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Distributed event-triggered consensus of multi-agent systems with measurement noise and guaranteed interval bounds

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Abstract: A distributed event-triggered method is developed to reach the consensus with bounded-error measurements. The approach is derived from an initial event-triggered consensus scheme developed in Seyboth et al. (2013). The strategies are presented for single and double-integrator models, considering a fully connected graph. Proofs of convergence to a ball centered at the average consensus value are given. The presence of Zeno behavior is excluded. Guaranteed bounds for consensus are characterized. Results are illustrated with numerical applications.

Keywords: multi-agent systems, consensus, event-triggered control, distributed, measurement noise, bounded-error measurement, guaranteed enclosure.

1. INTRODUCTION

Consensus determination constitutes a major topic of multi-agent coordination Olfati-Saber and Murray (2004); Ren and Atkins (2007). Moreover, due to exchange limitation, it is required that reaching the consensus should be performed using limited exchange of information. Event-triggered approaches have been designed to tackle this issue for simple (Dimarogonas et al. (2012); Seyboth et al. (2013)) or more general dynamics (Garcia et al. (2014); Viel et al. (2016)). Most of the developed techniques do not address the problem of noise in exchanged information. However, existing methods consist in designing proper stochastic approximation type consensus protocols to reduce the noise effect (Huang and Manton (2009)). This early work allows Hu et al. (2015) to introduce event-triggered mechanism for mean-consensus. In Ge et al. (2017), a set-membership leader-follower event-triggered consensus is described that provide bounding ellipsoidal set containing the states of all followers of a leader using recursive convex optimization.

In this paper, we consider bounded uncertainties on the state estimates of agents. An event-triggered strategy is described to enable the MAS to converge to a guaranteed consensus region that should be updated during the trajectory. This work is inspired from Seyboth et al. (2013). Section 3 presents the problem definition. Because of the noise, the self-error from true state is no more accessible for agent, then new *communication triggering conditions* (CTCs) based on error's bound are introduced for single-integrator (Section 4) and double-integrator (Section 5) dynamics. The presence of bounded noise results in defining a region of consensus contrary to classical noise free case. Thus, Sections 4.3 and 5.3 expose strategies to obtain guaranteed interval bounds for consensus values from the communication mechanism and using lower and upper bounding dynamical systems for the uncertain systems (Kieffer and Walter (2006)). Simulations of theoretical results are exposed for each type of dynamics. Finally, conclusions are drawn in Section 6.

2. PRELIMINARIES

2.1 Graph Theory

The interaction topology of a network of N agents is represented using a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with the set of nodes $\mathcal{V} = \{1, 2, ..., N\}$ and edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. $\mathcal{E}_{ij}$ is the edge between the nodes i and j, with $i, j = 1, ..., N$. The adjacency matrix A is defined by $a_{ij} = 1$ if i and j are adjacent and $a_{ij} = 0$ otherwise. The set of neighbors of a node i is $\mathcal{N}_i = \{j \in \mathcal{V} | (i, j) \in \mathcal{E}, i \neq j\}$ and N_i its cardinal number. If there is a path from i to j, then i and j are called connected. If all pairs of nodes in $\mathcal{G}$ are connected, then $\mathcal{G}$ is called *connected*. The degree matrix D of $\mathcal{G}$ is the diagonal matrix with elements d_i equal to the cardinality of node i's neighbor set N_i. The Laplacian matrix L of $\mathcal{G}$ is defined as $L = D - A$. L is symmetric and positive semi-definite iff $\mathcal{G}$ is undirected. Moreover, in that case, every row and every column of L sum to zero, which means L satisfies $L 1_N = 0$. L has only one null eigenvalue $\lambda_1(L)$, and all its other non-zero eigenvalues $\lambda_2(L) \leq \lambda_3(L) \leq \ldots \leq \lambda_N(L)$ are strictly positive.

Lemma 1. from Seyboth et al. (2013). Suppose L is the Laplacian of an undirected, connected graph $\mathcal{G}$. Then for all $t \geq 0$ and all vectors $v \in \mathbb{R}^N$ with $\mathbf{1}^T v = 0$, it holds that $\left\| e^{-Lt} v \right\| \leq e^{-\lambda_2(L)t} \|v\|$.

Lemma 2. from Seyboth et al. (2013). Suppose L is the Laplacian of an undirected, connected graph $\mathcal{G}$. Define $\mu > 0$ and $\Gamma = \begin{bmatrix} 0 & I_N \\ -L & -\mu L \end{bmatrix}$ Then, for all $t \geq 0$ and all vectors $v \in \mathbb{R}^{2N}$ with $[\mathbf{1}^T \, \mathbf{0}^T] v = [\mathbf{0}^T \, \mathbf{1}^T] v = 0$, it holds that $\left\| e^{\Gamma t} v \right\| \leq e^{Re(\lambda_3(\Gamma))t} c_v \|v\|$, with $c_v = \left\| V^{-1} \right\| \|V\|$ and where V is a non singular matrix defined as in Seyboth et al. (2013) from the eigen vectors of Γ.

2.2 Interval Analysis

A real interval $[x]$ is a connected subset of $\mathbb{R}$. The *lower bound* $lb([x])$ of an interval $[x]$, also denoted by $\underline{x}$, is defined as $\underline{x} = lb([x]) \stackrel{\Delta}{=} \sup \{a \in \mathbb{R} \cup \{-\infty, \infty\} \mid \forall x \in [x], \, a \leq x\}$. Its *upper bound* $ub([x])$], also denoted $\overline{x}$, is defined as

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$\overline{x} = ub([x]) \triangleq \inf\{b \in \mathbb{R} \cup \{-\infty, \infty\} \mid \forall x \in [x], x \leq b\}$. The *width* of any non-empty interval $[x]$ is $w([x]) = \overline{x} - \underline{x}$.

