Leader-follower Consensus of Unicycle-type Vehicles via Smooth Time-invariant Feedback

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***Abstract*— For a system of multiple vehicles with nonholonomic constraints and communicating under a directed spanning-tree graph, we solve the problem of full consensus, that is, convergence to a common unspecified value both in position and orientation. Remarkably, our controller is smooth time-invariant thanks to a polar-coordinates based model. Furthermore, the proposed control is quite simple, as it uses only relative information and achieves a more natural behaviour of the vehicles, making it well suited for practical applications. We establish (almost) global asymptotic convergence to the consensus manifold using the Lyapunov framework and cascaded systems theory.**

I. INTRODUCTION

Multiagent mobile-robot systems have received much attention due to their advantages over single-agent systems, such as their reduced cost, higher efficiency, robustness, and reconfigurability. For multiagent systems, the consensus problem, that is, the convergence of the states of all agents to a common unspecified value, constitutes the basis for the most common applications such as rendezvous, formation control, collaborative area coverage, flocking, *etc.* [1], [2]. In the literature of consensus of nonholonomic vehicles two main problems are addressed, position consensus, in which case all agents are to converge to the same position with arbitrary or predetermined orientation, and full consensus, in which case, agreement on the orientation is also achieved.

In the seminal paper [3], necessary conditions for asymptotic stabilisation of nonlinear systems via smooth time-invariant feedback are laid and in [4], similar conditions for set-point consensus of multiagent nonholonomic systems are given. As a consequence, much attention has been paid to the problem of designing time-varying or non-smooth controllers for set-point stabilisation of nonholonomic systems. In [5] a time-varying control is designed for feedback linearised nonholonomic systems over a directed-tree graph. A δ-persistency-of-excitation-based time-varying control is used in [6], [7], [8] that achieves full-consensus-based formation over undirected graphs whereas in [9] a time-varying controller is proposed for the consensus of nonholonomic systems in chained form, over a directed-tree graph. On the other hand, the authors in [10] propose a non-smooth feedback for position consensus of a multi-agent over undirected graphs.

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Thus, time-varying controllers have been proved effective to stabilise nonholonomic systems and also to achieve consensus, but they add a degree of complexity to the control design problem and to the stability analysis. Moreover, persistency-of-excitation-based controllers may induce undesirable oscillatory motions. These drawbacks may be overcome via smooth time-invariant feedback. For nonholonomic systems, such controllers may be designed without contradicting [3], [4] provided the system is singular precisely at the origin, as is the case of the model based on polar coordinates [11], [12]. Based on the latter, a smooth time-invariant controller is presented in [13] for consensus of nonholonomic agents over a directed spanning tree, albeit for a linearised system, hence, achieving only position consensus. In [14] a continuous time-invariant feedback for position consensus is proposed for multi-agent systems over undirected graphs. However, time delays have to be considered in order to avoid algebraic loops in the control. In [15] a smooth time-invariant controller is designed to achieve consensus, albeit only in position, for a system communicating over a directed graph containing a directed spanning tree.

The objective of this paper is to propose a solution for the problem of full consensus of a group of nonholonomic vehicles communicating over a directed spanning tree graph. If this problem has been previously studied from the time-varying perspective, to the authors' knowledge, it has not been addressed in a smooth time-invariant way. This is made possible using a polar coordinates representation. The practical advantage of the latter is that relying only on relative quantities such as distances and line-of-sight angles, it is more suited to practical implementations since, usually, only relative measurements are available on board. On the other hand, a smooth time-invariant control also allows for a more natural behaviour of the vehicles. Moreover, by using polar coordinates the consensus problem is transformed into the stabilisation of the origin, rendering it simpler to analyse from a stability-theory perspective.

Thus, in this paper a multi-agent model is proposed based on the polar transformation introduced in [11]. Then, distributed a smooth time-invariant controller is proposed based on backstepping control. Finally, (almost) global asymptotic convergence to the consensus manifold is showed using Lyapunov theory and cascaded-systems arguments. In Section II are presented the model and the problem statement. Our main results are presented in Section III; these are illustrated via numerical simulations in section IV. We close with some concluding remarks in Section V.

II. Model and Problem Statement

Notation. We denote by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ a digraph defined by a node set $\mathcal{V} := \{1, 2, \ldots, n\}$, corresponding to the labels of the nodes' states, and an edge set $\mathcal{E} \subseteq \mathcal{V}^2$ of cardinality m which characterises the network communication topology. A directed edge e_k, with $k \le m$, is an ordered pair $(i, j) \in \mathcal{E}$ if and only if a connection exists from node i to node j. A directed *tree* is a subgraph consisting in a root node, with no parent, and a set of nodes reachable from the root. A directed *spanning* tree $\mathcal{G}_\mathcal{T} \subset \mathcal{G}$ is a directed tree containing all the nodes in $\mathcal{G}$.

