Edge-based Strict Lyapunov Functions for Consensus with Connectivity Preservation over Directed Graphs

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Abstract

In this paper we address the edge-agreement problem with preserved connectivity for networks of first and second-order systems under proximity constraints and interconnected over a class of directed graphs. We provide a strict Lyapunov function that leads to establishing uniform asymptotic stability of the consensus manifold with guaranteed connectivity preservation. Furthermore, robustness of the edge-agreement protocol, in the sense of input-to-state stability with respect to external input disturbances, is also demonstrated. These results hold for directed-spanning-tree and directed-cycle topologies, which are notably employed, respectively, in leader-follower and cyclic-pursuit control.

Key words: Consensus, multi-agent systems, synchronisation, Lyapunov methods

1 Introduction

As it is well-known, the existence of a rooted directed spanning tree is a necessary condition for consensus over directed graphs (Ren, 2008). Yet, although necessary, this condition may as well be conservative in some cases. For instance, for networks of autonomous multi-vehicle systems that (can) communicate only if they are within range. The connectivity-preservation problem is typically addressed using gradient-type consensus algorithms, relying on so-called barrier functions. For undirected graphs see, *e.g.*, (Ji and Egerstedt, 2007; Boskos and Dimarogonas, 2017) for the case of first-order systems and (Verginis and Dimarogonas, 2019; Sun et al., 2017; Wen et al., 2012), for second-order systems. For systems interacting over directed graphs (digraphs), however, there are far fewer works. In (Sabattini et al., 2015; Poonawala and Spong, 2017; Cai et al., 2017) connectivity is achieved, but under somewhat conservative assumptions; it is assumed that the digraph is *strongly-connected* and, moreover, the controllers proposed therein rely on the estimation of the algebraic connectivity, which is a global parameter.

In recent years an alternative, *stability-oriented*, approach was proposed. This is the so-called *edge-agreement* representation, which defines the differences between pairs of neighbouring nodes, rather than the values of the nodes themselves, as the states of the resulting dynamical system (Zelazo et al., 2007; Mukherjee and Zelazo, 2019; Zeng et al., 2017). Within this framework consensus is assessed if the edge-state trajectories converge to zero. More precisely, the agreement problem may be reformulated as one of stabilisation of the origin for a dynamical system whose states represent the interconnection *edges*, as opposed to the nodes *cf.* Zeng et al. (2014).

This is well-suited for Lyapunov-based control and Lyapunov's direct method of analysis. In (Zeng et al., 2014) a Lyapunov function is given to establish consensus for quasi-strongly connected digraphs. In (Mukherjee and Zelazo, 2019) consensus of first and second-order systems over directed graphs is guaranteed, even in the presence of edge-weight uncertainties, by means of a strict Lyapunov function. In (Zeng et al., 2017), a strict Lyapunov function is used to establish consensus under dynamic quantisation of the communication, for second-order systems over digraphs. In (Zeng et al., 2016), input-to-state stability is established for the edge-agreement algorithm of second-order systems. In (Alvarez-Jarquín and Loría, 2014) consensus is established for the elementary directed-path topology, but under the assumption that it switches and the interconnections are time-varying. In (Chowdhury et al., 2018) a strict Lyapunov function is provided, albeit for undirected graphs with time-varying interconnections.

The connectivity problem, however, is not addressed in any of the previous works using the edge-representation

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framework. Alternatively, it is addressed in the literature within the classical nodes-representation framework and this, most often, for undirected connected graphs or strongly connected digraphs.

In this paper we address the consensus problem with preserved connectivity, for a network of first and second-order systems, using the edge-based representation framework. It is the outgrowth of the conference paper (Restrepo et al., 2020), on consensus of first-order systems interconnected over directed spanning trees.

We present consensus algorithms for two kinds of digraphs: directed spanning trees and directed cycles. Each of the two topologies studied presents difficulties and practical interests of its own, notably in formation control of autonomous vehicles using a leader-follower configuration (Consolini et al., 2008; Maghenem et al., 2020) and in the context of cyclic pursuit (Marshall et al., 2004). As stressed in Santilli et al. (2019), directed cyclic topologies naturally appear in multi-agent systems of vehicles equipped with field-of-view sensors.

Furthermore, we provide *strict* Lyapunov functions with which we are able to establish both connectivity and strong properties for the closed-loop system, such as uniform asymptotic stability of the consensus equilibrium and input-to-state stability.

Now, even though *strict* Lyapunov functions have been proposed earlier for consensus problems, for edge-based and node-based digraphs of both first and second-order systems, this is done without addressing the connectivity-preservation requirement, or viceversa. In Santilli et al. (2019), for instance, a non-strict Lyapunov function is provided for first-order systems with field-of-view constraints, but only boundedness of the trajectories is guaranteed. The difficulty of constructing a strict Lyapunov function is stressed therein.

Thus, our main statements potentially serve as basis to the solution of concrete control problems of nonlinear multi-agent systems under realistic constraints (actuator saturation, sensors' capabilities, energetic autonomy) that are required to undertake tasks in aerodynamically perturbed environments.

2 Model and problem formulation

2.1 Notations and preliminaries

We use $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ to denote a weighted digraph defined by a node set $\mathcal{V} = \{1, 2, \ldots, n\}$ with cardinality n and corresponding to the agents' states, an edge set $\mathcal{E} \subseteq \mathcal{V}^2$ with cardinality m and characterising the information exchange between agents, and a positive diagonal matrix $\mathcal{W} \in \mathbb{R}^{m\times m}$, whose entries represent the weights of the edges. A directed edge, e_k, is an ordered pair $(i, j) \in \mathcal{E}$ if and only if there exists a connection *from* node i *to* node j.