3. PROBLEM DEFINITION

Consider a multi-agent system (MAS) of N agents whose communication topology can be described by an *undirected fully-connected* and *time-invariant* graph $\mathcal{G}$. It is also assumed that there is *no communication delay*.
Let x_i the state of each Agent $i \in \mathcal{V}$ and u_i its control input. It is assumed that each Agent i is able to compute or has access to an estimate $\hat{x}_i$ of its own state x_i. Let t_k^i be the time instant at which the k-th message is broadcast by Agent i. This message is composed of its state's estimate:

$$\hat{x}_i(t) = x_i\left(t_k^i\right) + w_i(t_k^i), \quad \forall t \in \left[t_k^i, t_{k+1}^i\right[\tag{1}$$

where w_i is some additive bounded noise. $\hat{x}_i$ kept constant between two communication instants and equal to the last broadcast value. It is also assumed that Agent i is able to broadcast $\hat{x}_i$ to all other neighbor agents j, $j \in \mathcal{N}_i$. Due to the presence of noise in the state estimates, only a bounded consensus of the MAS can be obtained, that is $\exists \epsilon > 0$ s.t. $\lim_{t \to +\infty} \|x_i(t) - x_j(t)\| \leq \epsilon, \forall (i,j) \in \mathcal{E}$.
Define $\tilde{w}$ as:

$$\tilde{w} = \sup_{i \in \mathcal{V}} \left(\sup_{k \in \mathbb{N}^*} |w_i\left(t_k^i\right)| \right) \tag{2}$$

The trigger function $f_i(.)$ for Agent i takes values in $\mathbb{R}$ and relies on local information only. A communication is triggered when the CTC $f_i(.) > 0$ is satisfied. As soon as an agent sends its state value or receives other agent state value, it recomputes its control u_i immediately.

Problem: The problem addressed here is to design distributed control laws and CTCs requiring only local information available to allow the MAS to achieve a bounded consensus. It also provides boundary values on the consensus reflecting the bound on the noise and the broadcast strategy.

4. SINGLE-INTEGRATOR AGENTS

4.1 Consensus & communication triggering condition

Consider first the noise-free case *i.e* $w_i(t) = 0$, for agents with single-integrator dynamics described as

$$\dot{x}_i(t) = u_i(t) \tag{3}$$

with $i \in \mathcal{V}$, where $x_i(t) \in \mathbb{R}$ is the state of Agent i and $u_i(t) \in \mathbb{R}$ its control input. The dynamics of the MAS can be described in a matrix form as $\dot{x}(t) = u(t)$, with the state vector $x(t) = [x_1(t), ..., x_N(t)]^T$, $x(0) = x_0 \in \mathbb{R}^N$ and the control vector $u(t) = [u_1(t), ..., u_N(t)]^T$. Seyboth et al. (2013) proposed an event-based implementation,

$$u_i(t) = \sum_{j \in \mathcal{N}_i} (\hat{x}_j(t) - \hat{x}_i(t)), \tag{4}$$

or in matrix form $u(t) = -L\hat{x}(t)$, where Agent i does not use its true state $x_i(t)$ but the last broadcast estimate $\hat{x}_i(t)$ to guarantee the zero average of the control $u(t)$ so that a consensus can be achieved. As the MAS evolve, the state x_i will drift from the last broadcast value $\hat{x}_i$. Therefore, an estimation error e_i can be defined for Agent i as

$$e_i(t) = \hat{x}_i(t) - x_i(t). \tag{5}$$

Let $e(t) = [e_1(t), ..., e_N(t)]^T$ be the collection of MAS errors. Let us recall from Olfati-Saber and Murray (2004) the disagreement vector of the MAS $\delta(t)$:

$$\delta(t) = x(t) - a(t)\mathbf{1}_N, \tag{6}$$

with $a(t) = (1/N)\,\mathbf{1}_N^T x(t)$, so that $\mathbf{1}_N^T \delta(t) = 0$.
In case there are no measurement noises, a CTC is defined in *Theorem 3.2* of Seyboth et al. (2013) as

$$f_i(t, e_i(t)) = |e_i(t)| - \left(c_0 + c_1 e^{-\alpha t}\right) > 0, \tag{7}$$

with constants $c_0 \geqslant 0$, $c_1 \geqslant 0$, $c_0 + c_1 > 0$, and $0 < \alpha < \lambda_2(L)$. Then, for all initial condition $x_0 \in \mathbb{R}^N$, the disagreement vector δ of the closed-loop system converges to a ball centered at the origin with radius $r = \|L\| \sqrt{N} c_0 / \lambda_2(L)$. Moreover, the closed-loop system does not exhibit Zeno behavior.

4.2 Presence of unknown but bounded measurement noise

Consider now that Agent i's state estimate is defined as (1). Let us define a bound on the error, that can be evaluated by Agent i, as

$$\tilde{e}_i(t) = \tilde{w} + \int_{t_k^i}^{t} |u_i(s)|\, \mathrm{d}s, \quad \forall t \in \left[t_k^i, t_{k+1}^i\right[. \tag{8}$$

Based on this definition, let us also introduce the following new communication triggering condition

$$f_i(t, \tilde{e}_i(t)) = \tilde{e}_i(t) - \left(c_0 + c_1 e^{-\alpha t}\right) > 0 \tag{9}$$

with constants $c_0 > \tilde{w} \geqslant 0$, $c_1 \geqslant 0$, $c_0 + c_1 > 0$, and $0 < \alpha < \lambda_2(L)$.

Theorem 3. Consider the multi-agent system (3), with control law (4) and state estimate (1) for each Agent i of the MAS. If communications are triggered when the CTC (9) is verified then, for all initial conditions $x_0 \in \mathbb{R}^N$, a bounded consensus is obtained for the MAS and the disagreement vector δ converges to a ball centered at the origin with radius $r = \|L\| \sqrt{N} c_0 / \lambda_2(L)$. Moreover the MAS does not exhibit Zeno behavior.