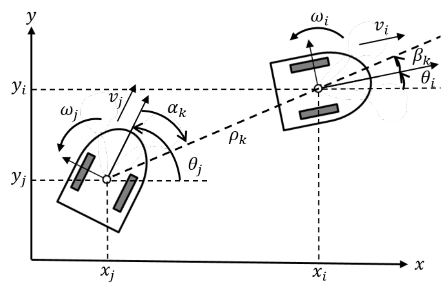


Fig. 1: Leader-follower scheme and polar-coordinates variables

We consider the consensus problem for swarms of autonomous unicycle-type vehicles communicating in a leader-follower fashion —see Fig. 1. That is, each robot is leader to one or several other vehicles called followers and each of the latter has only one leader, except for one swarm leader that moves freely. This interconnection topology may be described using a directed spanning-tree topology, as illustrated in Fig. 2.

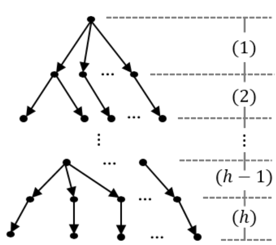


Fig. 2: Directed spanning tree $\mathcal{G}_\mathcal{T}$

Most commonly, unicycle vehicles are modelled using the so-called nonholonomic integrator. For an arbitrary agent labelled $i \le n$ this is

$$\dot{x}_i = v_i \cos\theta_i \tag{1a}$$
$$\dot{y}_i = v_i \sin\theta_i \tag{1b}$$
$$\dot{\theta}_i = \omega_i, \tag{1c}$$

where $q_i = [x_i, y_i]^\top \in \mathbb{R}^2$ denotes the Cartesian position, and $\theta_i \in (-\pi, \pi]$ denotes its orientation with respect to the axis of the abscissae. The control inputs $v_i \in \mathbb{R}$ and $\omega_i \in \mathbb{R}$ denote, respectively, the linear speed and the angular velocity.

It is known from the seminal paper [3] that, for (1), the origin is not stabilisable via time-invariant differentiable controls. The dual of such result for the problem of consensus, which is inherently a stabilisation one, appeared in [4]. Yet, in spite of the geometric obstructions to stabilise (1) that are described in [3], smooth time-invariant control of a unicycle is not impossible. In [11], [12] a controller is proposed for unicycles modelled by equations equivalent to (1), but expressed in polar coordinates with respect to the goal. Such model, being singular at the origin, is convenient for control purposes since it does not belong to the class of nonholonomic integrators characterised in [3] and, in addition, it has the practical advantage of relying on local relative measurements: distance and line of sight, which makes it attractive from a robotics standpoint.

With such motivations, following the polar transformation proposed in [11], we define a similar transformation for a multiagent unicycle system. Furthermore, rather than expressing the dynamics of a vehicle itself our model, describes the *relative* behaviour for any pair of vehicles.

For every pair of leader (labelled i) and follower (labelled j) we define

$$\rho_k := |q_i - q_j| \tag{2a}$$
$$\beta_k := \arctan\left(\frac{y_i - y_j}{x_i - x_j}\right) - \theta_i, \quad \forall \rho_k > 0 \tag{2b}$$
$$\alpha_k := \arctan\left(\frac{y_i - y_j}{x_i - x_j}\right) - \theta_j, \quad \forall \rho_k > 0 \tag{2c}$$

where ρ_k represents the distance between agents i and j, β_k is the angle between the line of sight and the direction of movement of the leader, agent i, and α_k is the angle between the line of sight and the direction of movement of the follower, agent j. Note that the three-dimensional space of (x_i, y_i, θ_i) is mapped into another space of dimension 3, corresponding to the *relative* coordinates $(\rho_k, \alpha_k, \beta_k)$ —see Figure 1 for an illustration.

Next, we differentiate on both sides of (2), and we use (1), to obtain the dynamics equations

$$\dot{\rho}_k = v_i \cos\beta_k - v_j \cos\alpha_k \tag{3a}$$
$$\dot{\beta}_k = \frac{1}{\rho_k}\left[-v_i \sin\beta_k + v_j \sin\alpha_k\right] - \omega_i \tag{3b}$$
$$\dot{\alpha}_k = \frac{1}{\rho_k}\left[-v_i \sin\beta_k + v_j \sin\alpha_k\right] - \omega_j \tag{3c}$$

with inputs v_i, v_j, ω_i, $\omega_j \in \mathbb{R}$.

Thus, Equations (3) represent the relative dynamics of an arbitrary leader-follower pair. In terms of graph theory, they represent the dynamics of the state $e_k := (i, j)$ for an edge-based representation of the graph [16] —see Figures 2 and 3. The advantage of the edge-based approach for the analysis of graphs is that consensus problems are naturally recast as problem of stabilisation of an equilibrium, as opposed to the stabilisation of an infinitely-dimensional manifold, which is the case when one uses the more common node-based representation. For the multiagent systems (3) in polar coordinates the full consensus problem (that is both in position and orientation) boils down to the stabilisation of the origin. Indeed, $\rho_k = 0$ is equivalent to $q_i = q_j$ and $(\beta_k, \alpha_k) = (0, 0)$ if and only if $\theta_i = \theta_j$, for all $k \le m$.