Fundamental in the edge-based framework is the so-called incidence matrix of a digraph, $E(\mathcal{G}) \in \mathbb{R}^{n\times m}$. This is a matrix with rows indexed by the nodes and columns indexed by the edges. Its (i, k)th entry is defined as follows: $[E]_{ik} := -1$ if i is the terminal node of edge e_k, $[E]_{ik} := 1$ if i is the initial node of edge e_k, and $[E]_{ik} := 0$ otherwise.

We recall from Zeng et al. (2017) that the incidence matrix corresponds to the sum of the so-called in-incidence and out-incidence matrices, denoted $E_\odot(\mathcal{G}) \in \mathbb{R}^{n\times m}$ and $E_\otimes(\mathcal{G}) \in \mathbb{R}^{n\times m}$ respectively. That is,

$$E = E_\odot + E_\otimes \tag{1}$$

and the elements of $E_\odot$ and $E_\otimes$ are defined as follows. $[E_\odot]_{ik} := -1$ if i is the terminal node of edge e_k and $[E_\odot]_{ik} := 0$ if otherwise, while $[E_\otimes]_{ik} := 1$ if i is the initial node of edge e_k and $[E_\otimes]_{ik} := 0$ if otherwise.

Then, the weighted Laplacian matrix $L(\mathcal{G}) \in \mathbb{R}^{n\times n}$ of a digraph $\mathcal{G}$ can be defined in terms of the incidence and the in-incidence matrices, as

$$L(\mathcal{G}) = E_\odot(\mathcal{G})\mathcal{W}E(\mathcal{G})^\top. \tag{2}$$

For digraphs, $L(\mathcal{G})$ has a simple zero eigenvalue and all other non-zero eigenvalues are in the open left-half complex plane, if and only if the digraph contains a spanning tree (Ren, 2008). In what follows, the argument $\mathcal{G}$ is dropped when clear from the context.

2.2 Directed edge-Laplacian and reduced-order system

Using an appropriate labelling of the edges (Mukherjee and Zelazo, 2019) the incidence matrix is expressed as

$$E = [\, E_t \quad E_c \,] \tag{3}$$

where $E_t \in \mathbb{R}^{n\times(n-1)}$ denotes the full-column-rank incidence matrix corresponding to an arbitrary spanning tree $\mathcal{G}_T \subset \mathcal{G}$ and $E_c \in \mathbb{R}^{n\times(m-n+1)}$ represents the incidence matrix corresponding to the remaining edges not contained in $\mathcal{G}_T$. The labelling is as follows: let the root node be labelled "1" and let the remaining nodes be labelled as follows. Any two nodes i and j belonging to a branch b_l of the tree are labelled such that if the path length from the root to i is shorter than the path length from the root to j, then $i < j$. Then, label the $n-1$ edges such that for any edge $e_k = (i, j)$, one has $j > k$.

Furthermore, as observed in (Zelazo et al., 2007), since E_t has full column-rank, the incidence matrices satisfy

$$E_t T = E_c \tag{4}$$

where $T := \left(E_t^\top E_t\right)^{-1} E_t^\top E_c$. Thus, defining

$$R := [\, I_{n-1} \quad T \,] \tag{5}$$

with I_{n-1} denoting the $n-1$ identity matrix, one obtains an alternative representation of the incidence matrix of the digraph that is given by

$$E = E_t R. \tag{6}$$

The identity (6) is useful to derive a reduced-order dynamic model corresponding to the dynamics of a network formed by a spanning-tree $\mathcal{G}_T \subset \mathcal{G}$, which is easier to analyse than the original multi-agent system and for which consensus is achieved if and only if it is also the case for $\mathcal{G}$. For illustration, let us consider a classic weighted consensus protocol for a group of n first-order systems of dimension N,

$$\dot{x}_i = u_i, \quad x_i,\, u_i \in \mathbb{R}^N \tag{7}$$

where $u_i = -\sum_{j=1}^{n} a_{ij}(x_i - x_j)$ for all $i \leq n$ corresponds to the control input for each agent and $a_{ij} \geq 0$ is strictly positive if and only if the ith and jth nodes are interconnected. In compact form, the systems' states are collected in the vector $x = \left[x_1^\top, \ldots, x_n^\top\right]^\top \in \mathbb{R}^{nN}$ and the control input is $u = \left[u_1^\top, \ldots, u_n^\top\right]^\top \in \mathbb{R}^{nN}$. Then, denoting by I_N the $N \times N$ identity matrix, the *nodes* dynamics for this multi-agent system is given by

$$\dot{x} = -[L \otimes I_N]x, \quad x \in \mathbb{R}^{nN}, \tag{8}$$

Now, following Zelazo et al. (2007) we introduce the following coordinate transformation that maps the nodes' space to that of the edges,

$$z := [E^\top \otimes I_N]x, \quad z := [z_1^\top \cdots z_k^\top \cdots z_m^\top]^\top. \tag{9}$$