Proof. From (6), one has $\dot{\delta}(t) = \dot{x}(t) - \dot{a}(t)\mathbf{1}_N$. Assuming that the same information is received by all its neighbor agents when each Agent i broadcasts a message (no time delay, no packet loss) and that the communication graph $\mathcal{G}$ is balanced, then it can be shown that $\sum_{i=0}^{N} u_i = 0$. In this case, $\dot{a}(t) = 0$ and $\dot{\delta}(t) = \dot{x}(t)$. Since $\dot{x}(t) = -L\hat{x}(t) = -L(x(t) + e(t))$, one obtains

$$\dot{\delta}(t) = -L\delta(t) - Le(t) \tag{10}$$

and thus $\delta(t) = e^{-Lt}\delta(0) - \int_0^t e^{-L(t-s)} Le(s)\, \mathrm{d}s$. Knowing the estimation error is bounded by the trigger function due to the CTC, $|e_i(t)| \leq \tilde{e}_i(t) \leq (c_0 + c_1 e^{-\alpha t})$ and using Lemma 1, an upper bound for the disagreement vector δ can be obtained similarly to Seyboth et al. (2013):

$$\begin{aligned} \|\delta(t)\| \leq & e^{-\lambda_2 t} \left(\|\delta(0)\| - \|L\| \sqrt{N} \left(\frac{c_0}{\lambda_2} + \frac{c_1}{\lambda_2 - \alpha} \right) \right) \\ & + e^{-\alpha t} \frac{\|L\| \sqrt{N} c_1}{\lambda_2 - \alpha} + \frac{\|L\| \sqrt{N} c_0}{\lambda_2} \\ \leq & \|\delta(0)\| + \|L\| \sqrt{N} \left(\frac{c_0}{\lambda_2} + \frac{c_1}{\lambda_2 - \alpha} \right) = \tilde{\delta} \end{aligned} \tag{11}$$

From (5), we have $e_i(t) = \hat{x}_i(t) - x_i(t) = x_i\left(t_k^i\right) + w_i\left(t_k^i\right) - x_i(t)$. The aim here is to show that the inter-event times can be lower bounded by a strictly positive constant τ. If i triggers at time $t^* > 0$, then $e_i(t^*) = w_i(t^*)$,

$\tilde{e}_i(t^*) = \tilde{w}$ and $f_i(t^*, \tilde{e}_i(t^*)) \le 0$. Knowing this and the time-derivative of $e_i(t)$ given by $\dot{e}_i(t) = -\dot{x}_i(t) = -u_i(t)$, for t between two event times,

$$|e_i(t)| \le |w_i(t^*)| + \int_{t^*}^{t} |u_i(s)| \,\mathrm{d}s \le \tilde{w} + \int_{t^*}^{t} |u_i(s)| \,\mathrm{d}s = \tilde{e}_i(t) \tag{12}$$

the control is bounded by (Seyboth et al. (2013))

$$|u_i(t)| \le \|u(t)\| \le \|L\| \left(\tilde{\delta} + \sqrt{N}(c_0 + c_1)\right) = \tilde{u}. \tag{13}$$

We assume here $c_0 \neq 0$. For $t \ge t^*$ and before the next event time, (12) and (13) gives $|e_i(t)| \le \tilde{e}_i(t) \le \tilde{w} + (t - t^*)\tilde{u}$. The event is triggered when CTC is satisfied *i.e.* not before $\tilde{e}_i(t) > c_0$, and therefore not before $\tilde{w} + (t - t^*)\tilde{u} > c_0$. For $c_0 > \tilde{w}$, the inter-event times are lower bounded by $\tau(t^*) = \frac{c_0 - \tilde{w}}{\tilde{u}} > 0$. This bound holds for all event times t^* and all agents i. Thus we get the condition

$$c_0 > \tilde{w} \ge 0 \tag{14}$$

for exclusion of the Zeno behavior. Moreover, since $e(t)$ is piecewise continuous and the right hand side of (10) is globally Lipschitz in δ, existence and uniqueness of the solution is guaranteed as said in Khalil (1996). Thus, $\|\delta(t)\|$ converges exponentially to a ball of radius $r = \|L\| \sqrt{N} \frac{c_0}{\lambda_2}$ as $t \to \infty$.

4.3 Guaranteed bounds on the state of the MAS and on the consensus

Assuming that the bound $\tilde{w}$ is known by all the agents, this section aims at exploiting this information along with the constraint imposed by the CTC to derive bounds on the state of the agents and on the consensus value. These bounds are computed in a distributed way by all the agents, and updated dynamically based on the application of Muller's theorem (Müller (1927)), by Kieffer and Walter (2006) between two triggering instants inside the MAS.

Let us also define by K the index of the K-th message broadcast in the MAS, whatever the broadcasting agent. Consider $t \in [t_K, t_{K+1}[$. Each Agent i of the MAS will compute a lower bound $\underline{x}^i(t)$ and an upper bound $\overline{x}^i(t)$ of $x(t)$ using

$$\dot{\underline{x}}^i(t) = -L\underline{x}^i(t), \quad \dot{\overline{x}}^i(t) = -L\overline{x}^i(t) \tag{15}$$

At the trigger instant t_K, Agent i updates its control input u_i and the j-th component of $\underline{x}^i$ and $\overline{x}^i$ by using the information $\hat{x}_j(t_K)$ sent by the Agent j that triggered the communication at t_K:

$$\underline{x}^i_j(t_K) = \hat{x}_j(t_K) - h(t_K), \quad \overline{x}^i_j(t_K) = \hat{x}_j(t_K) + h(t_K) \tag{16}$$

with $h(t_K) = c_0 + c_1 e^{-\alpha t_K}$. Since the graph $\mathcal{G}$ is fully connected and there is no communication delay, using this proposed protocol, all the agents update simultaneously the same component of the lower and upper bound of x. Therefore the $\underline{x}^i$ will be identical at t_K and $\forall t \in [t_K, t_{K+1}[$ for all the Agents i of the MAS. The same consideration stands for the upper bound $\overline{x}^i$. Let us denote by $\underline{x}(t)$ and $\overline{x}(t)$ these vectors over $[t_K, t_{K+1}[$. It is assumed that all agents trigger a communication at the initial instant $t = 0$. Consider the CTC (9) of triggering Agent j. At time t_K one has $\tilde{e}_j(t_K) - h(t_K) \le 0$. Since $|e_j(t)| \le \tilde{e}_j(t)$, one can deduce $-h(t_K) \le e_j(t_K) \le h(t_K)$. Recalling that $e_j(t) = \hat{x}_j(t) - x_j(t)$, one obtains

$$\underline{x}_j(t_K) = \hat{x}_j(t_K) - h(t_K) \le x_j(t_K) \le \hat{x}_j(t_K) + h(t_K) = \overline{x}_j(t_K) \tag{17}$$

Therefore

$$\underline{x}(t_K) \le x(t_K) \le \overline{x}(t_K). \tag{18}$$

In addition, it can be shown that (15) define a lower and upper dynamical system of $\dot{x} = -L\hat{x}$, in the sense that

$$\dot{\underline{x}} = -L\underline{x} \le \dot{x} = -L\hat{x} \le \dot{\overline{x}} = -L\overline{x} \tag{19}$$

where the inequalities are interpreted component by component. Using Muller's Theorem between two triggering instants, one can conclude that $x(t)$ is bounded by $\underline{x}(t)$ and $\overline{x}(t)$ as

$$\underline{x}(t) \le x(t) \le \overline{x}(t), \forall t \in [t_K, t_{K+1}[\,. \tag{20}$$

These bounds allow to find an estimation of the state of the MAS, which can be computed in a distributed way by each agent.

