Stability and Robustness of edge-agreement-based Consensus Protocols for undirected Proximity Graphs

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ABSTRACT
We address the consensus problem, with guarantee of connectivity maintenance, for multiagent systems of first and second order, communicating over a connected, undirected graph with proximity constraints. Our approach relies on so-called barrier Lyapunov functions, which encode the distance constraints between pairs of agents. Beyond the consensus algorithms, our primary contribution is to provide *strict* barrier Lyapunov functions, *i.e.*, positive definite and with negative definite derivative in the agreement subspace. Thus, uniform asymptotic stability of the consensus manifold and the maintenance of the connectivity are guaranteed. Furthermore, with the said barrier functions, we demonstrate the robustness of consensus protocols by establishing global input-to-state stability.

KEYWORDS
Multiagent systems; Constrained consensus; Edge agreement; Lyapunov methods

1. Introduction

The consensus control problem continues to receive great attention since it constitutes the basis for applications such as rendezvous, formation control, flocking, *etc.* —see Cortés and Egerstedt (2017); Olfati-Saber, Fax, and Murray (2007); Ren (2006). For undirected graphs it is well-known that a necessary and sufficient condition for consensus is that the communication graph must be connected (every agent communicates with one another). Although necessary, however, *assuming* connectivity may result conservative in various practical situations. For instance, in mobile robotics, the agents can exchange information only if they remain within a certain relative range.

Several works in the literature address the problem of connectivity maintenance. In Panagou, Stipanovic, and Voulgaris (2016), so-called barrier functions (see Section 2) are used to guarantee that all agents remain inside a given region, but without considering the communication topology. In Ji and Egerstedt (2007) barrier functions, as well as properties of the graph Laplacian matrix, are used to show consensus and preservation of connectivity. A general framework for connectivity maintenance, also using barrier functions, is proposed in Dimarogonas and Kyriakopoulos (2008) for both static and dynamic graphs. In Boskos and Dimarogonas (2017) robustness with respect to additional bounded inputs is also demonstrated. In Poonawala and Spong

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(2017) and Gasparri, Sabattini, and Ulivi (2017) potential functions and estimation of the algebraic connectivity are used to show global connectivity maintenance. In Yoo (2018) connectivity is guaranteed via a nonlinear interconnection that is implemented using a nonlinear transformation of the consensus errors.

Most currently in the literature, consensus is studied relying on node-based models of graphs. The control design and analysis of such models heavily rely on linear algebra and other tools tailored for, and limited to, linear time-invariant systems. In recent years, following Zelazo, Rahmani, and Mesbahi (2007), the so-called *edge-agreement* approach has emerged. Further developed in Zelazo and Mesbahi (2011) and Zelazo, Schuler, and Allgöwer (2013), among others, the edge-based analysis framework presents significant advantages over the node-based perspective since consensus is recast as a problem of attractivity of the origin (in the edges space) —*cf.* Alvarez-Jarquín and Loría (2014), a property that is well understood in control theory. In particular, relying on an edge-based representation naturally allows for the use of Lyapunov stability theory. From a practical viewpoint, edge-based representations implicitly rely on relative, rather than absolute, measurements; this makes it more attractive in many applications. In addition, other relaxations, such as to the case when measurements are quantized, are made simpler under the edge-based approach (Dimarogonas and Johansson (2010)).

Works on edge-based consensus include diverse scenarii and contributions: in Nguyen (2017) a consensus controller in the presence of disturbances and uncertainties is designed –see also Nguyen, Narikiyo, and Kawanishi (2018) for an optimal controller design; in Zhao, Liu, Wen, Ren, and Chen (2018) finite-time agreement is achieved for second order systems using edge-based notions and in Chowdhury, Sukumar, Maghenem, and Loría (2018) convergence rates are given for edge-Laplacian-based consensus of first-order multiagent systems with time-varying interconnections. Notably, in the latter a strict Lyapunov function is constructed, which leads to estimating the convergence rate. In Mukherjee and Zelazo (2018) the edge agreement protocol is extended to directed graphs and robustness of consensus is shown for second-order systems with respect to edge-weight disturbances. In Zeng, Wang, and Zheng (2016) consensus over directed graphs containing spanning trees is shown with a strict Lyapunov function. However, the control is designed based on the small-gain theorem, which greatly restricts the control and hence avoids the direct extension of this methodology for connectivity maintenance. Thus, in none of these references, connectivity maintenance is addressed.

Thus, while edge-based agreement is beginning to gain interest for the control design of multiagent systems, state-dependent constrained consensus, such as the connectivity maintenance problem, has received limited attention so far. Moreover, in general, the works focusing on connectivity maintenance rely either on global information, which must be estimated, or on the construction of non-strict Lyapunov functions. Hence, asymptotic stability is shown using auxiliary theorems that do not allow to assert stronger properties in terms of robustness and uniformity.

In this paper we present distributed consensus controllers that guarantee consensus and connectivity maintenance for undirected connected graphs, even in the presence of disturbances. Contrary to Dimarogonas and Johansson (2010) where consensus is only guaranteed when the graph is a spanning tree, we use, based on the edge-agreement framework, a reduced-order model (see Section 2) which allows us to design a barrier-function-based control to achieve consensus with connectivity maintenance for any connected graph and for both first and second-order systems. Beyond the controllers themselves, our primary contribution is to provide constructive proofs of our main statements. That is, we contribute with *strict* Lyapunov functions. Hence, we establish uniform asymptotic and input-to-state stability of the closed-loop system, simultaneously with connectivity maintenance. Our results hold for first and second-order systems interacting through any connected graph.

The rest of the paper is organized as follows: in Section 2 we recall some material on the edge-based graph theory and we formulate the problem that we address. In Section 3 we present our main results on consensus with connectivity maintenance for first and second-order systems. In Section 4 we provide an analysis of robustness in terms of input-to-state stability, followed by some simulation results in Section 5. Concluding remarks are given in Section 6.