That is, $z_k := x_i - x_j$ where $i,\, j \in \mathcal{V}$ and $k \leq m$. Therefore, it follows from (9) that the agreement condition $\{x_i = x_j,\ \forall\,(i,j) \in \mathcal{V}^2\}$ is equivalent to $\{z = 0\}$. This is significant because in the edge-variables' representation, consensus may be reformulated as a stabilisation problem of the origin for the system

$$\dot{z} = -[E^\top E_\odot \mathcal{W} \otimes I_N]z, \tag{10}$$

which is obtained by differentiating both sides of (9) and using (2) and (8). Eq. (10) is defined in function of the so-called edge Laplacian matrix $L_e(\mathcal{G}) \in \mathbb{R}^{m\times m}$,

$$L_e := E^\top E_\odot \mathcal{W}, \tag{11}$$

which lies at the basis of the edge-representation framework. As the dual of L, the edge Laplacian, L_e, has the same non-zero eigenvalues as L; hence, $\text{rank}(L_e) = \text{rank}(L) = n-1$ —see (Zeng et al., 2017). Next, as in the latter, we split the edges' states. Let

$$z = \left[z_t^\top \;\; z_c^\top\right]^\top, \quad z_t \in \mathbb{R}^{(n-1)N},\, z_c \in \mathbb{R}^{(m-n+1)N} \tag{12}$$

where z_t are the state variables corresponding to the edges of an arbitrary directed spanning tree $\mathcal{G}_T$ and z_c denote the state of the remaining edges, $\in \mathcal{G}\backslash\mathcal{G}_T$. Thus, after (9), (12), and (3), we see that

$$z_t := \left[E_t^\top \otimes I_N\right]x. \tag{13}$$

Furthermore, after (4) it is readily seen that the states of the arcs not contained in the tree $\mathcal{G}_T$, z_c, satisfy

$$z_c = \left[T^\top \otimes I_N\right]z_t. \tag{14}$$

Another useful identity that stems from (5) and (14) is

$$z = \left[R^\top \otimes I_N\right]z_t \tag{15}$$

which, together with (6) and (10), implies that

$$\dot{z}_t = -\left[E_t^\top E_\odot \mathcal{W} R^\top \otimes I_N\right]z_t. \tag{16}$$

Even though this equation has a reduced dimension with respect to (10), in view of the presence of R on the right-hand side, it also takes into account the effect of the arcs z_c onto the states z_t, so it captures the behaviour of the overall network. In particular, consensus is achieved if and only if $z_t \to 0$. This problem, with the weight matrix $\mathcal{W} = I_m$ and with linear interconnections, has been widely studied in the literature, including using Lyapunov's direct method —see Mukherjee and Zelazo (2019); Zeng et al. (2017); Chowdhury et al. (2018), but rarely with nonlinear interconnections, as in this paper.

2.3 Connectivity maintenance

Besides consensus (i.e. $z_t \to 0$) in this paper we address the problem of guaranteeing that a graph initially connected remains so. We define such property as follows.

Definition 1 (Connectivity maintenance) *For each $k \leq m$, let $\Delta_k > 0$ denote the maximal distance between the nodes i and j such that the communication between them, through the arc $e_k = (i,j)$, is reliable. We say that the graph's connectivity is maintained (hence, the proximity constraint holds) if the set*

$$\mathcal{J} := \left\{z \in \mathbb{R}^{mN} : |z_k| < \Delta_k,\ \forall\, k \leq m\right\}, \tag{17}$$

where $z_k = x_i - x_j$, is forward invariant. That is, $|z_k(0)| < \Delta_k$ implies that $z(t) \in \mathcal{J}$ for all $t \geq 0$.

The controller guaranteeing connectivity maintenance is designed as a gradient law based on so-called Barrier Lyapunov functions. These are reminiscent of Lyapunov functions taking values in open subsets of the Euclidean space and such that they grow unboundedly as z_k approaches the border of the open set. The control law induces a nonlinear *connectivity potential* thereby rendering the interconnections nonlinear.

Definition 2 (Connectivity potential) *Let $p_0 \in \mathbb{R}$ and, for each $k \leq m$, let $\mathcal{B}_{\Delta_k} := \{z_k \in \mathbb{R}^N : |z_k| < \Delta_k\}$. Let $\alpha_k : \left[0, \Delta_k^2\right) \to \mathbb{R}_{\geq 0}$, $s \mapsto \alpha_k(s)$, be $\mathcal{C}^1$ and non-decreasing on $\left[0, \Delta_k^2\right)$, such that $\alpha_k(s) \to \infty$ as $s \to \Delta_k^2$, and $p_k : \mathcal{B}_{\Delta_k} \to \mathbb{R}_{>0}$, defined as*

$$p_k(z_k) := \frac{\partial \alpha_k}{\partial s}(|z_k|^2),$$

is also non-decreasing, $p(z_k) \geq p_0 > 0$ for all $|z_k| < \Delta_k$, and $p_k(z_k) \to \infty$ as $|z_k| \to \Delta_k$. Then, we define the connectivity potential $P(z) := diag\left[p_k(z_k)\right] \in \mathbb{R}^{m\times m}$.

Also, we define the *Barrier* function $U_k : \mathcal{B}_{\Delta_k} \to \mathbb{R}_{\geq 0}$ as

$$U_k(z_k) := \alpha_k(|z_k|^2), \quad \forall\, k \leq m. \tag{18}$$

Similarly defined Barrier Lyapunov functions, but taking values in the *nodes* space, are used *e.g.*, in Ji and Egerstedt (2007), Boskos and Dimarogonas (2017), and Tang et al. (2013). In Ji and Egerstedt (2007); Boskos and Dimarogonas (2017) the so-called "edge-tension" function was designed for the consensus with connectivity preservation over undirected graphs.