Based on this knowledge, we derive bounds on the value of the consensus. As shown in the proof of Theorem 3, for $t \in [t_K, t_{K+1}[$, one has $\dot{a}(t) = 0$. Then the agreement value a of the consensus remains invariant between two triggering instants and can be defined as $a(t) = a(t_K) = (1/N)\mathbf{1}_N^T x(t_K)$. Extension of the proof results to $\overline{a}(t)$ and $\underline{a}(t)$ is straight forward and we can define $\underline{a}(t_K) = (1/N)\,\mathbf{1}_N^T \underline{x}(t_K)$ and $\overline{a}(t_K) = (1/N)\,\mathbf{1}_N^T \overline{x}(t_K)$. From (18), one has

$$\underline{a}(t_K) \le a(t_K) \le \overline{a}(t_K). \tag{21}$$

Let us assume that for each agent of the MAS, the realization of its corresponding noise at each triggering instant is such that there exists at least one agent of the MAS for which the control input, as computed by (4), is non zero [1]. For such an Agent i, one has therefore $|u_i(t_K)| > 0$. Then, as t increases, $\tilde{e}_i(t)$ evaluated from (12) also increases and reaches the broadcast threshold $h(t)$ which is decreasing with time. The CTC (9) is hence verified and another communication is triggered. Therefore the number K of triggered communications inside the MAS increases with time t. As $t \to \infty$, $h(t) \to c_0$, and one can find some integer $m > 0$ such that $\forall K \ge m, h(t_K) \sim c_0$. In this case, using (17), one has

$$\underline{a}(t_K) = \frac{1}{N}\mathbf{1}_N^T \underline{x}(t_K) \sim \frac{1}{N}\mathbf{1}_N^T \hat{x}(t_K) - c_0 \tag{22}$$

$$\overline{a}(t_K) = \frac{1}{N}\mathbf{1}_N^T \overline{x}(t_K) \sim \frac{1}{N}\mathbf{1}_N^T \hat{x}(t_K) + c_0 \tag{23}$$

It can be concluded that, as $t \to \infty$, the width of the interval $[\underline{a}(t_K), \overline{a}(t_K)]$ tends to $2c_0$.

4.4 Simulation example

A simulation example is proposed in this section to illustrate the proposed approach. A network of five agents with a fully-connected communication graph $\mathcal{G}$ is considered. Initial conditions are chosen randomly. An uniformly distributed random noise within $[-\tilde{w}, \tilde{w}]$ is used for the perturbation w_i affecting the state estimate of each Agent i, with $\tilde{w} = 9.10^{-5}$. Triggering function parameters are set as $c_0 = 0.0001 = 1.1\tilde{w}$, $c_1 = 0.2499$, $\alpha = 0.9\lambda_2(\mathcal{G}) = 0.45$. Figure 1 shows evolution of agents' states to the consensus, and the triggering instants defined from the evaluation of the distributed CTC (9). A set of 200 simulations gives an average number of communications of 8.64% using the proposed new CTC and in presence of noise - with 100% being the percentage in case periodic communications at each sampling period. Figure 2 presents the evolution of

[1] In practice, this assumption is likely to be verified, e.g. if the w_i are independent randomly distributed variables.

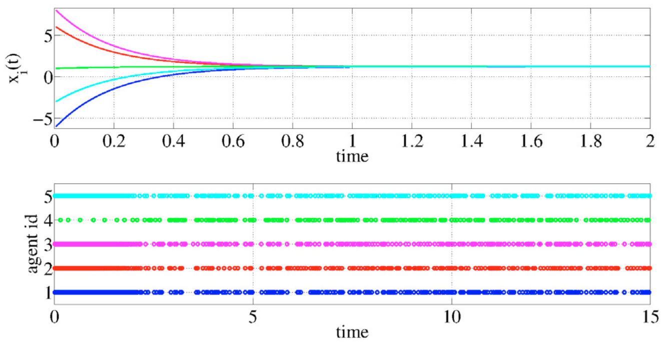

Figure 1. Evolution of the agents' states to the consensus and their triggering instants.

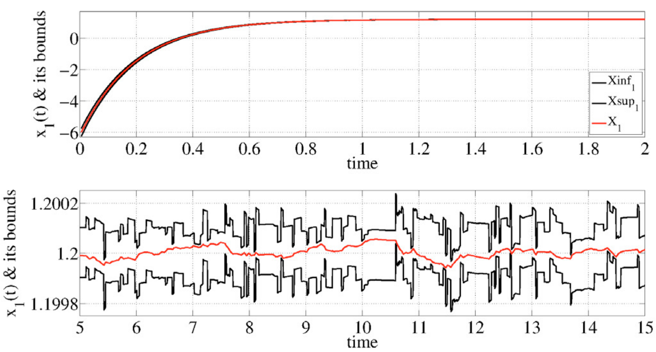

Figure 2. Evolution of $x_1(t)$, $\underline{x}_1(t)$ and $\overline{x}_1(t)$.

the bounds $\underline{x}_1$ and $\overline{x}_1$ on the state of Agent 1. Similar behaviour is obtained for all the agents. Figure 3 shows

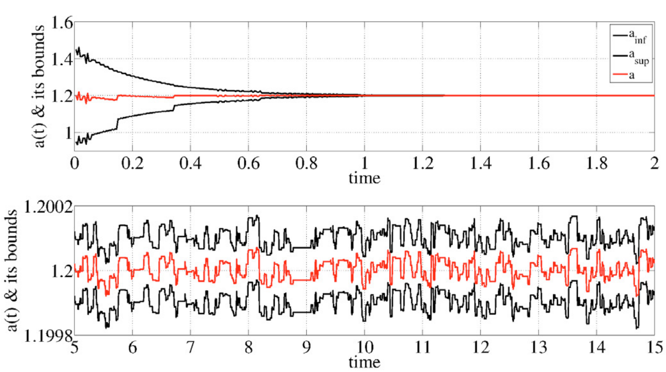

Figure 3. Evolution of $a(t)$, $\underline{a}(t)$ and $\overline{a}(t)$.

the evolution of the consensus value and its bounds. As can be observed, the bounds on the consensus value tend to an interval of width close to $2c_0$.