III. Main results

A. Control Design

The control approach for (3) follows the same rationale as in [17] for the stabilisation of a single unicycle vehicle in polar coordinates and may be explained with reference to Figure 1. In the stabilisation-of-one-vehicle scenario it is desired to asymptotically stabilise the position and orientation of agent j, so the "leader" is a static point ($v_i = \omega_i = 0$) and the goal is to steer $q_j \to q_i$ and $\theta_j \to \theta_i$. Hence, the control strategy consists in steering first the agent j so that it is aligned in orientation with the agent i, *i.e.*, on the line of sight connecting both agents. In other words, we seek to make $\alpha_k \to 0$ and $\beta_k \to 0$, as this guarantees that $\theta_j \to \theta_i$. Then, once the agent j is aligned with its leader i, as $\rho_k \to 0$ we have $q_j \to q_i$.

To accomplish such tasks, we may exploit the natural cascaded structure of the system. Indeed, note that for this system, with $v_i = \omega_i = 0$, the input ω_j only directly affects the state α_k. Hence, following a backstepping-like procedure we first design a virtual input $\alpha_k^* : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$; $(\rho_k, \beta_k) \mapsto \alpha_k^*(\rho_k, \beta_k)$ such that $\alpha_k^*(0,0) = 0$, and input v_j in order make $\rho_k(t) \to 0$ and $\beta_k(t) \to 0$ asymptotically. Then, ω_j is designed so that $\alpha_k \to \alpha_k^*$. In other words, defining $\tilde{\alpha}_k := \alpha_k - \alpha_k^*$, it is desired to design ω_j so that $\tilde{\alpha}_k(t) \to 0$ as $t \to \infty$. Then, since $\alpha_k^*(0,0) = 0$, we have $\alpha_k(t) \to 0$ as $t \to \infty$.

This approach is suitable whether the leader is static or in motion. Moreover, it may be applied recursively for any pair of moving agents, modelled by the equations (3). The stabilisation problem is similar, only the system's dimension increases. Thus, inspired by the control design in [17] for the stabilisation of one robot, we define the decentralised leader-follower consensus control inputs

$$v_i = -c_1 \sum_{k \le m} a_{ik} \sqrt{1 + (c_3 \beta_k)^2}\, \rho_k \tag{4a}$$

$$\begin{aligned}\omega_i = -\sum_{k \le m} a_{ik} \Bigg[& c_2 \tilde{\alpha}_k + \psi_k \sum_{j \le n} a_{jk} v_j \\ & + \left[1 + \frac{c_3}{1 + (c_3 \beta_k)^2}\right] \frac{\sin(\tilde{\alpha}_k + \alpha_k^*)}{\rho_k} \sum_{j \le n} a_{jk} v_j \Bigg]\end{aligned} \tag{4b}$$

where c_1, c_2, $c_3 > 0$ are design constants, the coefficients a_{ik} are given by

$$a_{ik} := \begin{cases} -1 & \text{if edge } e_k \text{ is incident on node } i \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

$$\begin{aligned}\psi_k \;\; := \;\; & -\left(\frac{\rho_k}{\tilde{\alpha}_k}\right) [\cos(\tilde{\alpha}_k + \alpha_k^*) - \cos \alpha_k^*] \\ & + \left(\frac{\beta_k}{\rho_k \tilde{\alpha}_k}\right) [\sin(\tilde{\alpha}_k + \alpha_k^*) - \sin \alpha_k^*],\end{aligned} \tag{6}$$

and the virtual control α_k^* is defined as

$$\alpha_k^* := \arctan(-c_3 \beta_k). \tag{7}$$

Remark 1: Note that the control law (4b) is well posed since $\lim_{\tilde{\alpha}_k \to 0} \frac{\cos(\tilde{\alpha}_k + \alpha_k^*) - \cos \alpha_k^*}{\tilde{\alpha}_k} = \sin(-\alpha_k^*)$ and $\lim_{\tilde{\alpha}_k \to 0} \frac{\sin(\tilde{\alpha}_k + \alpha_k^*) - \sin \alpha_k^*}{\tilde{\alpha}_k} = \cos(\alpha_k^*)$

In what follows, we show through a simple case study of reduced dimension, how the controller above achieves consensus for (3). In Section III-C we present our main results, for swarms of arbitrary dimension.

B. Case study: 4 nonholonomic agents

Consider a multi-agent system composed of four vehicles subject to nonholonomic constraints, with motions defined by (3), and a communication topology represented by a directed spanning tree labelled as in Fig. 3.