3 Main results

3.1 Consensus of first-order systems

Our first statement, on consensus of first-order systems interconnected over directed spanning-tree graphs, is a foundation block, but it has interest of its own. Consider a network of n dynamical systems, (7), expressed in edge coordinates,

$$\dot{z} = \big[E^\top \otimes I_N\big]u, \tag{19}$$

in closed-loop with the gradient control law

$$u(z) = -c_1\left[E_\odot P(z) \otimes I_N\right]z, \tag{20}$$

where $c_1 > 0$ is the network connectivity strength and the matrix $P(z) := \mathrm{diag}\big[p_k(z_k)\big]$ models the interconnections —see Definition 2. From this point on it is assumed, without loss of generality, that the weight matrix $\mathcal{W} = I_m$. We emphasise that each component of u depends only on local information since $E_\odot$ represents the incoming edges on each node, that is, the available information to each agent as defined by the digraph.

Replacing (20) into (19) and, akin to (16), we obtain the reduced-dimension closed-loop system

$$\dot{z}_t = -c_1\left[E_t^\top E_\odot \tilde{P}(z_t)R^\top \otimes I_N\right]z_t, \tag{21}$$

where, for consistency in the notation, we introduced $\tilde{P}(z_t) := P(\big[R^\top \otimes I_N\big]\, z_t)$, but we stress that $\tilde{P}(z_t) \in \mathbb{R}^{m\times m}$ and $P(z) \in \mathbb{R}^{m\times m}$ are identical.

Remark 1 *Equation* (21) *highlights another perk of using the edge-based representation when considering nonlinear interconnections. Note that, from* (19)-(20)*, the closed-loop system is given by*

$$\dot{z} = -c_1\left[E^\top E_\odot P(z) \otimes I_N\right]z =: -c_1\big[L_e P(z) \otimes I_N\big]z,$$

whereas, in the nodes' representation, the same closed-loop system yields,

$$\dot{x} = -c_1\left[E_\odot P(x) E^\top \otimes I_N\right]x =: -c_1\big[L(x) \otimes I_N\big]x.$$

Note that using the representation in terms of the nodes, the graph Laplacian $L(x)$ is state-dependent. Therefore, one cannot rely on eigenvalue analysis for the stability analysis. On the other hand, using the edge-based representation it is possible to dissociate the interaction topology, represented by the (unweighted) edge Laplacian L_e, and the nonlinear interconnections given by the diagonal matrix $P(z)$. Hence, despite the nonlinear weights, it is possible to use the eigenvalue analysis of the edge Laplacian in order to prove asymptotic stability of the consensus manifold with guaranteed connectivity by means of a strict Lyapunov function. This is presented in the proofs of Propositions 1 and 2.

Proposition 1 *Consider n systems as in* (7) *with limited communication ranges and interconnected through a digraph $\mathcal{G}$ which is either a directed spanning tree or a directed cycle. Then, for any initial conditions satisfying $z(0) \in \mathcal{J}$ the control law* (20) *guarantees that $z_k \to 0$ for all $k \leq m$, and preserves connectivity of $\mathcal{G}$, that is, the set $\mathcal{J}$ as defined in* (17) *is forward invariant for the closed-loop trajectories. Moreover, the function*

$$V(z_t) = \sum_{k\leq m} \gamma_k U_k(z_k), \qquad \gamma_k > 0, \tag{22}$$

where U_k is defined in (18)*, is a strict Lyapunov function for the closed-loop system* (21) *on its domain, which is*

$$\mathcal{J}_t := \big\{z_t \in \mathbb{R}^{(n-1)N} : |z_k| < \Delta_k,\ \forall\, k \leq m\big\}.$$

Proof. Using

$$\frac{\partial U_k}{\partial z_k} = 2p_k(z_k)z_k$$

and defining $\Gamma := \mathrm{diag}\,[\gamma_k]$ with $\gamma_k > 0$ yet to be determined, we obtain

$$\frac{\partial V}{\partial z_t} = 2\left[R\Gamma\tilde{P}(z_t)R^\top \otimes I_N\right]z_t.$$

Hence, the derivative of $V(z_t)$ along (21) is

$$\dot{V}(z_t) = -2\,c_1\,z_t^\top\left[R\tilde{P}(z_t)\Gamma R^\top E_t^\top E_\odot \tilde{P}(z_t)R^\top \otimes I_N\right]z_t. \tag{23}$$

The previous equation holds regardless of the graph topology; next, we analyse the two cases under consideration.

Case 1 (Directed spanning tree). We have $\mathcal{G} = \mathcal{G}_T$. Therefore, $z = z_t$, $E = E_t$, and $E_\odot = E_{\odot t}$. In turn, from the latter and (15), we have $R = I_{n-1}$. Now, akin to (11) albeit with an abuse of notation, we define the edge-Laplacian matrix of a directed spanning tree as $L_{et} := E_t^\top E_{\odot t} I_{n-1} \in \mathbb{R}^{(n-1)\times(n-1)}$. Hence, (23) becomes

$$\dot{V}(z_t) = -c_1\,z_t^\top\left[\tilde{P}(z_t)\big(\Gamma L_{et} + L_{et}^\top\Gamma\big)\tilde{P}(z_t) \otimes I_N\right]z_t. \tag{24}$$

Next, we show that for an in-incidence matrix constructed using the labelling approach of Mukherjee and Zelazo (2019) previously mentioned, the right hand side of (24) is negative definite. Indeed, in this case,

$$E_{\odot t} = \begin{bmatrix}0_{n-1\times 1} & -I_{n-1}\end{bmatrix}^\top. \tag{25}$$