5. DOUBLE-INTEGRATOR AGENTS

5.1 Consensus & communication triggering condition

Consider first the noise-free case for agents with double-integrator dynamics described as

$$\dot{x}_i(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_i(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i(t), \qquad (24)$$

where $x_i(t) = [\xi_i(t)\ \zeta_i(t)]^T \in \mathbb{R}^2$ is the state of Agent i and $u_i \in \mathbb{R}$ its control input. Seyboth et al. (2013) proposed the event-based implementation

$$u(t) = -L\left(\hat{\xi}(t) + diag(t - t_k^1, ..., t - t_k^N)\hat{\zeta}(t) + \mu\hat{\zeta}(t)\right) \qquad (25)$$

with $\mu > 0$ and with the stack vectors $u = [u_1, \dots, u_N]^T$, $\hat{\xi} = [\hat{\xi}_1, \dots, \hat{\xi}_N]^T$ and $\hat{\zeta} = [\hat{\zeta}_1, \dots, \hat{\zeta}_N]^T$ and where estimates $\hat{\xi}_i(t)$ and $\hat{\zeta}_i(t)$ are defined between two triggering instants by $\hat{\xi}_i(t) = \xi_i(t_k^i)$ and $\hat{\zeta}_i(t) = \zeta_i(t_k^i)$, $\forall t \in [t_k^i, t_{k+1}^i[$. Introducing the measurement errors

$$\begin{aligned} e_\xi(t) &= \hat{\xi}(t) + diag(t - t_k^1, ..., t - t_k^N)\hat{\zeta}(t) - \xi(t) \\ e_\zeta(t) &= \hat{\zeta}(t) - \zeta(t) \end{aligned} \qquad (26)$$

allows to re-write (25) as $u(t) = -L(\xi(t) + \mu\zeta(t) + e_\xi(t) + \mu e_\zeta(t))$. Using this control law, the closed-loop dynamics of the second-order multi-agent system are

$$\begin{bmatrix} \dot{\xi} \\ \dot{\zeta} \end{bmatrix} = \Gamma \begin{bmatrix} \xi \\ \zeta \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ L & L \end{bmatrix} e(t) \qquad (27)$$

with $e = [e_\xi^T\ \mu e_\zeta^T]^T$ and Γ defined as in Lemma 2.
A CTC is defined in *Th. 5.2* of Seyboth et al. (2013) as

$$f_i(t, e_{\xi,i}(t), e_{\zeta,i}(t)) = \left\| \begin{matrix} e_{\xi,i}(t) \\ \mu e_{\zeta,i}(t) \end{matrix} \right\| - \left(c_0 + c_1 e^{-\alpha t}\right) \qquad (28)$$

where $e_{\xi,i}$ and $e_{\zeta,i}$ respectively stand for the i-th component of e_ξ and e_ζ and with constants $c_0 \geq 0$, $c_1 \geq 0$, $c_0 + c_1 > 0$ and $0 < \alpha < |Re(\lambda_3(\Gamma))|$. Using this CTC, *Th. 5.2* of Seyboth et al. (2013) ensures that the disagreement vector δ of the closed loop system converges to a ball centered at the origin with radius $r = c_0 c_v \sqrt{2N}\,\|L\| / |Re(\lambda_3(\Gamma))|$, for all initial conditions $\xi_0, \zeta_0 \in \mathbb{R}^N$. Moreover, the MAS does not exhibit Zeno behavior. This result is obtained in the noise-free case.

5.2 Presence of unknown but bounded measurement noise

Considering now that state estimates are defined as

$$\begin{aligned} \hat{\xi}_i(t) &= \xi_i(t_k^i) + w_{\xi,i}(t_k^i), \\ \hat{\zeta}_i(t) &= \zeta_i(t_k^i) + w_{\zeta,i}(t_k^i) \end{aligned} \qquad (29)$$

$\forall t \in \left[t_k^i, t_{k+1}^i\right[$ for each Agent i of the MAS, where $w_{\xi,i}$ and $w_{\zeta,i}$ are some additive unknown but bounded noises. Let us denote $\tilde{w}_\xi = \sup_{i\in\mathcal{V}}(\sup_{k\in\mathbb{N}^*} |w_{\xi,i}(t_k^i)|)$ and $\tilde{w}_\zeta = \sup_{i\in\mathcal{V}}(\sup_{k\in\mathbb{N}^*} |w_{\zeta,i}(t_k^i)|)$ the respective bounds on all the possible realizations of $w_{\xi,i}$ and $w_{\zeta,i}$ for all agents of the MAS and all communication times.
To introduce a new CTC for this problem, let us first define

$$|U_i(t)| = \mu \left| \int_{t_k^i}^{t} -u_i(s)ds \right|, \quad |V_i(t)| = \tilde{w}_\zeta(t - t_k^i) + \int_{t_k^i}^{t} |U_i(s)|ds, \qquad (30)$$

$e'_{\zeta,i}(t) = \tilde{w}_\zeta + |U_i(t)|$, $e'_{\xi,i}(t) = \tilde{w}_\xi + |V_i(t)|$ and $e'_i = [e'_{\xi,i}, \mu e'_{\zeta,i}]$. Based on these notations, a new CTC to be evaluated by Agent i is introduced:

$$f_i(t, e'_i(t)) = \|e'_i(t)\|_\infty - \left(c_0 + c_1 e^{-\alpha t}\right) > 0 \qquad (31)$$

with constants $c_0 > \max(\tilde{w}_\xi, \tilde{w}_\zeta) \geq 0$, $c_1 \geq 0$, $c_0 + c_1 > 0$ and $0 < \alpha < |Re(\lambda_3(\Gamma))|$.