2. Preliminaries

Notation. The real n-coordinate space, with $n \in \mathbb{N}$, is denoted as $\mathbb{R}^n$; $\mathbb{R}^n_{\geq 0}$ and $\mathbb{R}^n_{>0}$ are the sets of real n-vectors with all elements non-negative and positive, respectively. The notation $|x|$ is used for the Euclidean norm of a vector $x \in \mathbb{R}^n$. We use $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ to denote a weighted graph defined by a node set $\mathcal{V} = \{1, 2, \dots, n\}$ with cardinality n and corresponding to the agents' states, an edge set $\mathcal{E} \subseteq \mathcal{V}^2$ with cardinality m and characterizing the information exchange between agents, and a positive diagonal matrix $\mathcal{W} \in \mathbb{R}^{m \times m}$, whose entries represent the weights of the edges. An edge, e_k, is an ordered pair $(i, j) \in \mathcal{E}$ if and only if there exists a connection *from* node i *to* node j. In an undirected graph $(i, j) \in \mathcal{E}$ implies $(j, i) \in \mathcal{E}$. An *orientation* of an undirected graph $\mathcal{G}$ is the assignment of directions to its edges. An undirected graph is said to be *connected* if there is an undirected path between every pair of distinct nodes. A *tree* is a subgraph in which every node has exactly one parent except for one node, called the root, which has no parent and which has a path to every other node. A *spanning tree* is a tree subgraph containing all nodes in $\mathcal{V}$.

2.1. Motivation

In the study of consensus of large-scale interconnected dynamical systems, it is typical to turn to graph theory for a mathematical representation of the overall network. For first-order systems, $\dot{x}_i = u_i$, $x_i \in \mathbb{R}^N$, $u_i \in \mathbb{R}^N$, $i \leq n$, under the action of the classical consensus algorithm, $u_i = -\sum_{j=1}^{n} a_{ij}(x_i - x_j)$ with $a_{ij} \geq 0$, the problem boils down to study the behaviour of the system

$$\dot{x} = -\left[L \otimes I_N\right] x, \quad [L]_{ij} := \begin{cases} -a_{ij} & \text{if } i \neq j \\ \displaystyle\sum_{k \leq n} a_{ik} & \text{otherwise} \end{cases} \tag{1}$$

where $x^\top := [x_1^\top \cdots x_n^\top]$, '$\otimes$' denotes the Kronecker product, I_N the identity matrix of dimension N, L is the so-called weighted graph Laplacian matrix $L \in \mathbb{R}^{n \times n}$. For undirected connected graphs with constant affine interconnections, L is a symmetric positive semi-definite matrix and has zero as a simple eigenvalue with associated eigenvector $\mathbf{1} := [1 \cdots 1]^\top$ —see *e.g.*, Merris (1994). This well-known statement of linear algebra is at the basis of numerous works on consensus. However, if the network's interconnections are not constant or affine, the use of linear algebra and graph theory may appear limited to study dynamical behaviour of the interconnected systems. Alternatively, one may turn to stability theory and, more particularly, to the use of Lyapunov's method. In that regard, for the system (1) consensus means that the manifold

$$\mathcal{S} := \{x \in \mathbb{R}^{nN} \,:\, x_1 = x_2 = \cdots = x_n\}$$

is attractive. Now, while studying the consensus problem relying on Lyapunov's method allows for generalizations otherwise impossible using tools for linear systems, set-stability analysis poses other considerable difficulties. To overcome the latter, in this paper we appeal to an alternative representation of networked systems, based on the dynamics of the *edges*, as opposed to that of the *nodes*. For clarity of exposition, we start by recalling some notions related to the edge-based representation; we direct the reader to Zelazo et al. (2007) for greater detail.

2.2. Edge-representation and reduced-order dynamics

To introduce the edge-representation, following Zelazo et al. (2013), we start by stressing that the Laplacian of a connected undirected graph admits the natural factorization $L := E\mathcal{W}E^\top$ where $E \in \mathbb{R}^{n\times m}$ is the so-called incidence matrix and its elements are defined as follows. $[E]_{ik} = 1$ if i is the initial node of edge e_k, $[E]_{ik} = -1$ if i is the terminal node of edge e_k, and $[E]_{ik} = 0$ otherwise. Then, the edge state variables are defined as

$$z := \big[E^\top \otimes I_N\big]x, \quad z \in \mathbb{R}^{mN}. \tag{2}$$

That is, the vector $z := \big[z_1^\top \,\cdots\, z_k^\top \,\cdots\, z_m^\top\big]^\top$, represents differences between any pair of nodes. More precisely, for each $k \le m$, and $i, j \in \mathcal{V}$, $z_k := x_i - x_j$. In these coordinates the networked systems' dynamics, equation (1), is replaced by

$$\dot{z} = -\left[L_e \otimes I_N\right] z, \quad L_e := E^\top E\,\mathcal{W}. \tag{3}$$

The weighted *edge* Laplacian matrix $L_e \in \mathbb{R}^{m\times m}$ is the 'edge dual' of L and, as such, it has the same non-zero eigenvalues as L hence, rank(L_e) =rank$(L) = n-1$.

Now, as it is well known, consensus holds if and only if $\mathcal{G}$ contains at least one spanning tree. This suggests that the graph dynamics may be studied by concentrating on that of a reduced-order system, whose states correspond exclusively to those of the arcs in a tree. Indeed, following an appropriate labelling of the edges (see Zelazo et al. (2013) for details) we may partition the edge states, and correspondingly the incidence matrix E, as

$$z = \begin{bmatrix} z_t^\top & z_c^\top \end{bmatrix}^\top \quad \text{and} \quad E = [\, E_t \quad E_c \,]\,. \tag{4}$$