Then, defining $B := -E_{\otimes t}^\top E_{\odot t}$ and using (1), as well as the fact that from (25) $E_{\odot t}^\top E_{\odot t} = I_{n-1}$, we see that the edge Laplacian of a directed spanning tree satisfies

$$L_{et} = E_t^\top E_{\odot t} = E_{\odot t}^\top E_{\odot t} + E_{\otimes t}^\top E_{\odot t} =: I - B. \tag{26}$$

Now, since $[E_{\otimes t}]_{ij} = 1$ implies that $[E_{\odot t}]_{ij} = 0$ and, in view of the previously mentioned labelling, $[E_{\otimes t}^{\top}]_{ij} = 0$ for $i < j$, it follows that B is a lower triangular matrix with zero diagonal and all other elements either equal to 0 or 1. Moreover, for a directed spanning tree, $\mathrm{rank}(L_{et}) = n-1$ and all the eigenvalues of L_{et} lie on the open left-hand complex plane; indeed, they coincide with the eigenvalues of the graph's Laplacian L. Thus, from the latter and (26), we conclude that L_{et} is a non-singular M-matrix (Plemmons, 1977), that is, a real matrix with positive diagonal, non-positive off-diagonal elements, and eigenvalues with strictly positive real parts. Now, after Plemmons (1977), every non-singular M-matrix is diagonally stable, that is, for any $Q = Q^{\top} > 0$, L_{et} admits a diagonal solution $\Gamma :=\mathrm{diag}\big[\gamma_k\big]$, to the Lyapunov inequality

$$\Gamma L_{e_t} + L_{e_t}^{\top}\Gamma \geq Q. \tag{27}$$

Therefore, redefining γ_k in (22), if necessary, so that (27) holds, and since $\tilde{P}(z_t) > 0$, we have

$$\dot{V}(z_t) \leq\ -c_1' \left|\big[\tilde{P}(z_t) \otimes I_N\big] z_t\right|^2 \tag{28}$$

where $c_1' := c_1\,\lambda_{min}(Q)$ and $\lambda_{min}(\,\cdot\,)$ denotes the smallest eigenvalue of $(\,\cdot\,)$. Hence, $\dot{V}(z_t)$ is negative definite on $\mathcal{J}_t$.

Case 2 (Directed cycle). Setting $\gamma_k = 1$ for all $k \leq m$, and using (6), equation (23) becomes

$$\dot{V}(z_t) = -c_1 z_t^{\top}\big[R\tilde{P}(z_t)\big(E^{\top}E_{\odot}{+}E_{\odot}^{\top}E\big)\tilde{P}(z_t)R^{\top}{\otimes}I_N\big]z_t.$$

Then, from (1), we have

$$E^{\top}E_{\odot} + E_{\odot}^{\top}E = E^{\top}E + E_{\odot}^{\top}E_{\odot} - E_{\otimes}^{\top}E_{\otimes} \tag{29}$$

and following the same labelling rules mentioned above, the in-incidence and out-incidence matrices become

$$E_{\odot} = \begin{bmatrix} 0_{1\times n-1} & -1 \\ -I_{n-1} & 0_{n-1\times 1}\end{bmatrix}, \quad E_{\otimes} = I_{n\times n}.$$

Hence, we have $E_{\odot}^{\top}E_{\odot} = I_{n\times n}$ and $E_{\otimes}^{\top}E_{\otimes} = I_{n\times n}$. Consequently, using (29) and (6), again, we obtain

$$\dot{V}(z_t) = -c_1\; z_t^{\top}\big[R\tilde{P}(z_t)R^{\top}E_t^{\top}E_tR\tilde{P}(z_t)R^{\top} \otimes I_N\big]z_t,$$

where $E_t^{\top}E_t$ is a positive-definite matrix corresponding to the edge Laplacian of an undirected tree (Zelazo et al. (2007)). Then, since R has full row-rank, we have

$$\dot{V}(z_t) \leq -c_1' \; \left|\big[R\tilde{P}(z_t)R^{\top} \otimes I_N\big]z_t\right|^2 \tag{30}$$

where $c_1' := c_1\,\lambda_{min}(E_t^{\top}E_t)$ with $\lambda_{min}(E_t^{\top}E_t)$ being the smallest eigenvalue of $E_t^{\top}E_t$, so $\dot{V}(z_t)$ is negative definite on $\mathcal{J}_t$. From (28) and (30), V as defined in (22) is a strict Lyapunov function for (21).

Now we establish connectivity of the set $\mathcal{J}$. To that end, we remark that $z_t \in \mathcal{J}_t$ implies that $z \in \mathcal{J}$ and we show that $\mathcal{J}_t$ is forward invariant. We proceed by contradiction. Assume that there exists $T > 0$ such that for all $t \in [0, T)$, $z_t(t) \in \mathcal{J}_t$ and $z_t(T) \notin \mathcal{J}_t$. More precisely, we have $|z_k(t)| \to \Delta_k$ as $t \to T$ for at least one $k \leq m$. From the definition of V, this implies that $V(z_t(t)) \to \infty$ as $t \to T$ which is in contradiction with (28) and (30). We conclude that $V(z_t(t))$ is bounded, *i.e.*, $V(z_t(t)) \leq V(z_t(0)) < \infty$ for all $t \geq 0$. Connectivity preservation follows.