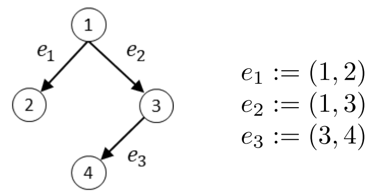


Fig. 3: Directed spanning tree for 4 agents

For the purpose of analysis, we define a multivariable model containing the three states of all the vehicles, by labelling the levels of the tree based on the distance to the root, as in Figure 2. To that end, we assign the state variable

$$\xi^{(p)\top} := \left[\, \rho^{(p)\top} \;\; \beta^{(p)\top} \;\; \tilde{\alpha}^{(p)\top} \,\right] \qquad p \le h,$$

to each level in the tree. Thus, for the graph in Figure 3 let $\xi^{(1)\top} := [\rho^{(1)\top} \; \beta^{(1)\top} \; \tilde{\alpha}^{(1)\top}]$ collect the states corresponding to the edges e_1 and e_2 and let $\xi^{(2)\top} := [\, \rho^{(2)} \; \beta^{(2)} \; \tilde{\alpha}^{(2)} \,]$ contain those relative to the edge e_3. That is, $\rho^{(1)\top} := [\, \rho_1 \;\; \rho_2 \,]$, $\beta^{(1)\top} := [\, \beta_1 \;\; \beta_2 \,]$, and $\tilde{\alpha}^{(1)\top} := [\tilde{\alpha}_1 \;\; \tilde{\alpha}_2]$, whereas $\rho^{(2)} = \rho_3$, $\beta^{(2)} = \beta_3$, and $\tilde{\alpha}^{(2)} = \tilde{\alpha}_3$. Note, after (4) and (5), that since the first node ($i = 1$) is the root, which does not have incident edges, we have $a_{ik} = 0$ hence, $v_1 = \omega_1 = 0$. Furthermore, since every node in a tree has only one incident edge, we have that each v_i depends only on the state of edge e_k where, e_k is incident on node i. The same applies for ω_i.

With these notations the system (3) in closed loop with (4), for the considered graph, can be written in the compact cascaded-system form,

$$\dot{\xi}^{(2)} \;= f^{(2)}(\xi^{(2)}) + g^{(2)}(\xi^{(1)}, \xi^{(2)}) \tag{8a}$$

$$\dot{\xi}^{(1)} \;= f^{(1)}(\xi^{(1)}), \tag{8b}$$

in which the nominal systems, $\dot{\xi}^{(p)} = f^{(p)}(\xi^{(p)})$ with $p \le 2$, take the form

$$\begin{aligned}\dot{\rho}^{(p)} = & -c_1 \rho^{(p)} - c_1 \Big[\mathrm{Cos}(\tilde{\alpha}^{(p)} + \alpha^{*(p)}) \\ & - \mathrm{Cos}(\alpha^{*(p)}) \Big] D(\beta^{(p)}) \rho^{(p)}\end{aligned} \tag{9a}$$

$$\begin{aligned}\dot{\beta}^{(p)} = & -c_1 c_3 \beta^{(p)} + c_1 \mathrm{diag}\left(\frac{1}{\rho_k^{(p)}}\right) \Big[\mathrm{Sin}(\tilde{\alpha}^{(p)} + \alpha^{*(p)}) \\ & - \mathrm{Sin}(\alpha^{*(p)}) \Big] D(\beta^{(p)}) \rho^{(p)}\end{aligned} \tag{9b}$$

$$\dot{\tilde{\alpha}}^{(p)} = -c_2 \tilde{\alpha}^{(p)} - c_1 \Psi^{(p)} D(\beta^{(p)}) \rho^{(p)} \tag{9c}$$

where $\mathrm{Cos}(s) := \mathrm{diag}(\cos(s_k))$, $\mathrm{Sin}(s) := \mathrm{diag}(\sin(s_k))$, $\Psi^{(p)} := \mathrm{diag}(\psi_k)$, and $D(\beta) := \mathrm{diag}(\sqrt{1 + c_3^2\beta_k^2})$.

Remark 2: Note that to obtain (9) we used the identities

$$\sin(\arctan(s)) = \frac{s}{\sqrt{1+s^2}}, \quad \cos(\arctan(s)) = \frac{1}{\sqrt{1+s^2}}.$$