Theorem 4. Consider the multi-agent system (24) with the control law (25) and state estimates (29). If communications are triggered when the CTC (31) is verified then, for all initial conditions $\xi_0, \zeta_0 \in \mathbb{R}^N$, a bounded consensus is obtained for the MAS and the disagreement vector δ converges to a ball centered at the origin with radius $r = c_0 c_v \sqrt{2N}\,\|L\| / |Re(\lambda_3(\Gamma))|$, where c_v is defined as in Lemma 2. Moreover the MAS does not exhibit Zeno behavior.

Proof. Similarly to the case with single-integrator agents, the disagreement dynamics is defined as

$$\dot{\delta}(t) = \Gamma\delta(t) - \begin{bmatrix} 0 & 0 \\ L & L \end{bmatrix} e(t) \tag{32}$$

and its analytical solution is $\delta(t) = e^{\Gamma t}\delta(0) - \int_0^t e^{\Gamma(t-s)} \begin{bmatrix} 0 & 0 \\ L & L \end{bmatrix} e(s)\, ds$. Similarly to Seyboth et al. (2013), Lemma 2 yields $\|\delta(t)\| \leq k_1 + k_2 e^{-\alpha t} + k_3 e^{Re(\lambda_3(\Gamma))t}$, with $Re(\lambda_3(\Gamma)) < -\alpha < 0$ and $k_1 = \frac{c_0 c_v \sqrt{2N}\|L\|}{|Re(\lambda_3(\Gamma))|}$, $k_2 = \frac{c_1 c_v \sqrt{2N}\|L\|}{|Re(\lambda_3(\Gamma))+\alpha|}$, $k_3 = c_v \|\delta(0)\|$ positive constants. An upper bound $\|u(t)\|$ of the control $u(t)$ is obtained as (see Seyboth et al. (2013)): $\|u(t)\| \leq \sqrt{1+\mu^2}\,\|L\|(k_1+k_2+k_3) + \sqrt{2}\,\|L\|(c_0+c_1) = \tilde{u}$. Again we have to prove that the Zeno behavior is excluded. For that purpose, let us compute a bound on $\|e_i(t)\|_\infty$ for each Agent i and for t between two triggering instants. Denote by t^* the last triggering instant.

$$\|e_i(t)\|_\infty = \max\left(\left|\int_{t^*}^t \dot{e}_{\xi,i}(s)\, ds\right|, \mu\left|\int_{t^*}^t \dot{e}_{\zeta,i}(s)\, ds\right|\right) \tag{33}$$

Using the fact that $\mu\dot{e}_\zeta(t) = -\mu u(t)$, one has

$$\begin{aligned} \mu\left|\int_{t^*}^t \dot{e}_{\zeta,i}(s)\right| ds &= \mu\left|\int_{t^*}^t -u_i(s)\, ds\right| \\ \mu|e_{\zeta,i}(t) - w_{\zeta,i}(t^*)| &= \mu\,|U_i(t)| \\ \mu|e_{\zeta,i}(t)| &\leq \mu\left(\tilde{w}_\zeta + |U_i(t)|\right) \leq \mu\left(\tilde{w}_\zeta + \tilde{u}(t-t^*)\right) \end{aligned} \tag{34}$$

Let us define $e'_{\zeta,i}(t) = \tilde{w}_\zeta + |U_i(t)|$ and $\tilde{e}_{\zeta,i}(t) = \tilde{w}_\zeta + \tilde{u}(t-t^*)$. Using the fact that $\dot{e}_{\xi,i}(t) = e_{\zeta,i}(t)$,

$$\begin{aligned} \left|\int_{t^*}^t \dot{e}_{\xi,i}(s)\, ds\right| &= \left|\int_{t^*}^t e_{\zeta,i}(s)\, ds\right| \leq \int_{t^*}^t \tilde{w}_\zeta\, ds + \int_{t^*}^t |U_i(s)|\, ds \\ |e_{\xi,i}(t) - w_{\xi,i}(t^*)| &\leq |V_i(t)| \\ |e_{\xi,i}(t)| &\leq \tilde{w}_\xi + |V_i(t)| \leq \tilde{w}_\xi + (c_0+c_1)(t-t^*) \end{aligned} \tag{35}$$

Let us introduce $e'_{\xi,i}(t) = \tilde{w}_\xi + |V_i(t)|$ and $\tilde{e}_{\xi,i}(t) = \tilde{w}_\xi + (c_0+c_1)(t-t^*)$. We get from (33):

$$\|e_i(t)\|_\infty \leq \max\left(e'_{\xi,i}(t), \mu e'_{\zeta,i}(t)\right) \leq \max\left(\tilde{e}_{\xi,i}(t), \mu\tilde{e}_{\zeta,i}(t)\right)$$

Here, we assume $c_0 \neq 0$. The event is triggered when CTC is satisfied *i.e.* not before $\|e_i(t)\|_\infty > c_0$, and therefore not before $\max(\tilde{e}_{\xi,i}, \mu\tilde{e}_{\zeta,i}) > c_0$. The inter-event times are lower bounded by $\tau(t^*) = \max\left(\frac{c_0-\tilde{w}_\xi}{c_0+c_1}, \frac{c_0-\mu\tilde{w}_\zeta}{\mu\tilde{u}}\right) > 0$. This bound holds for all event times t^* and all agents i. Thus we get the condition $c_0 > \max(\tilde{w}_\xi, \mu\tilde{w}_\zeta)$ for exclusion of the Zeno behavior. Moreover, $e(t)$ is piecewise continuous and the right hand side of (32) is globally Lipschitz in δ, existence and uniqueness of the solution is guaranteed as said in Khalil (1996). Thus, $\|\delta(t)\|$ converges exponentially to the ball $\|\delta\| < k_1$ as $t \to \infty$.