It is left to show that the set $\mathcal{J}$ corresponds to the domain of attraction for the closed-loop system. This follows by showing that all solutions of (21) starting in $\mathcal{J}_t$ converge to the origin. To that end, for any $\varepsilon \in (0, \Delta_k)$, consider a subset $\mathcal{J}_{\varepsilon t} \subset \mathcal{J}_t$ defined as $\mathcal{J}_{\varepsilon t} := \{z_t \in \mathbb{R}^{(n-1)N} : |z_k| < \Delta_k - \varepsilon,\ \forall\, k \leq m\}$ and let $\bar{\mathcal{J}}_{\varepsilon t}$, denote the closure of $\mathcal{J}_{\varepsilon t}$. From Definition 2 and (18) it follows that $V(z_t)$ is positive definite on $\bar{\mathcal{J}}_{\varepsilon t}$ and it satisfies the bounds $\beta|z_t|^2 \leq V(z_t) \leq h(|z_t|)$, where $\beta > 0$ and $h(\cdot)$ is defined and strictly increasing everywhere in $\bar{\mathcal{J}}_{\varepsilon t}$, $h(s) > 0$ for all $s > 0$, and $h(0) = 0$. This means that $V(z_t) \to 0$ as $z_t \to 0$. Therefore, from (28), (30), and standard Lyapunov theory it follows that all trajectories of (21) starting in $\mathcal{J}_{\varepsilon t}$ converge to the origin. The previous arguments hold for any $\varepsilon \to 0$, so the origin is attractive for all trajectories $z_t(t)$ starting in $\mathcal{J}_t$, that is, for all trajectories $z(t)$ starting in $\mathcal{J}$. ∎

3.2 *Output consensus of second-order systems*

The significance of Proposition 1 resides in the potential use of a strict Lyapunov function for other consensus control problems. For instance, V may be used in Lyapunov-based control design, such as backstepping, to achieve output consensus. We illustrate this fact here for second-order systems, but the method extends to higher relative-degree systems.

Consider the second-order system

$$\dot{x}_i = v_i \tag{31a}$$
$$\dot{v}_i = u_i \tag{31b}$$

where $x_i \in \mathbb{R}^N$ and $v_i \in \mathbb{R}^N$ are respectively, the position and the velocity of agent $i \leq n$, and $u_i \in \mathbb{R}^N$ is the control input. The control goal is to achieve output consensus, where the outputs correspond to the variables x_i; hence, it is required to steer $v_i \to 0$ for all $i \leq n$.

We collect the states in the vectors $x = \big[x_1^{\top} \cdots x_n^{\top}\big]^{\top} \in \mathbb{R}^{nN}$ and $v = \big[v_1^{\top} \cdots v_n^{\top}\big]^{\top} \in \mathbb{R}^{nN}$ and the inputs into $u = \big[u_1^{\top} \cdot u_n^{\top}\big]^{\top} \in \mathbb{R}^{nN}$. Then, applying the edge transformation (9), the position consensus problem may be reformulated as the stabilisation of the origin for

$$\dot{z} = \big[E^{\top} \otimes I_N\big]v \tag{32a}$$
$$\dot{v} = u. \tag{32b}$$

We follow a standard backstepping procedure. First we design a virtual input $z \mapsto v^*(z)$, satisfying $v^*(0) = 0$, to stabilise the origin for the subsystem (32a). Next, the input u is designed so that $v(t) \to v^*(z(t))$ as $t \to \infty$.

The virtual control v^* is defined using (20), that is,

$$v^*(z) := -c_1\big[E_\odot P(z)\otimes I_N\big]z. \tag{33}$$

Then, we define $\tilde v := v - v^*$ and we use $v = \tilde v + v^*$ and (33) in (32) to rewrite the latter equations as

$$\begin{aligned}\dot z =&\ -c_1\big[E^\top E_\odot P(z)\otimes I_N\big]z + \big[E^\top\otimes I_N\big]\tilde v && (34a)\\ \dot{\tilde v} =&\ u + c_1\big[E_\odot P(z)E^\top\otimes I_N\big](\tilde v + v^*)\\ &+ c_1\big[E_\odot \dot P(z)\otimes I_N\big]z. && (34b)\end{aligned}$$

Thus, using the feedback-linearizing control law

$$\begin{aligned}u(z,\tilde v) :=&\ -c_1\big[E_\odot P(z)E^\top\otimes I_N\big](\tilde v + v^*)\\ &- c_1\big[E_\odot \dot P(z)\otimes I_N\big]z - c_2\tilde v\end{aligned} \tag{35}$$

with $c_2 > 0$, we obtain the following.

Proposition 2 *Consider n systems as in (31) with limited communication ranges and interconnected through a digraph $\mathcal{G}$ which is either a directed spanning tree or a directed cycle. Then, for any initial conditions satisfying $z(0)\in\mathcal{J}$ the control law (35) guarantees that $z_k\to 0$ for all $k\le m$, $v_i\to 0$ for all $i\le n$, and preserves the connectivity of $\mathcal{G}$, that is, the set $\mathcal{J}$ as defined in (17) is forward invariant. Furthermore the function $V:\mathcal{J}_t\times\mathbb{R}^{nN}\to\mathbb{R}_{\ge 0}$, defined as*

$$V(z_t,\tilde v) = \frac{1}{2}\sum_{k\le m}\gamma_k U_k(z_k) + \frac{c_3}{2}|\tilde v|^2, \tag{36}$$

where $\gamma_k > 0$ and $c_3 > 0$ are design parameters, and the functions U_k are defined in (18), is a strict Lyapunov function for the closed-loop system (37). □

Proof. The closed-loop equation is computed by replacing (35) into (34). Now, in view of the tree-cycle dichotomy of (12), together with (6) and (15), we obtain the reduced-order closed-loop dynamics

$$\begin{aligned}\dot z_t =&- c_1\big[E_t^\top E_\odot \tilde P(z_t)R^\top\otimes I_N\big]z_t + \big[E_t^\top\otimes I_N\big]\tilde v && (37a)\\ \dot{\tilde v} =&- c_2\tilde v && (37b)\end{aligned}$$

where we recall that $\tilde P(z_t) := P(\big[R^\top\otimes I_N\big]\,z_t)$.