Furthermore, the interconnection term $g^{(2)}$ takes the form

$$g^{(2)}\left(\xi^{(1)}, \xi^{(2)}\right) = \begin{bmatrix} c_1 \cos\beta_3 \sqrt{1+(c_3\beta_2)^2}\rho_2 \\ \tilde{g}_\beta\left(\xi^{(1)}, \xi^{(2)}\right) \\ \tilde{g}_\alpha\left(\xi^{(1)}, \xi^{(2)}\right) \end{bmatrix},$$

where

$$\begin{aligned} \tilde{g}_\beta\left(\xi^{(1)}, \xi^{(2)}\right) &:= -c_1 \frac{\sin\beta_3}{\rho_3}\sqrt{1+(c_3\beta_2)^2}\rho_2 \\ &- c_2\tilde{\alpha}_2 - c_1\psi_2\sqrt{1+(c_3\beta_2)^2}\rho_2 \\ &- c_1\left(1 + \frac{c_3}{1+(c_3\beta_2)^2}\right)\sqrt{1+(c_3\beta_2)^2}\sin(\tilde{\alpha}_2 + \alpha_2^*) \end{aligned}$$

and

$$\begin{aligned} \tilde{g}_\alpha\left(\xi^{(1)}, \xi^{(2)}\right) &:= -c_1 \frac{\sin\beta_3}{\rho_3}\sqrt{1+(c_3\beta_2)^2}\rho_2 \\ &- c_2c_3\sqrt{1+(c_3\beta_2)^2}\tilde{\alpha}_2 - c_1\psi_2\sqrt{1+(c_3\beta_2)^2}\rho_2 \\ &- c_1\left(1 + \frac{c_3}{1+(c_3\beta_2)^2}\right)\sqrt{1+(c_3\beta_2)^2}\sin(\tilde{\alpha}_2 + \alpha_2^*). \end{aligned}$$

The cascade structure of the system (8) captures well the fact that the dynamics of the edges in the first level of the tree, e_1 and e_2 are autonomous while the dynamics of e_3 is driven by the former. Moreover, there is a considerable amount of literature on cascaded systems to rely on. For instance, according to [18], asymptotic stability of the origin of a nonlinear time-varying cascaded system of the form (8) follows if (the respective origins for) the nominal systems $\dot{\xi}^{(1)} = f^{(1)}(\xi^{(1)})$ and $\dot{\xi}^{(2)} = f^{(2)}(\xi^{(2)})$ are asymptotically stable and the solutions of (8) are bounded.

The first condition may be asserted, for each $p \leq 2$, using the Lyapunov function candidate

$$V_p(\xi^{(p)}) = \frac{1}{2}\left|\xi^{(p)}\right|^2, \tag{10}$$

whose total derivative along the trajectories of (9) —see also (6), satisfies

$$\begin{aligned} \dot{V}_p(\xi^{(p)}) &= -c_1|\rho^{(p)}|^2 - c_1c_3|\beta^{(p)}|^2 - c_2|\tilde{\alpha}^{(p)})|^2 \\ &\leq -c'|\xi^{(p)}|^2 < 0 \end{aligned} \tag{11}$$

where $c' := \min\{c_1,\ c_1c_3,\ c_2\}$. Global exponential stability of the origin for the nominal subsystems (9) follows.

Now we establish boundedness of the solutions of (8). To that end, we stress that the interconnection term $g^{(2)}\left(\xi^{(1)}, \xi^{(2)}\right)$ may be upper-bounded as

$$g^{(2)}\left(\xi^{(1)}, \xi^{(2)}\right) \leq \begin{bmatrix} \gamma_\rho(|\xi^{(1)}|) \\ \max\left\{1, \frac{1}{\rho_3}\right\}\gamma_\beta(|\xi^{(1)}|) \\ \max\left\{1, \frac{1}{\rho_3}\right\}\gamma_\alpha(|\xi^{(1)}|) \end{bmatrix} \tag{12}$$

where $\gamma_\rho(s), \gamma_\beta(s), \gamma_\alpha(s) \in \mathcal{K}_\infty$. Therefore, in view of (11) and (12), the total derivative of the quadratic Lyapunov function in (10), with $p = 2$, along the trajectories of (8a) satisfies

$$\begin{aligned} \dot{V}_2(\xi^{(2)}) &\leq -c_1|\rho_3|^2 - c_1c_3|\beta_3|^2 - c_2|\tilde{\alpha}_3|^2 + |\rho_3|\gamma_\rho(|\xi^{(1)}|) \\ &+ \max\left\{1, \frac{1}{\rho_3}\right\}\Big(|\beta_3|\gamma_\beta(|\xi^{(1)}|) + |\tilde{\alpha}_3|\gamma_\alpha(|\xi^{(1)}|)\Big). \end{aligned} \tag{13}$$