The states $z_t \in \mathbb{R}^{(n-1)N}$, which correspond to the first $(n-1)N$ elements of z, denote the states of the edges forming an arbitrary spanning tree contained in a connected graph $\mathcal{G}$, while the states z_c correspond to the states of the arcs not in the tree. The states z and z_t are correlated as follows,

$$z = \Big[R^\top \otimes I_N\Big]\, z_t, \quad R := [\, I \quad T \,]\,, \tag{5}$$

where $T := \left(E_t^\top E_t\right)^{-1} E_t^\top E_c$, while z_t and z_c satisfy

$$z_c = \left[T^\top \otimes I_N\right] z_t. \tag{6}$$

Correspondingly, the incidence matrices E, E_t and E_c satisfy

$$E_t T = E_c \tag{7}$$

and

$$E = E_t R. \tag{8}$$

Thus, differentiating on both sides of the first equation in (5) and using (3), (5) again, and (8), we obtain

$$\dot{z}_t = -\left[E_t^\top E_t R \mathcal{W} R^\top \otimes I_N\right] z_t. \tag{9}$$

The latter equation is remarkable because even though it is of reduced dimension ($z_t \in \mathbb{R}^{(n-1)N}$), it completely captures the behaviour of the whole system. In particular, consensus for (1) holds if and only if the origin $\{z_t = 0\}$ is attractive for the solutions of (9).

In this paper, we demonstrate consensus via Lyapunov's direct method, by constructing strict Lyapunov functions for (9). By recasting the problem as one of stability of the origin, we establish stronger properties than mere non-uniform convergence to the manifold $\mathcal{S}$; we establish robustness vis-a-vis of bounded disturbances and uniform asymptotic stability. In that regard, it is fitting to remark that such properties are harder to obtain using the node-based representation (1). As a matter of fact, apart from (Restrepo, Loría, Sarras, & Marzat, 2020), where directed-spanning-tree graphs are considered, we are unaware of strict Lyapunov functions for (1) in the literature, when the Laplacian, $L(x)$, is state dependent, as is the case for proximity graphs.

2.3. Connectivity maintenance

As we mentioned, connectivity is a necessary and sufficient condition for consensus. Yet, assuming that this condition holds may result conservative in concrete applications, such as those involving autonomous vehicles. Therefore, in this paper, in addition to consensus, we address the following problem.

Definition 2.1 (Connectivity maintenance)**.** Let $\Delta > 0$ denote the maximal distance between any pair of nodes i and j such that the communication between them, through the arc $e_k = (i, j)$, is reliable. We say that the graph's connectivity is maintained (hence, the proximity constraint holds) if the set

$$\mathcal{J} := \left\{z \in \mathbb{R}^{mN} : |z_k| < \Delta,\ \forall\, k \leq m\right\}, \tag{10}$$

where $z_k = x_i - x_j$, is forward invariant. That is, if $|z_k(0)| < \Delta$ implies that $z(t) \in \mathcal{J}$ for all $t \geq 0$.

In order to design a decentralized controller to guarantee consensus with connectivity maintenance, we rely on Dimarogonas and Kyriakopoulos (2008) to design a gradient-type control law that we derive using a *connectivity potential*, which is defined as follows.

Definition 2.2 (Connectivity potential)**.** Let $p_0 \in \mathbb{R}$, $\mathcal{B}_\Delta := \{z_k \in \mathbb{R}^N \, : \, |z_k| < \Delta\}$ and, for each $k \leq m$, let $\alpha_k : \big[0, \Delta^2\big) \to \mathbb{R}_{\geq 0}$, $s \mapsto \alpha_k(s)$, be $\mathcal{C}^1$ and non-decreasing on $\big[0, \Delta^2\big)$, such that $\alpha_k(s) \to \infty$ as $s \to \Delta^2$, and $p_k : \mathcal{B}_\Delta \to \mathbb{R}_{>0}$, defined as

$$p_k(z_k) := \frac{\partial \alpha_k}{\partial s}(|z_k|^2), \tag{11}$$

is also non-decreasing, $p(z_k) \geq p_0 > 0$ for all $|z_k| < \Delta$, and $p_k(z_k) \to \infty$ as $|z_k| \to \Delta$. Then, we define the connectivity potential $P(z) := \mathrm{diag}\big[p_k(z_k)\big] \in \mathbb{R}^{m \times m}$.

The connectivity potential is naturally derived from a so-called *Barrier* function $U_k : \mathcal{B}_\Delta \to \mathbb{R}_{\geq 0}$, defined as

$$U_k(z_k) := \alpha_k(|z_k|^2). \tag{12}$$

Indeed, note that

$$\frac{\partial U_k}{\partial z_k} = 2p_k(z_k)z_k \tag{13}$$

For instance,

$$U_k(z_k) = \ln\left(\frac{\Delta^2}{\Delta^2 - |z_k|^2}\right)$$

is a Barrier Lyapunov function.

Remark 1. Modulo performing a change of coordinates to the nodes space, an example of barrier functions satisfying the previous definition is the so-called "edge tension" function used in Ji and Egerstedt (2007) and Boskos and Dimarogonas (2017). Similarly, the Barrier Lyapunov Functions used in Tang, Keng, and He (2013) and references therein, are also examples of barrier functions as per the previous definition.

3. Consensus in the edges space

3.1. First-order system

Consider n systems evolving in an N-dimensional workspace,

$$\dot{q}_i = u_i, \quad q_i \in \mathbb{R}^N, \tag{14}$$

where q_i and u_i denote the position and control input of each agent, respectively. In compact form, the systems' states are collected in the vector $q = \big[q_1^\top \ \cdots \ q_n^\top\big]^\top \in \mathbb{R}^{nN}$ and the control inputs into $u = \big[u_1^\top \ \cdots \ u_n^\top\big]^\top \in \mathbb{R}^{nN}$.