Furthermore, in view of (30), the total derivative of $V(z_t,\tilde v)$ along the trajectories of (37) satisfies

$$\begin{aligned}\dot V(z_t,\tilde v) \le&- c_1'\,\big|\big[R\tilde P(z_t)R^\top\otimes I_N\big]z_t\big|^2 - c_2c_3|\tilde v|^2\\ &+ z_t^\top\big[R\tilde P(z_t)\Gamma R^\top E_t^\top\otimes I_N\big]\tilde v.\end{aligned} \tag{38}$$

Note that this bound holds indistinctly for directed-cycle topologies and, with $R = I_{n-1}$, for directed-spanning-tree graphs.

Now, given c_1', $\gamma_{max} := \max_{k\le m}\{\gamma_k\}$, and c_3, let $\delta > 0$ be such that $c_1'' := c_1' - \frac{1}{2}\delta\gamma_{max}\lambda_{max}(E_t^\top E_t)$ and $c_2' := c_2c_3 - \frac{1}{2\delta}$ are positive. Then, after applying Young's inequality to the third term in the right-hand side of (38), we obtain

$$\dot V(z_t,\tilde v) \le -c_1''\,\big|\big[R\tilde P(z_t)R^\top\otimes I_N\big]z_t\big|^2 - c_2'|\tilde v|^2. \tag{39}$$

Thus, $\dot V(z_t,v) < 0$ for all $(z_t,v)\in\{\mathcal{J}_t\times\mathbb{R}^{nN}\}\backslash\{(0,0)\}$ and V in (36) is a strict Lyapunov function for (37). Forward invariance of the set $\mathcal{J}_t$, hence of $\mathcal{J}$, follows from the same arguments as in the proof of Proposition 1. Consequently, the connectivity of $\mathcal{G}$ is preserved for any $z(0)\in\mathcal{J}$ and for any $v(0)$. Finally, note that

$$\beta_1|z_t|^2 + \beta_2|\tilde v|^2 \le V(z_t,\tilde v) \le h(|z_t|) + \beta_3|\tilde v|^2 \tag{40}$$

where β_1,β_2,β_3 are positive constants and $h(\cdot)$ is defined and strictly increasing everywhere in $\bar{\mathcal{J}}_{\varepsilon t}$ and satisfies $h(0) = 0$. Thus, following the same arguments as in the proof of Proposition 1, we have asymptotic stability of the origin for all trajectories starting in $\mathcal{J}$. ■

3.3 Robustness of the edge-consensus algorithm

In this section we use the strict Lyapunov functions previously constructed to analyse the robustness of the edge consensus with connectivity preservation. In particular we establish input-to-state stability with respect to a bounded input perturbation.

Consider a multi-agent system with an additive disturbance, that is, in edge-coordinates

$$\begin{aligned}\dot z &= \big[E^\top\otimes I_N\big]v && (41a)\\ \dot v &= u + d && (41b)\end{aligned}$$

where $d := \big[d_1^\top,\ldots,d_n^\top\big]^\top\in\mathbb{R}^{nN}$ is the bounded input disturbance. Then defining v^* as in (33) and u as (35), akin to (37), the reduced-order closed-loop error system takes the form

$$\begin{aligned}\dot z_t =&- c_1\big[E_t^\top E_\odot\tilde P(z_t)R^\top\otimes I_N\big]z_t + \big[E_t^\top\otimes I_N\big]\tilde v && (42a)\\ \dot{\tilde v} =&- c_2\tilde v + d. && (42b)\end{aligned}$$

Proposition 3 *The multi-agent system (41) under proximity constraints, with a communication topology defined by a digraph $\mathcal{G}$ which is either a directed spanning tree or a directed cycle, in closed loop with the controller (35) is input-to-state stable with respect to the disturbance d. Furthermore, the digraph remains connected for all $t\ge 0$.* □

Proof. Take the Lyapunov function defined in (36). Then, from (39) and (42), we have

$$\dot V(z_t,\tilde v) \le -c_1''\,\big|\big[R\tilde P(z_t)R^\top\otimes I_N\big]z_t\big|^2 - c_2'|\tilde v|^2 + c_3\tilde v^\top d. \tag{43}$$

Next, given c_2' and c_3 let $\delta' > 0$ be such that $c_2'' := c_2' - c_3/(2\delta') > 0$. Applying Young's inequality to the third term on the right-hand side of (43), we obtain

$$\dot V(z_t,\tilde v) \le -c_1''\,\big|\big[R\tilde P(z_t)R^\top\otimes I_N\big]z_t\big|^2 - c_2''|\tilde v|^2 + \frac{c_3\delta'}{2}|d|^2, \tag{44}$$

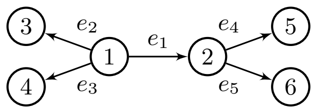
so the system (42) is input-to-state stable.