Because of the max function in (13) we consider two scenarii. First, let $\rho_3 \gg 1$ so that $\max\{1, 1/\rho_3\} = 1$ and define $\lambda_1, \lambda_2, \lambda_3 > 0$ sufficiently large so that $c_1' := c_1 - \frac{1}{2\lambda_1} > 0$, $c_2' := c_1c_3 - \frac{1}{2\lambda_2} > 0$, and $c_3' := c_2 - \frac{1}{2\lambda_3} > 0$. Applying Young's inequality, (13) we obtain

$$\begin{aligned} \dot{V}_2(\xi^{(2)}) \leq &- c_1'|\rho_3|^2 - c_2'|\beta_3|^2 - c_3'|\tilde{\alpha}_3|^2 + \frac{\lambda_1}{2}\gamma_\rho(|\xi^{(1)}|)^2 \\ &+ \frac{\lambda_2}{2}\gamma_\beta(|\xi^{(1)}|)^2 + \frac{\lambda_3}{2}\gamma_\alpha(|\xi^{(1)}|)^2 \\ \leq &- c'|\xi^{(2)}|^2 + \gamma(|\xi^{(1)}|) \end{aligned} \tag{14}$$

where $c' := \min\{c_1', c_2', c_3'\}$ and $\gamma(|\xi^{(1)}|) := \frac{\lambda_1}{2}\gamma_\rho(|\xi^{(1)}|)^2 + \frac{\lambda_2}{2}\gamma_\beta(|\xi^{(1)}|)^2 + \frac{\lambda_3}{2}\gamma_\alpha(|\xi^{(1)}|)^2$.

Now consider the case where $\max\{1, 1/\rho_3\} = 1/\rho_3$ and note that for any $\delta > 0$ and for any $\rho_3 \geq \delta$, $\max\{1, 1/\rho_3\} \leq 1/\delta$. Also, define $\lambda_4, \lambda_5, \lambda_6 > 0$ such that $c_1'' := c_1 - \frac{1}{2\lambda_4\delta} > 0$, $c_2'' := c_1c_3 - \frac{1}{2\lambda_5\delta} > 0$, and $c_3'' := c_2 - \frac{1}{2\lambda_6\delta} > 0$. Then, applying Young's inequality in (13), we obtain

$$\dot{V}_2(\xi^{(2)}) \leq -c''|\xi^{(2)}|^2 + \frac{1}{\delta}\gamma(|\xi^{(1)}|) \tag{15}$$

where, $c'' := \min\{c_1'', c_2'', c_3''\}$.

From (14) and (15), we have that subsystem (8a) is input-to-state stable with respect to $\xi^{(1)}$ for any $\delta > 0$ and for any $\rho_3 \geq \delta$. Moreover, from (11), $\xi^{(1)}$ converges to 0 exponentially and is bounded. Hence, the solutions of (8) are bounded and converge to zero.

Remark 3: The previous rationale is valid on the domain of definition of the closed-loop system, (8), which corresponds to $\cup_{k\leq m}\{\rho_k > 0\} \cap \{(\alpha_k,\ \beta_k) \in \mathbb{R}^2\}$. The latter also corresponds to the domain of attraction.

For clarity of exposition, so far we considered the simple graph in Figure 3. However, the previous arguments hold for any value of p, hence for a tree of any dimension. This is the rationale of our main statement, which we present next.

C. Multiagent systems of any dimension

Consider a swarm of n unicycles evolving on the plane. Consider further that the communication topology is represented by a directed spanning tree with h levels as in Figure 2. From (4) we have that each edge' state in level p depends only on its own and on the state of its parent edge in level $p-1$, $p \leq h$. Therefore, extending the arguments used in the previous section, for an arbitrary directed spanning tree with h levels, the system (3) in closed loop with (4) can be expressed as a nested cascaded system of the form

$$\begin{cases} \dot{\xi}^{(h)} &= f^{(h)}(\xi^{(h)}) + g^{(h)}(\xi^{(h-1)}, \xi^{(h)}), \\ &\vdots \\ \dot{\xi}^{(p)} &= f^{(p)}(\xi^{(p)}) + g^{(p)}(\xi^{(p-1)}, \xi^{(p)}) \\ &\vdots \\ \dot{\xi}^{(2)} &= f^{(2)}(\xi^{(2)}) + g^{(2)}(\xi^{(1)}, \xi^{(2)}) \\ \dot{\xi}^{(1)} &= f^{(1)}(\xi^{(1)}) \end{cases} \tag{16}$$

where the nominal systems $\dot{\xi}^{(p)} = f^{(p)}(\xi^{(p)})$ are as in (9) and, as for $g^{(2)}$, the interconnection terms $g^{(p)}(\xi^{(p-1)}, \xi^{(p)})$ can be bounded by

$$g^{(p)}\left(\xi^{(p-1)}, \xi^{(p)}\right) \leq \begin{bmatrix} \gamma_\rho(|\xi^{(p-1)}|) \\ \max\left\{1, \frac{1}{\bar{\rho}^{(p)}}\right\} \gamma_\beta(|\xi^{(p-1)}|) \\ \max\left\{1, \frac{1}{\bar{\rho}^{(p)}}\right\} \gamma_\alpha(|\xi^{(p-1)}|) \end{bmatrix}, \tag{17}$$

with $\bar{\rho}^{(p)} := \min\{\rho^{(p)}\}$, for any $2 \leq p \leq h$. Then, we have the following.