It is also assumed that the systems communicate according to a connected, undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, I_m)$ and are subject to the proximity constraint that two agents communicate if and only if they are within a zone in which the communication of any pair of agents is guaranteed. More precisely, $(i,j) \in \mathcal{E}$ if and only if $|q_i - q_j| < \Delta$, where Δ is given *a priori*. Hence, the objective is to design decentralized control laws u_i such that the agents converge to the same position, while connectivity as per in Definition 2.1 is maintained. To deal with this problem, we start by rewriting the system in edge-coordinates

$$z := \big[E^\top \otimes I_N\big] q. \tag{15}$$

Then, differentiating on both sides of (15) using $\dot{q} = u$ —*cf.* (14), we obtain

$$\dot{z} \;=\; \big[E^\top \otimes I_N\big] u. \tag{16}$$

Next, let the control law be given by

$$u(z) = -c_1 \big[E P(z) \otimes I_N\big] z, \tag{17}$$

where $c_1 > 0$ is the interconnection strength. Replacing (17) into (16) and using (3) with $\mathcal{W} = I_m$, we obtain

$$\dot{z} \;=\; -c_1 \big[L_e P(z) \otimes I_N\big] z, \tag{18}$$

so, proceeding as for Equation (9), we obtain the corresponding reduced-order system,

$$\dot{z}_t = -c_1 \big[E_t^\top E_t R \tilde{P}(z_t) R^\top \otimes I_N\big] z_t, \tag{19}$$

where, for consistency of notation, we defined

$$\tilde{P}(z_t) := P\big([R^\top \otimes I_N] z_t\big). \tag{20}$$

Notice that, from (5), $\tilde{P}(z_t) \equiv P(z)$. For this system we have the following.

Proposition 3.1. *Consider the system* (16) *in closed-loop with* (17)*. Assume that the graph is connected. Then, the controller* (17) *guarantees consensus with connectivity maintenance. Furthermore the function* $V : \mathcal{J}_t \to \mathbb{R}_{\geq 0}$*, where*

$$\mathcal{J}_t := \big\{ z_t \in \mathbb{R}^{(n-1)N} : |z_k| < \Delta_k, \; \forall\, k \leq m \big\},$$

defined as

$$V(z_t) := \frac{1}{2} \sum_{k \leq m} U_k(z_k) \tag{21}$$

with U_k *given in* (12)*, is a strict Lyapunov function for the closed-loop system* (19)*.*

Proof. To obtain the total derivative of the function $z_t \mapsto V(z_t)$, we start by computing its gradient. To that end, we use Eq. (5) to recognize that the right-hand side of (21) may be denoted using a function $z \mapsto \tilde{V}(z)$, so that $V(z_t) =: \tilde{V}([R^\top \otimes I_N] z_t)$.

Then, using (13), we obtain

$$\frac{\partial V}{\partial z_t} \quad = \quad \big[R \otimes I_N\big]\big[\tilde{P}(z_t) \otimes I_N\big]z = \big[R\tilde{P}(z_t)R^\top \otimes I_N\big]z_t. \tag{22}$$

Hence, the derivative of $V(z_t)$ along the trajectories of (19) is

$$\dot{V}(z_t) = -c_1 z_t^\top \big[R\tilde{P}(z_t)R^\top E_t^\top E_t R\tilde{P}(z_t)R^\top \otimes I_N\big]z_t.$$

By definition, the entries in the diagonal of $\tilde{P}(z_t)$ are positive, hence, the matrix $R\tilde{P}(z_t)R^\top$ is symmetric positive-definite. Furthermore, the matrix $E_t^\top E_t =: L_{et}$ corresponds to the Laplacian of a spanning tree contained in $\mathcal{G}$ and it is symmetric positive definite. Therefore, defining $\lambda_{min}(L_{et})$ as the smallest eigenvalue of L_{et}, let $c_1' = c_1\lambda_{min}(L_{et}) > 0$, so we obtain

$$\dot{V}(z_t) \le -c_1' z_t^\top \big[(R\tilde{P}(z_t)R^\top)^2 \otimes I_N\big]z_t. \tag{23}$$

Thus, $\dot{V}(z_t) < 0$ for all $z_t \in \mathcal{J}_t \backslash \{0\}$ and V in (21) is a strict Lyapunov function for the closed-loop system (19).

Next we establish connectivity maintenance, or equivalently forward invariance of the set $\mathcal{J}$. We proceed by contradiction. Suppose that there exists $T > 0$ such that for all $t \in [0, T)$, $z(t) \in \mathcal{J}$ and $z(T) \notin \mathcal{J}$. That is, we have $|z_k(t)| \to \Delta$ as $t \to T$ for at least one $k \le m$. Consequently, from the definition of V, we have $V(z_t(t)) \to \infty$ as $t \to T$. This, however, is in contradiction with (23), which implies that the Lyapunov function $V(z_t(t))$ is bounded, *i.e.*, $V(z_t(t)) \le V(z_t(0)) < \infty$ for all $t \ge 0$. Thus, connectivity is preserved.

Now, we show that $\mathcal{J}$ corresponds to the domain of attraction of (19). To that end, define the subset $\mathcal{J}_\varepsilon \subset \mathcal{J}$ as

$$\mathcal{J}_\varepsilon := \{z \in \mathbb{R}^{mN} : |z_k| < \Delta - \varepsilon, \quad \forall k \le m\} \tag{24}$$

where $\varepsilon \in (0, \Delta)$ is an arbitrarily small constant. From Definition 2.2 and (12) it follows that $V(z_t)$ is positive definite for all z_t making part of z contained in the closure $\bar{\mathcal{J}}_\varepsilon$ of $\mathcal{J}_\varepsilon$ and it can be bounded as

$$\beta|z_t|^2 \le V(z_t) \le h(|z_t|), \tag{25}$$

where β is a positive constant and $h(\cdot)$ is a positive strictly increasing function defined everywhere in $\bar{\mathcal{J}}_\varepsilon$ and $h(0) = 0$. This means that $V(z_t) \to 0$ as $z_t \to 0$. Therefore, from (23) it follows that for all trajectories of the closed-loop system starting in $\mathcal{J}_\varepsilon$, the origin is uniformly asymptotically stable. Moreover, since ε can be chosen arbitrarily small, taking the limit $\varepsilon \to 0$, we have uniform asymptotic stability of the origin of the closed-loop system for all trajectories starting in $\mathcal{J}$. Thus, consensus is guaranteed with preserved connectivity. □

3.2. Second-order systems

Let us consider now second-order systems,

$$\dot{q}_i = v_i \tag{26a}$$
$$\dot{v}_i = u_i \tag{26b}$$

where $u_i \in \mathbb{R}^N$ corresponds to the control input. As in the previous scenario let the communication topology of the system be represented by an undirected, connected graph under proximity constraints.

