < t_{k+1} \) for \( k \geq 0 \) is given by

\[ T_{\sigma(\cdot)}(t) = T_{\sigma_k}(t - t_k)T_{\sigma_{k-1}}(t_k - t_{k-1})...T_{\sigma_0}(t_1 - t_0) \quad t \in [t_k, t_{k+1}). \]

The evolution is then given by the following switched system

\[ (\Sigma) \rightarrow (\Sigma_s) : \]

\[ x_t = T_{\sigma(\cdot)}(t)x_0, \]

\[ x_0 = \varphi \in X. \]
Theorem (F.M. Hante and M. Sigalotti)


The conditions

(i) there exist \(M \geq 1\) and \(w > 0\) such that

\[
\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq Me^{wt}, \quad t \geq 0, \quad \sigma(\cdot)\text{-uniformly},
\]

(ii) there exists a function \(V : X \to [0, \infty)\) such that \(\sqrt{V(\cdot)}\) is a norm on \(X\),

\[
V(\varphi) \leq c\|\varphi\|^2_X
\]

for some constant \(c > 0\) and

\[
\overline{D}_{\sigma}V(\varphi) \leq -\|\varphi\|^2_X, \quad \sigma \in \mathcal{S}, \varphi \in X.
\]

are necessary and sufficient for the existence of constants \(K \geq 1\) and \(\mu > 0\) such that

\[
\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq Ke^{-\mu t}, \quad t \geq 0, \quad \sigma(\cdot)\text{-uniformly}.
\]
**Uniform exponential boundedness**

**Lemma**

Suppose that condition (K) holds. If \( X = C([-r, 0], \mathbb{R}^n) \) or \( H^1([-r, 0], \mathbb{R}^n) \) then the solutions of \((\Sigma_s)\) are \( \sigma(\cdot) \)-uniformly exponentially bounded.

**Proof.**

1. **case** \( X = C([-r, 0], \mathbb{R}^n) \).

   By integrating system \((\Sigma)\) and using equation (K), one has for every \( t \geq 0 \)

   \[
   \|x_t\|_C \leq \|\varphi\|_C + m \int_0^t \|x_s\|_C ds.
   \]

   Thanks to Gronwall’s Lemma, we have

   \[
   \|x_t\|_C \leq \|\varphi\|_C e^{mt}. \tag{1}
   \]

2. **case** \( X = H^1([-r, 0], \mathbb{R}^n) \).

   Same reasoning + Poincaré Inequality.
First converse theorem

**Theorem**

Suppose that condition \((K)\) holds. System \((\Sigma_s)\) is uniformly exponentially stable in \(X\), if and only if there exists a function \(V : X \rightarrow [0, \infty)\) such that \(\sqrt{V(\cdot)}\) is a norm on \(X\),

\[
V(\varphi) \leq c\|\varphi\|_X^2,
\]

for some constant \(c > 0\) and

\[
D_\sigma V(\varphi) \leq -\|\varphi\|_X^2, \quad \sigma \in S, \varphi \in X.
\]
Lemma (F.M. Hante and M. Sigalotti$^a$)


Assume that

(i) there exist $M \geq 1$ and $w > 0$ such that

$$
\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq Me^{wt}, \quad t \geq 0, \ \sigma(\cdot)\text{-uniformly},
$$

(ii) there exist $c \geq 0$ and $p \in [1, +\infty)$ such that

$$
\int_0^{+\infty} \|T_{\sigma(\cdot)}(t)x\|_X^p \leq c\|x\|_X^p, \ \sigma(\cdot)\text{-uniformly},
$$

for every $x \in X$.

Then there exist $K \geq 1$ and $\mu > 0$ such that

$$
\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq Ke^{-\mu t}, \quad t \geq 0, \ \sigma(\cdot)\text{-uniformly}.
$$
Theorem

Suppose that condition \((K)\) holds. Then system \((\Sigma_s)\) is uniformly exponentially stable in \(X\) if and only if there exists a continuous function \(V : X \to [0, +\infty)\) such that
\[
V(\varphi) \leq c\|\varphi\|_X^2,
\]
for some constant \(c > 0\) and
\[
\mathcal{D}_\sigma V(\varphi) \leq -|\varphi(0)|^2, \sigma \in \mathbb{S}, \varphi \in X.
\]
Proof

1. \[ V(x_t) - V(x_0) \leq - \int_0^t |x_s(0)|^2 ds \]

2. \[ \int_0^\infty |x_s(0)|^2 ds \leq c \| \varphi \|^2_X \]

3. \[ \int_0^t \| x_s \|^2_{H^1} ds \leq c_1 \int_0^t |x_s(0)|^2 ds + c_2 \| \varphi \|^2_{H^1} ds, \]

4. \[ \int_0^\infty \| x_t \|^2_{H^1} ds \leq c_0 \| \varphi \|^2_{H^1}, \]
Extension to measurable cases

\[ Q := \{ L_\sigma \in \mathcal{L}(X, \mathbb{R}^n) \mid \sigma \in \mathcal{S} \}. \]

**Theorem**

*System* (\( \Sigma \)) *is uniformly exponentially stable* for \( L : [0, +\infty) \to Q \) *such that* \( L(\cdot)\varphi \) *is measurable* for any \( \varphi \in X \) *if and only if it is uniformly exponentially stable* for \( L \in PC([0, +\infty), Q) \).
Proof

Lemma

System $(\Sigma)$ is uniformly exponentially stable for $L : [0, +\infty) \rightarrow Q$ such that $L(\cdot)\varphi$ is measurable for any $\varphi \in C([-r, 0], \mathbb{R}^n)$ if and only if it is uniformly exponentially stable for $L \in PC ([0, +\infty), Q)$.

Lemma

Suppose that condition $(K)$ holds. The following two statements are equivalent:

(i) System $(\Sigma)$ is uniformly exponentially stable in $C([-r, 0], \mathbb{R}^n)$.

(ii) System $(\Sigma)$ is uniformly exponentially stable in $H^1([-r, 0], \mathbb{R}^n)$. 
In this work we give a collection of converse Lyapunov–Krasovskii theorems for uncertain retarded functional differential equations.

The first converse Theorem shows that the existence of a squared norm $V(\cdot)$ on $C([-r,0],\mathbb{R}^n)$ is a necessary and sufficient condition for the uniform exponential stability of system ($\Sigma$).

By the second converse theorem the assumption that $V(\cdot)$ is a squared norm is dropped.

One of the novelties of our results is that these functionals may not have a strictly positive norm-dependent lower bound, in contrast with what is known in the literature.
Thank you for your attention