Proposition 1 (Main result): Consider n agents with non-holonomic constraints (1) and communicating over a directed spanning tree $\mathcal{G}_\mathcal{T}(\mathcal{V}, \mathcal{E})$. The smooth time-invariant controller (4) achieves full consensus, *i.e.*, $q_i \to q_j$ and $\theta_i \to \theta_j$, for all $(i, j) \in \mathcal{E}$ and for all inital conditions in the set

$$\mathcal{D} := \bigcup_{k \leq m} \{\rho_k > 0\} \cap \{(\alpha_k, \beta_k) \in \mathbb{R}^2\}$$

Sketch of Proof. The proof follows similar arguments as for the previous case-study.

First, for the nominal systems $\dot{\xi}^{(p)} = f^{(p)}(\xi^{(p)})$ with $p \leq h$, we use the candidate Lyapunov function $V_p(\xi^{(p)})$ defined in (10). Evaluating its total time derivative along the trajectories of the closed-loop system (9), we obtain, using also (6),

$$\begin{aligned} \dot{V}_p(\xi^{(p)}) = &- c_1|\rho^{(p)}|^2 - c_1c_3|\beta^{(p)}|^2 - c_2|\tilde{\alpha}^{(p)})|^2 \\ &\leq -c'|\xi^{(p)}|^2 < 0, \end{aligned} \tag{18}$$

which implies exponential stability of the origin for all initial conditions in $\mathcal{D}$.

Now, consider the last two equations in (16). Using (11) and (17), we obtain

$$\begin{aligned} \dot{V}_2(\xi^{(2)}) \leq &-c_1|\rho^{(2)}|^2 - c_1c_3|\beta^{(2)}|^2 - c_2|\tilde{\alpha}^{(2)}|^2 \\ +|\rho^{(2)}|\gamma_\rho(|\xi^{(1)}|) &+ \max\left\{1, \frac{1}{\bar{\rho}^{(2)}}\right\}\left(|\beta^{(2)}|\gamma_\beta(|\xi^{(1)}|)\right. \\ &\left. +|\tilde{\alpha}^{(2)}|\gamma_\alpha(|\xi^{(1)}|)\right). \end{aligned} \tag{19}$$

Let $\delta > 0$ be such that $\bar{\rho}^{(2)} \geq \delta$. Furthermore, define $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 > 0$ such that $c_1' := c_1 - \frac{1}{2\lambda_1} > 0$, $c_2' := c_1c_3 - \frac{1}{2\lambda_2} > 0$, $c_3' := c_2 - \frac{1}{2\lambda_3} > 0$, $c_1'' := c_1 - \frac{1}{2\lambda_4\delta} > 0$, $c_2'' := c_1c_3 - \frac{1}{2\lambda_5\delta} > 0$, and $c_3'' := c_2 - \frac{1}{2\lambda_6\delta} > 0$. Then, for any $\delta > 0$ such that $\bar{\rho}^{(2)} \geq \delta$, we have

$$\dot{V}_2(\xi^{(2)}) \leq -\bar{c}|\xi^{(2)}|^2 + \max\{1, \frac{1}{\delta}\}\gamma(|\xi^{(1)}|) \tag{20}$$

where $\bar{c} := \min\{c_1', c_2', c_3', c_1'', c_2'', c_3''\}$. From (20), we conclude that system $\dot{\xi}^{(2)}$ is input to state stable with respect to $|\xi^{(1)}|$, for any $\delta > 0$ such that $\bar{\rho}^{(2)} \geq \delta$. Therefore, since from (18), $\xi^{(1)} \to \mathbf{0}$ exponentially, it follows that the solutions $(\xi^{(1)}(t), \xi^{(2)}(t))$ are bounded, which implies that the origin $(\xi^{(1)}, \xi^{(2)}) = (\mathbf{0}, \mathbf{0})$ is attractive for al initial conditions on $\mathcal{D}$ —see Remark 3.

Proceeding recursively up to the first equation in (16) boundedness of $\xi^{(p)}(t)$ follows, for each $\delta > 0$ such that $\bar{\rho}^{(p)} \geq \delta$, and for all $3 \leq p \leq h$. Hence we conclude that the origin of the system (16) is attractive for al initial conditions in $\mathcal{D}$, which implies that full consensus is achieved.