Applying the edge-transformation (15) on $q \in \mathbb{R}^{nN}$ and using (26), we obtain

$$\dot{z} = \big[E^\top \otimes I_N\big] v \tag{27a}$$
$$\dot{v} = u \tag{27b}$$

where $v = \big[v_1^\top \ \cdots \ v_n^\top\big]^\top \in \mathbb{R}^{nN}$.

For this system the objective is to design decentralized control laws u_i guaranteeing that $z_k \to 0$ $\forall k \leq m$ and $v_i \to 0$ $\forall i \in \mathcal{V}$ as $t \to \infty$ while ensuring that the graph $\mathcal{G}$ remains connected for all time, *i.e.*, $|z_k(t)| < \Delta$, $\forall t \geq 0$.

Akin to (17) we introduce the control law

$$u := -c_1\big[EP(z) \otimes I_N\big] z - c_2 v \tag{28}$$

where c_1, $c_2 > 0$ and $P(z) := \mathrm{diag}\,[p_k(z_k)]$ for all $k \leq m$. Then, we have the following.

Proposition 3.2. *Consider the system* (27) *in closed-loop with* (28)*. Assume that the graph is connected. Then, the controller* (28) *guarantees position consensus with connectivity maintenance. Furthermore, the function*

$$V(z_t, v) = \frac{c_1}{2}\sum_{k\leq m} U_k(z_k) + \frac{1}{2}|v|^2 + c_3\, z_t^\top \big[L_{et}^{-1}E_t^\top \otimes I_N\big] v + \frac{c_2 c_3}{2} z_t^\top \big[L_{et}^{-1} \otimes I_N\big] z_t, \tag{29}$$

where $c_1 > 0$*,* $c_2 > c_3 > 0$*,* $L_{et} := E_t^\top E_t$*, and* U_k *is defined in* (12)*, is a strict Lyapunov function for the closed-loop system.*

Proof. First, note that $\sum_{k\leq m} U_k(z_k)$ is a positive definite and strictly increasing function on $\mathcal{J}$. Moreover, V can also be written as

$$V(z_t, v) = \frac{c_1}{2}\sum_{k\leq m} U_k(z_k) \ + \frac{1}{2}\begin{bmatrix} z_t \\ v \end{bmatrix}^\top \left[\begin{bmatrix} c_2 c_3 L_{et}^{-1} & c_3 L_{et}^{-1} E_t^\top \\ c_3 E_t L_{et}^{-1} & I \end{bmatrix} \otimes I_N\right] \begin{bmatrix} z_t \\ v \end{bmatrix}. \tag{30}$$

Recall first that under the assumption that the graph is connected, the matrix L_{et}^{-1}, which is the inverse of the edge Laplacian for spanning tree, exists and is positive definite. Then, using the Schur complement condition on the second term of the right-hand side of (30), positive-definiteness of V in z_t and v follows. Now, taking the time derivative of (29) and noting that

$$\frac{\partial}{\partial z_t}\sum_{k\leq m} U_k(z_k) = 2\big[R\tilde{P}(z_t)R^\top \otimes I_N\big] z_t \tag{31}$$

where R is defined in (5), we obtain

$$\begin{aligned}\dot V(z_t,v) &= c_1 z_t^\top\big[R\tilde P(z_t)R^\top E_t^\top\otimes I_N\big]v - c_1 z_t^\top\big[R\tilde P(z_t)R^\top E_t^\top\otimes I_N\big]v - c_2 v^\top v\\ &\quad - c_1c_3 z_t^\top\big[L_{et}^{-1}E_t^\top E_t R\tilde P(z_t)R^\top\otimes I_N\big]z_t - c_2c_3 z_t^\top\big[L_{et}^{-1}E_t^\top\otimes I_N\big]v\\ &\quad + c_3\, v^\top\big[E_tL_{et}^{-1}E_t^\top\otimes I_N\big]v + c_2c_3\, z_t^\top\big[L_{et}^{-1}E_t^\top\otimes I_N\big]v\\ &= - c_1c_3 z_t^\top\big[L_{et}^{-1}E_t^\top E_t R\tilde P(z_t)R^\top\otimes I_N\big]z_t - v^\top\big[(c_2 I - c_3E_tL_{et}^{-1}E_t^\top)\otimes I_N\big]v.\end{aligned}$$

Since the non-zero eigenvalues of L and of L_{et} coincide, $\lambda_{max}(E_tL_{et}^{-1}E_t^\top) = \lambda_{max}(E_t^\top E_tL_{et}^{-1}) = 1$. Then, letting $c_1' := c_1c_3$ and $c_2' := (c_2 - c_3)$, we have

$$\dot V(z_t,v) = - c_1' z_t^\top\big[R\tilde P(z_t)R^\top\otimes I_N\big]z_t - c_2'|v|^2. \tag{32}$$

Since R is full row rank and, by Definition 2.2, $\tilde P(z_t)$ is a diagonal matrix with strictly positive entries for all z_t such that $z\in\mathcal J$, we have that $\dot V(z_t,v)<0$ in $\mathcal J\times\mathbb R^{nN}\backslash\{0,0\}$.