5.3 Guaranteed bounds on the state of the MAS and on the consensus

As for the single-integrator case, one assumes the graph $\mathcal{G}$ is fully connected and there is no communication delay. Denote by K the index of the K-th message broadcast in the MAS, whatever the broadcasting Agent. Consider $t \in [t_K, t_{K+1}[$. Each Agent i of the MAS will compute lower bounds $\underline{\xi}(t)$, $\underline{\zeta}(t)$ and upper bounds $\overline{\xi}(t)$, $\overline{\zeta}(t)$ of $\xi(t)$, $\zeta(t)$, which are identical for all agents, using

$$\begin{bmatrix} \dot{\underline{\xi}} \\ \dot{\underline{\zeta}} \end{bmatrix} = \Gamma \begin{bmatrix} \underline{\xi} \\ \underline{\zeta} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ L & L \end{bmatrix} H(t), \quad \begin{bmatrix} \dot{\overline{\xi}} \\ \dot{\overline{\zeta}} \end{bmatrix} = \Gamma \begin{bmatrix} \overline{\xi} \\ \overline{\zeta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ L & L \end{bmatrix} H(t) \tag{36}$$

with $H(t)$ a vector of size $2N$ which all components are $h(t)$. And because $\mathcal{G}$ is fully-connected, (36) becomes

$$\begin{bmatrix} \dot{\underline{\xi}} \\ \dot{\underline{\zeta}} \end{bmatrix} = \Gamma \begin{bmatrix} \underline{\xi} \\ \underline{\zeta} \end{bmatrix}, \quad \begin{bmatrix} \dot{\overline{\xi}} \\ \dot{\overline{\zeta}} \end{bmatrix} = \Gamma \begin{bmatrix} \overline{\xi} \\ \overline{\zeta} \end{bmatrix} \tag{37}$$

At the trigger instant t_K, Agent i updates its control input u_i and the j-th component of $\underline{\xi}$, $\underline{\zeta}$ and $\overline{\xi}$, $\overline{\zeta}$ by using the information $\hat{\xi}_j(t_K)$, $\hat{\zeta}_j(t_K)$ sent by the Agent j that triggered the communication at t_K:

$$\begin{aligned} \underline{\xi}_j(t_K) &= \hat{\xi}_j(t_K) - h(t_K), & \overline{\xi}_j(t_K) &= \hat{\xi}_j(t_K) + h(t_K) \\ \underline{\zeta}_j(t_K) &= \hat{\zeta}_j(t_K) - h(t_K)/\mu, & \overline{\zeta}_j(t_K) &= \hat{\zeta}_j(t_K) + h(t_K)/\mu \end{aligned} \tag{38}$$

with $h(t_K) = c_0 + c_1 e^{-\alpha t_K}$. It is assumed that all the agents trigger a communication at the initial instant $t = 0$. Consider the CTC (31) of triggering Agent j. At time t_K, $\max(|e'_{\xi,j}(t_K)|, \mu|e'_{\zeta,j}(t_K)|) - h(t_K) \leq 0$. Since $\max(|e_{\xi,j}(t_K)|, \mu|e_{\zeta,j}(t_K)|) \leq \max(|e'_{\xi,j}(t_K)|, \mu|e'_{\zeta,j}(t_K)|)$, one can deduce $-h(t_K) \leq e_{\xi,j}(t_K) \leq h(t_K)$ and $-h(t_K) \leq \mu e_{\zeta,j}(t_K) \leq h(t_K)$. Therefore

$$\begin{aligned} \underline{\xi}_j(t_K) &= \hat{\xi}_j(t_K) - h(t_K) \leq \xi_j(t_K) \leq \overline{\xi}_j(t_K) = \hat{\xi}_j(t_K) + h(t_K) \\ \underline{\zeta}_j(t_K) &= \hat{\zeta}_j(t_K) - h(t_K)/\mu \leq \zeta_j(t_K) \leq \overline{\zeta}_j(t_K) = \hat{\zeta}_j(t_K) + h(t_K)/\mu \end{aligned} \tag{39}$$

so $\underline{\xi}(t_K) \leq \xi(t_K) \leq \overline{\xi}(t_K)$ and $\underline{\zeta}(t_K) \leq \zeta(t_K) \leq \overline{\zeta}(t_K)$. In addition, it can be shown that (37) define a lower and upper dynamical system for (27), in the sense that

$$\begin{bmatrix} \dot{\underline{\xi}} \\ \dot{\underline{\zeta}} \end{bmatrix} = \Gamma \begin{bmatrix} \dot{\underline{\xi}} \\ \dot{\underline{\zeta}} \end{bmatrix} \leq \begin{bmatrix} \dot{\xi} \\ \dot{\zeta} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ L & L \end{bmatrix} e(t) \leq \begin{bmatrix} \dot{\overline{\xi}} \\ \dot{\overline{\zeta}} \end{bmatrix} = \Gamma \begin{bmatrix} \dot{\overline{\xi}} \\ \dot{\overline{\zeta}} \end{bmatrix} \tag{40}$$

where the inequalities are interpreted component by component. Using Muller's Theorem between two triggering instants, one can conclude that

$$\begin{bmatrix} \underline{\xi} \\ \underline{\zeta} \end{bmatrix} \leq \begin{bmatrix} \xi \\ \zeta \end{bmatrix} \leq \begin{bmatrix} \overline{\xi} \\ \overline{\zeta} \end{bmatrix}, \ \forall t \in [t_K, t_{K+1}[\tag{41}$$

These bounds allow to find an estimation of the state of the MAS, which can be computed in a distributed way by each agent.

Based on this knowledge, one would like now to derive some bounds on the value of the consensus. From Seyboth et al. (2013)), for t between two events t_K and t_{K+1}, the state vector can be expressed $\xi(t) = a(t)\mathbf{1}_N + b(t)t\mathbf{1}_N + \delta_\xi(t)$ and $\zeta(t) = b(t)\mathbf{1}_N + \delta_\zeta(t)$ where δ_ξ and δ_ζ are block vectors composing δ such that $\mathbf{1}^T\delta_\xi(t) = 0$ and $\mathbf{1}^T\delta_\zeta(t) = 0$, and where $b(t) = b(t_K) = (1/N)\mathbf{1}_N^T\zeta(t_K)$ and $a(t) = a(t_K) = (1/N)\mathbf{1}_N^T\xi(t_K)$ are invariant quantities *i.e* for $t \in [t_K, t_{K+1}[$, $\zeta_i(t) \sim b(t_K)$ and $\xi_i(t) \sim a(t_K) + b(t_K)t$ as $t \to \infty$. Using the same scheme as in Section 4.3, it can be shown from (39) that:

$$\begin{aligned} \underline{a}(t_K) &\leq a(t_K) \leq \overline{a}(t_K) \\ \underline{b}(t_K) &\leq b(t_K) \leq \overline{b}(t_K). \end{aligned} \tag{42}$$

and therefore

$$\begin{aligned} \underline{b}(t_K) &= \frac{1}{N}\mathbf{1}_N^T\underline{\zeta}(t_K) \sim \frac{1}{N}\mathbf{1}_N^T\hat{\zeta}(t_K) - c_0/\mu \\ \overline{b}(t_K) &= \frac{1}{N}\mathbf{1}_N^T\overline{\zeta}(t_K) \sim \frac{1}{N}\mathbf{1}_N^T\hat{\zeta}(t_K) + c_0/\mu \end{aligned} \tag{43}$$

$$\begin{aligned} \underline{a}(t_K) &= \frac{1}{N}\mathbf{1}_N^T\underline{\xi}(t_K) \sim \frac{1}{N}\mathbf{1}_N^T\hat{\xi}(t_K) - c_0 \\ \overline{a}(t_K) &= \frac{1}{N}\mathbf{1}_N^T\overline{\xi}(t_K) \sim \frac{1}{N}\mathbf{1}_N^T\hat{\xi}(t_K) + c_0 \end{aligned} \tag{44}$$

So as $t \to \infty$, the width of intervals $[\underline{b}(t_K), \overline{b}(t_K)]$ tends to $2c_0/\mu$ and $[\underline{a}(t_K), \overline{a}(t_K)]$ tends to $2c_0$.

5.4 Simulation example

A simulated example has been designed to illustrate the proposed approach. Initial conditions are chosen randomly. Uniformly distributed random noises within $[-\tilde{w}_\xi, \tilde{w}_\xi]$ and $[-\tilde{w}_\zeta, \tilde{w}_\zeta]$ are respectively used for the perturbation $w_{\xi,i}$ and $w_{\zeta,i}$ affecting the state estimate of each Agent i, with $\tilde{w}_\xi = 9.10^{-5}$ and $\tilde{w}_\zeta = 4.5.10^{-5}$. Triggering function parameters are set to $c_0 = 0.0001$, $c_1 = 0.2499$, $\alpha = 0.95|Re(\lambda(\Gamma))| = 0.523$. Figure 4 shows that the

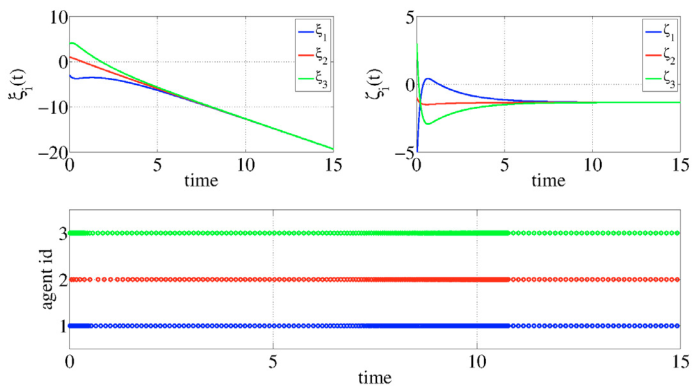

Figure 4. Simulation result for single-integrator agents.

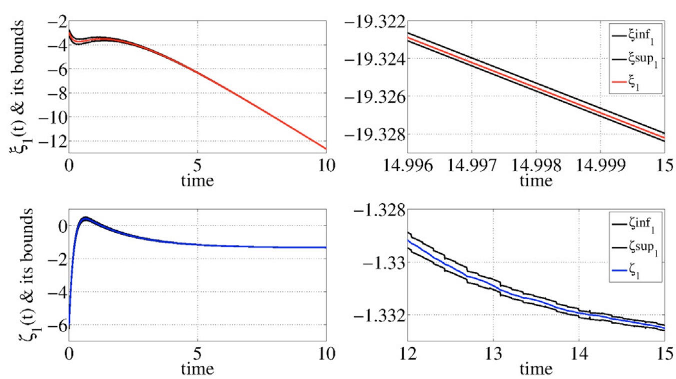

Figure 5. Evolution of $\xi_1(t)$, $\underline{\xi}_1(t)$, $\overline{\xi}_1(t)$ and $\zeta_1(t)$, $\underline{\zeta}_1(t)$, $\overline{\zeta}_1(t)$.

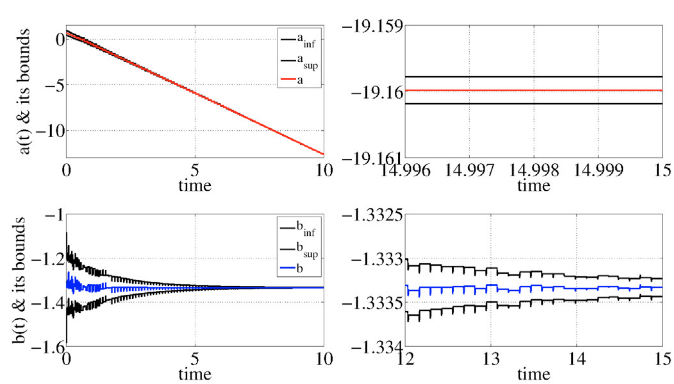

Figure 6. Evolution of $b(t)$, $\underline{b}(t)$, $\overline{b}(t)$ and $a(t)$, $\underline{a}(t)$, $\overline{a}(t)$.

triggering function (31) allows to reach the consensus and that communications are triggered regularly. The average rate of communication is of 9.99% for 3 agents. Figure 5 presents the evolution of the bounds $\underline{\xi}_1$, $\overline{\xi}_1$ and $\underline{\zeta}_1$, $\overline{\zeta}_1$ on the state of Agent 1. Similar behaviour is obtained for all the agents. Figure 6 shows evolution of the consensus values and their bounds.

6. CONCLUSIONS

A new distributed event-triggered communication method based on an existing work allowing multi-agent systems to reach the consensus in case where agent's state measurements are subject to unknown but bounded noise has been introduced. Based on bound on measurement error, conditions for design of a new CTC has been determined. From this mechanism, guaranteed bounding for consensus has been obtained using upper and lower dynamical systems. Simulations have shown the efficiency of the new strategy.

Further works include considering more complex CTCs or other state estimators as in Viel et al. (2016).

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