IV. Simulation Results

To illustrate our theoretical statements, we performed some numerical simulations using six robots interconnected in a directed spanning tree, as depicted in Figure 4. The controller parameters are fixed to the values

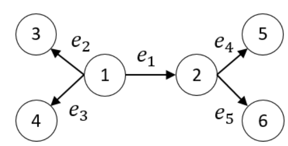
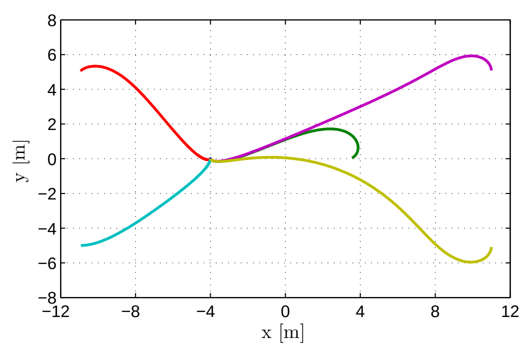
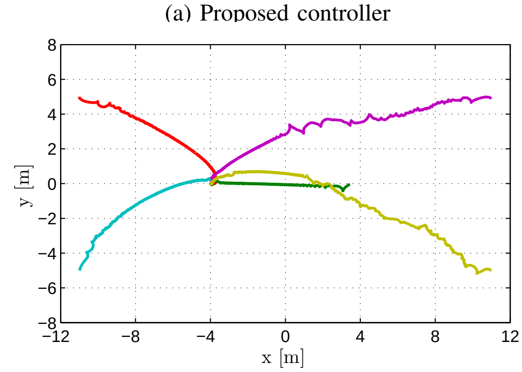


Fig. 4: Communication topology

$c_1 = 1$, $c_2 = 2$ and $c_3 = 0.5$. Furthermore, the initial conditions were set to $[x_1(0), y_1(0), \theta_1(0)] = [-4, 0, \pi/2]$, $[x_2(0), y_2(0), \theta_2(0)] = [3.5, 0, \pi/6]$, $[x_3(0), y_3(0), \theta_3(0)] = [-11, 5, \pi/4]$, $[x_4(0), y_4(0), \theta_4(0)] = [-11, -5, \pi/4]$, $[x_5(0), y_5(0), \theta_5(0)] = [11, 5, \pi/2]$, and $[x_6(0), y_6(0), \theta_6(0)] = [11, -5, -\pi/2]$.



(a) Proposed controller



(b) Time-varying controller [6]

Fig. 5: Leader-follower consensus simulation

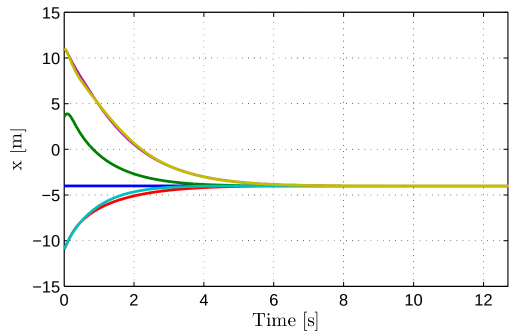


Fig. 6: Simulation results – x coordinates

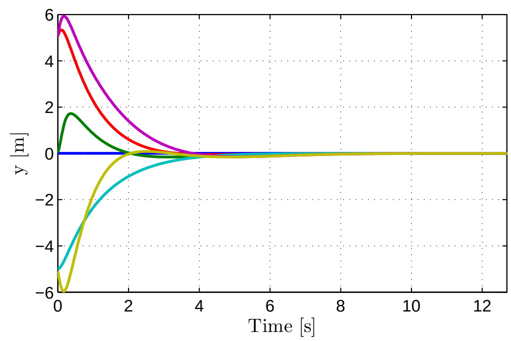


Fig. 7: Simulation results – y coordinates

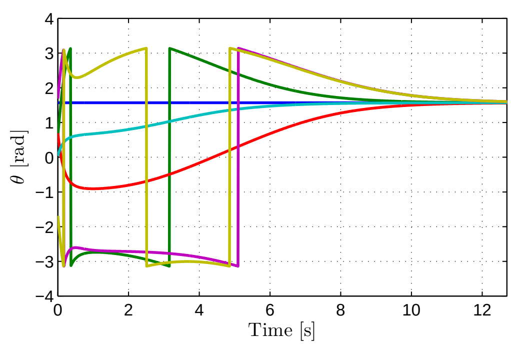


Fig. 8: Simulation results – orientation

The simulation results are presented in Figures 5a and 6-8. From figures 6-8 it is clear that asymptotic convergence to the agreement manifold is achieved both in position and in orientation, hence full consensus is achieved. For comparison, Figure 5b shows the simulation results for a smooth time-varying feedback taken from [6] based on the concept of δ-persistency of excitation. Although full consensus is also achieved in this case, the performance is somewhat degraded by the oscillations induced by the controller, whereas, for our controller, smooth motion is achieved.

V. Conclusions

We presented a solution to the full consensus problem for swarms of unicycles under a directed-spanning-tree communication topology using a distributed and smooth time-invariant feedback control law. The control methodology is based on a polar-coordinates-based model. Furthermore, it relies on the edge-based-graphs approach, which allows to express the consensus problem as a stabilisation problem, so it may be analysed through classical Lyapunov theory. Moreover, our controller is simple and provides a smooth motion of the agents, relative to persistency-of-excitation designs, and only uses relative information. Thus, it is well suited to practical applications.

Current work focuses on more general digraph topologies, higher-order systems, obstacle avoidance, and 3D autonomous vehicles.

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