Forward invariance of $\mathcal J$ is inferred using the same arguments as in the proof of Proposition 3.1. Thus, connectivity is preserved for any initial conditions $(z(0),v(0))\in\mathcal J\times\mathbb R^{nN}$. Finally, note that $V(z_t,v)$ is positive definite for all $v\in\mathbb R^{nN}$ and all z_t part of z such that $(z,v)\in\bar{\mathcal J}_\varepsilon\times\mathbb R^{nN}$ and satisfies

$$\alpha_1|z_t|^2+\beta_1|v|^2\le V(z_t,v)\le h(|z_t|)+\beta_2|v|^2 \tag{33}$$

where α_1, β_1, and β_2 are positive constants and $h(\cdot)$ is a positive strictly increasing function defined everywhere in $\bar{\mathcal J}_\varepsilon$ and $h(0)=0$. This means that $V(z_t,v)\to 0$ as $(z_t,v)\to(0,0)$. Therefore, from (32) we have that for all trajectories of the closed-loop system starting in $\mathcal J_\varepsilon\times\mathbb R^{nN}$, the origin is uniformly asymptotically stable, i.e., $v_i(t)\to 0\ \forall i\le n$ and $z_k(t)\to 0\ \forall k\le m$, or equivalently $q_i\to q_j\ \forall i,j\in\mathcal V$, as $t\to\infty$. Moreover, since ε can be chosen arbitrarily small, taking the limit $\varepsilon\to 0$, we have uniform asymptotic stability of the origin of the closed-loop system for all trajectories starting in $\mathcal J\times\mathbb R^{nN}$. Thus, consensus is achieved with preserved connectivity. □

4. Robustness Analysis

In this section we use the strict Lyapunov functions previously constructed to analyse the robustness of the edge consensus with connectivity maintenance. For each case, we establish input-to-state stability.

4.1. First-order systems

Consider first the case of a single-integrator system with an external bounded input, that is,

$$\dot q_i = u_i + d_i. \tag{34}$$

Applying the edge transformation (15) and control law (17), the reduced order system in closed loop becomes

$$\dot z_t = -c_1\big[E_t^\top E_t R\tilde P(z_t)R^\top\otimes I_N\big]z_t + \big[E_t^\top\otimes I_N\big]d \tag{35}$$

where $d := \left[d_1^\top \cdots d_n^\top\right]^\top \in \mathbb{R}^{nN}$ —*cf.* (19). The result is stated in the following Proposition.

Proposition 4.1. *Consider a multiagent system with a communication topology given by the initially connected graph $\mathcal{G}$ and described by the reduced order system* (35)*. For any bounded external input d, the graph remains connected, that is, $|z_k(t)| < \Delta$, $\forall k \leq m$ and $\forall t \geq 0$. Furthermore, the system is input-to-state stable with respect to d.*

Proof. Define the following Lyapunov function

$$V(z_t) = \frac{1}{2} \sum_{k \leq m} U_k(z_k). \tag{36}$$

Differentiating V with respect to time we obtain

$$\dot{V}(z_t) = -c_1 z_t^\top \left[R\tilde{P}(z_t)R^\top E_t^\top E_t R\tilde{P}(z_t)R^\top \otimes I_N\right] z_t + z_t^\top \left[R\tilde{P}(z_t)R^\top E_t^\top \otimes I_N\right] d. \tag{37}$$

Now, given c_1 let $\delta > 0$ be such that $c_1' := \left(c_1 - \frac{1}{2\delta}\right) \lambda_{min}(E_t^\top E_t) > 0$. Then, using Young's inequality on the second term of the right-hand side of (37) we have

$$\dot{V}(z_t) \leq -\, c_1' z_t^\top \left[(R\tilde{P}(z_t)R^\top)^2 \otimes I_N\right] z_t + \frac{\delta}{2}|d|^2. \tag{38}$$

In order to show connectivity maintenance it suffices to show that in the proximity of the limits of the connectivity region, that is, as $|z_k| \to \Delta$ for any $k \leq m$, the negative definite term in z_t in equation (38) dominates the second term, which is bounded by assumption. More precisely, let $\bar{d} := \sup_{t \geq 0} |d(t)|$ and $\varepsilon \in (0, \Delta)$ be an arbitrarily small constant. Let z_t be such that, there exists at least one $k \leq m$ such that $|z_k| \geq (\Delta - \varepsilon)$. Then, $|z_t| \geq \Delta - \varepsilon$, so from (38), the definition of $\tilde{P}(z_t)$ and Definition 2.2, we have

$$\dot{V}(z_t) \leq -c_1' \left[\frac{\partial \alpha_k}{\partial z_k}((\Delta - \varepsilon)^2)(\Delta - \varepsilon)\right]^2 + \frac{\delta}{2}|d|^2.$$

In turn from Definition 2.2 we have that $\frac{\partial \alpha_k}{\partial s}(s)$ is continuous, non-decreasing, and $\frac{\partial \alpha_k}{\partial s}(s) \to \infty$ as $s \to \Delta^2$. Then, there exists $\varepsilon^*(\bar{d}) > 0$ such that for all $\varepsilon < \varepsilon^*$, $\dot{V}(z_t) \leq 0$. Hence, connectivity maintenance follows from the same arguments as in Proposition 3.1. Furthermore, from Definition 2.2 and (38) we have

$$\dot{V}(z_t) \leq -c_1' p_0 |z_t|^2 + \frac{\delta}{2}|\bar{d}|^2. \tag{39}$$

Thus, the system (35) is input-to-state stable with respect to bounded external inputs. □

4.2. Second-order systems

In a similar way as for the first-order systems, consider a second-order system with an bounded external input, i.e.,

$$\dot{q}_i = v_i \tag{40a}$$
$$\dot{v}_i = u_i + d_i \tag{40b}$$

Applying the edge transformation (15) and control law (28), the reduced order system in closed loop becomes

$$\dot{z}_t = \left[E_t^\top \otimes I_N\right] v \tag{41a}$$
$$\dot{v} = -c_1\left[E_t R P(z_t) R^\top \otimes I_N\right] z_t - c_2 v + d \tag{41b}$$

where $d := \left[d_1^\top \ \cdots \ d_n^\top\right]^\top \in \mathbb{R}^{nN}$. Then, we have the following.

Proposition 4.2. *Consider a multiagent system with a communication topology given by the initially connected graph $\mathcal{G}$ and described by the reduced order system (41). For any bounded external input d, the graph remains connected, that is, $|z_k(t)| < \Delta$, $\forall k \leq m$ and $\forall t \geq 0$. Furthermore, the system is input-to-state stable with respect to d provided that $c_1 > 0$ and $c_2 > c_3 > 0$.*

Proof. Taking the Lyapunov function $V(z_t, v)$ from (29) and differentiating with respect to time we obtain

$$\begin{aligned}
\dot{V}(z_t, v) = & \, c_1 z_t^\top\left[R\tilde{P}(z_t)R^\top E_t^\top \otimes I_N\right]v - c_2 v^\top v - c_1 z_t^\top\left[R\tilde{P}(z_t)R^\top E_t^\top \otimes I_N\right]v \\
& - c_1 c_3 z_t^\top\left[L_{et}^{-1}E_t^\top E_t R\tilde{P}(z_t)R^\top \otimes I_N\right]z_t - c_2 c_3 z_t^\top\left[L_{et}^{-1}E_t^\top \otimes I_N\right]v + v^\top d \\
& + c_3 z_t^\top\left[L_{et}^{-1}E_t^\top \otimes I_N\right]d + c_3 v^\top\left[E_t L_{et}^{-1}E_t^\top \otimes I_N\right]v + c_2 c_3 z_t^\top\left[L_{et}^{-1}E_t^\top \otimes I_N\right]v \quad (42)\\
= & - c_1 c_3 z_t^\top\left[L_{et}^{-1}E_t^\top E_t R\tilde{P}(z_t)R^\top \otimes I_N\right]z_t - v^\top\left[(c_2 I - c_3 E_t L_{et}^{-1}E_t^\top) \otimes I_N\right]v \\
& + v^\top d + c_3 z_t^\top\left[L_{et}^{-1}E_t^\top \otimes I_N\right]d,
\end{aligned}$$

where we recall that $L_{et}^{-1} := (E_t^\top E_t)^{-1}$ exists and is positive definite under the assumption that the graph is connected.

Given c_2 and c_3, let $\delta > 0$ be such that $c_2' := \left(c_2 - c_3 - \frac{1}{2\delta}\right) > 0$. Then, using Young's inequality on the third and fourth terms of the right-hand side of (42), we have

$$\begin{aligned}
\dot{V}(z_t, v) \leq & - c_1 c_3 z_t^\top\left[R\tilde{P}(z_t)R^\top \otimes I_N\right]z_t - v^\top\left[\left(\left(c_2 - \frac{1}{2\delta}\right)I - c_3 E_t L_{et}^{-1}E_t^\top\right) \otimes I_N\right]v \\
& + \frac{c_3^2}{2\delta} z_t^\top\left[(E_t^\top E_t)^{-1} \otimes I_N\right]z_t + \delta|d|^2 \\
\leq & - c_1 c_3 z_t^\top\left[R\tilde{P}(z_t)R^\top \otimes I_N\right]z_t + \frac{c_3^2}{2\delta}\lambda_{max}(L_{et}^{-1})|z_t|^2 - c_2'|v|^2 + \delta|d|^2
\end{aligned} \tag{43}$$

Similarly to the case of first-order systems we need to show that as $|z_k| \to \Delta$ for any $k \leq m$, $\dot{V}(z_t) \leq 0$. More precisely, let $\bar{d} := \sup_{t\geq 0}|d(t)|$ and $\varepsilon \in (0, \Delta)$ be an arbitrarily small constant. Let z_t be such that, for at least one $k \leq m$ such that $|z_k| \geq (\Delta - \varepsilon)$.

Then, $|z_t| \geq \Delta - \varepsilon$, so from (43), the definition of $\tilde{P}(z_t)$ and Definition 2.2, we have

$$\dot{V}(z_t, v) \leq -c_1 c_3 \frac{\partial \alpha_k}{\partial z_k}((\Delta - \varepsilon)^2)(\Delta - \varepsilon)^2 + \frac{c_3^2}{2\delta}\lambda_{max}(L_{et}^{-1})(\Delta - \varepsilon)^2 - c_2'|v|^2 + \delta|d|^2 \tag{44}$$

Since we know from definition that, $\frac{\partial \alpha_k}{\partial s}(s)$ is continuous, non-decreasing, and $\frac{\partial \alpha_k}{\partial s}(s) \to \infty$ as $s \to \Delta^2$. Then, there exists $\varepsilon^*(\bar{d}) > 0$ such that for all $\varepsilon < \varepsilon^*$, $\dot{V}(z_t, v) \leq 0$. Connectivity maintenance follows from the same arguments as in Proposition 3.1. Furthermore, from Definition 2.2 and (44) we have

$$\dot{V}(z_t, v) \leq -\, c_1'|z_t|^2 - c_2'|v|^2 + \delta|d|^2 \tag{45}$$

where $c_1' := c_3\left(c_1 p_0 - \frac{c_3^2}{2\delta}\lambda_{max}\left(L_{et}^{-1}\right)\right)$. Note that c_1' can be made positive choosing δ sufficiently large. Now, defining $\zeta := \left[z_t^\top\ v^\top\right]^\top$, the derivative of V becomes

$$\dot{V}(\zeta) \leq -c'|\zeta|^2 + \delta|d|^2 \tag{46}$$

where $c' := \min\{c_1', c_2'\}$. Thus, the system (41) is input-to-state stable. □

5. Simulation Results

In this section, we present some simulation results that demonstrate the performance of the consensus algorithms with connectivity maintenance analysed in the previous sections. We considered a multiagent system of double-integrators composed of six agents interconnected through a communication topology represented by the connected graph in Figure 1.

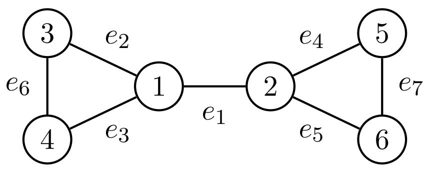


Figure 1. Connected graph

For the simulations, we considered that each agent is subject to an input perturbation which takes its maximal value at $t = 0$s and vanishes after $t = 15$s. The perturbations were modelled as $d_i(t) = \sigma_i(t)\,[1\ 1]^\top$, with

$$\sigma_i(t) = \begin{cases} -1.2(\tanh(2(t-15)) - 1),\ i = \{3, 5\} \\ 1.2(\tanh(2(t-15)) - 1),\ i = \{2\} \\ 0,\ i = \{1, 4, 6\}. \end{cases} \tag{47}$$

Two scenarios were considered. For the first scenario, we used the gradient control law proposed in (17), where the barrier function was defined as

$$U_k(z_k) = |z_k|^2 + \ln\left(\frac{\Delta^2}{\Delta^2 - |z_k|^2}\right) \tag{48}$$

and we set $c_1 = 2$ and $c_2 = 1.5$. Hence, for each agent, the control input u_i is

$$u_i = -c_1 \sum_{k \leq m} [E]_{ik} \left(1 + \frac{1}{\Delta^2 - |z_k|^2}\right) z_k - c_2 \, v_i. \tag{49}$$

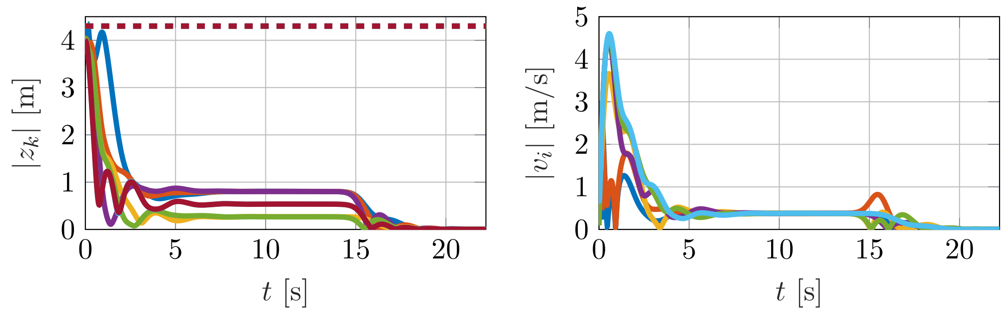
For the second scenario, we used a linear consensus algorithm as in Ren (2006), which in the edge-variables takes the form —see (2),

$$u_i = -c_1 \sum_{k \leq m} [E]_{ik} z_k - c_2 \, v_i. \tag{50}$$

For both scenarios, the initial conditions appear in Table 1 and the proximity constraint was set to $\Delta = 4.3$m. In Figure 2 we show the evolution of the edge states for the system with the proposed controller (49). It is clear from the Figure that once the disturbance vanishes, the edge states z_k converge to the origin, which implies that position consensus is achieved with zero velocity. Moreover, the distance constraints (dashed lines) are always respected, even in the presence of the disturbance $d(t)$. On the contrary, it can be seen from Figure 3 that the consensus algorithm (50) does not guarantee connectivity maintenance, hence consensus is not achieved.

Index	x **[m]**	y **[m]**	v_x **[m/s]**	v_y **[m/s]**
1	2.0	0.0	0.6	0.0
2	-2.0	0.0	-0.3	0.0
3	5.5	2.0	1.3	0.0
4	5.5	-2.0	0.1	0.0
5	-5.5	2.0	0.0	0.0
6	-5.5	-2.0	-0.8	0.0

Table 1. Initial Conditions (identical for the two scenarios)

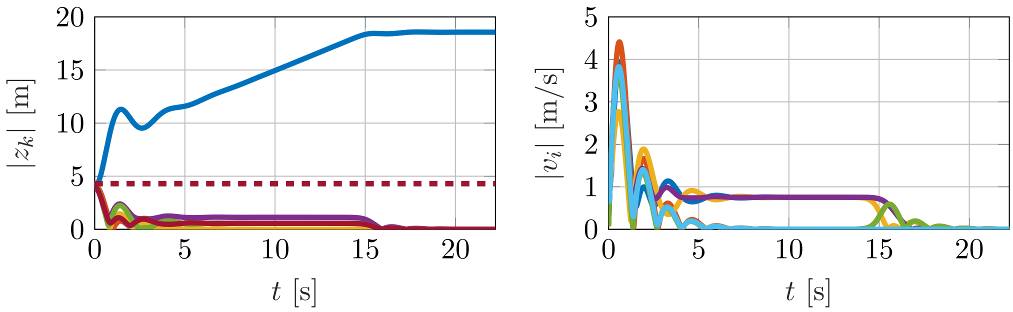


(a) Trajectories of the norm of the relative positions. Dashed lines: distance constraints.

(b) Trajectories of the norm of the agents' velocities.

Figure 2. Consensus with preserved connectivity under control law (49).

It is worth mentioning that, from a practical point of view, the input disturbance d could be considered as a bounded additional control input aiming to achieve a secondary task (Boskos and Dimarogonas (2017)). Therefore, as can be concluded from the theoretical analysis and the simulation results, the controller (49) guarantees consensus with preserved connectivity even in the presence of additional tasks more challenging from a connectivity maintenance point of view.



(a) Trajectories of the norm of the relative positions. Dashed lines: distance constraints.

(b) Trajectories of the norm of the agents' velocities.

Figure 3. Consensus without preserved connectivity under control law (50).

6. Conclusions

Analysing the behaviour of multiple interconnected systems using edge-based models, rather than the more usual node-based approach, has the technical advantage of naturally recasting the problem as that of stability of the *origin*, as opposed to a problem of stability of a manifold, which is the case under the nodes perspective. This facilitates considerably the enhancement of traditional consensus controllers by incorporating modifications such as nonlinear interconnections to cope with proximity constraints.

Thus, using an edge-based representation of undirected proximity graphs for networks of first and second-order multiagent systems, we constructed strict Lyapunov functions with which we established uniform asymptotic stability, robustness with respect to external disturbances, and connectivity maintenance. These are results without precedent in the literature, which open unexplored paths towards the solution of other difficult consensus-related problems, based on stability theory rather than linear algebra.

Ongoing research focuses on exploiting the results contributed here to consider additional inter-agent or information constraints such as collision/obstacle avoidance and quantized measurements in problem settings involving autonomous vehicles. On the other hand, extending this analysis to multiagent systems interconnected over directed graphs represents another interesting line of work.

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