To assert connectivity preservation in presence of additive disturbances, we show that in the proximity of the limits of the connectivity region the first term on the right-hand side of (44) dominates over the bounded disturbance. To that end, let $\bar{d} := \sup_{t\geq 0} |d(t)|$ and let $\varepsilon \in (0, \Delta_k)$ be a small constant to be determined. Let $z_t \in \mathbb{R}^{N(n-1)}$ be such that for some $k \leq m$ we have $|z_k| \geq \Delta_k - \varepsilon$. Then, $|z_t| \geq \Delta_k - \varepsilon$, so from (44), the definition of $\tilde{P}(z_t)$, and Definition 2, we have

$$\dot{V}(z_t, \tilde{v}) \leq -c_1'' \frac{\partial \alpha_k}{\partial s}\left([\Delta_k - \varepsilon]^2\right)[\Delta_k - \varepsilon]^2 - c_2''|\tilde{v}|^2 + \frac{c_3 \delta'}{2}\bar{d}^2.$$

Since $\frac{\partial \alpha_k}{\partial s}$ is continuous, non-decreasing, and $\frac{\partial \alpha_k}{\partial s}(s) \to \infty$ as $s \to \Delta_k^2$ it follows that there exists $\varepsilon^*(\bar{d})$ such that for all $\varepsilon \leq \varepsilon^*$, $\dot{V}(z_t, \tilde{v}) < 0$. The latter holds along trajectories starting from any initial conditions $z(0) \in \mathcal{J}$ which implies that $z(t)$ cannot approach the boundary of $\mathcal{J}$ so connectivity is preserved for all $t \geq 0$. ■

4 Numerical example



$$E_{\odot t} = \begin{bmatrix} 0_{1\times(n-1)} \\ -I_{n-1} \end{bmatrix}$$

Fig. 1. Directed spanning tree for 6 agents

We consider a network of six second-order systems interconnected over the spanning-tree digraph showed in Fig. 1, above. The systems are subject to smooth inverted-step-like vanishing disturbances defined as $d_i(t) = -\sigma(t)\,[1\ 1]^\top$, where $\sigma(t) = 2.4\big[\tanh(2(t - 15)) - 1\big] - [t+10]^{-1}$ for $i \in \{3, 5\}$, $d_2(t) = \sigma(t)\,[1\ 1]^\top$, and $d_i(t) = 0$ for $i \in \{1, 4, 6\}$. That is, d_i takes its maximal value at $t = 0$ and it smoothly vanishes around $t = 15$s. The Barrier functions are defined as $U_k(z_k) = |z_k|^2 + \ln\left(\frac{\Delta_k^2}{\Delta_k^2 - |z_k|^2}\right)$. Consequently, the gradient control law takes the form (35), with $c_1 = 3$, $c_2 = 2.5$, and $p_k(z_k) = 1 + [\Delta_k^2 - |z_k|^2]^{-1}$.

The agents' initial positions were set to $x_1(0) = [2.4,\ 0]$, $x_2(0) = [-0.58,\ -0.9]$, $x_3(0) = [4.5,\ 2]$, $x_4(0) = [5,\ -2]$, $x_5(0) = [-4.2,\ -0.45]$, and $x_6(0) = [-2,\ -4.2]$; the initial velocities were set to $v_1(0) = [-5,\ 0]$, $v_2(0) = [0,\ 0]$, $v_3(0) = [3,\ 0]$, $v_4(0) = [2,\ 0]$, $v_5(0) = [0,\ 0]$, and $v_6(0) = [0,\ 0]$. The initial conditions satisfy $z(0) \in \mathcal{J}$ with the set $\mathcal{J}$ given in (17) with the radii of the connectivity regions, Δ_k, set to 2.5, 3.2, 3.8, 3.5, 3.7, and 4 respectively.

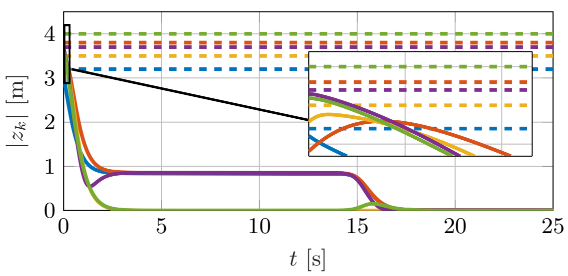


Fig. 2. Trajectories of the norm of the edges' states. Dashed lines: distance constraints.

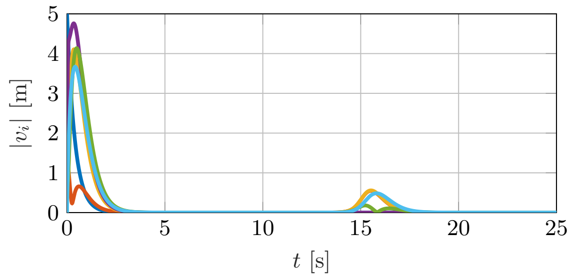


Fig. 3. Trajectories of the norm of the nodes' velocities.

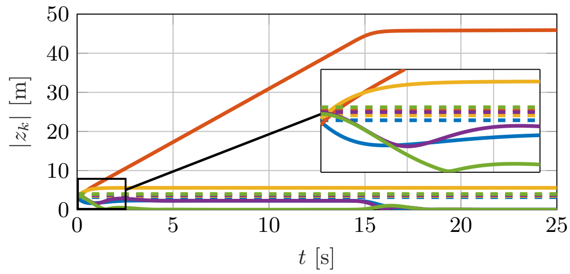


Fig. 4. Trajectories of the norm of the edges' states for a linear controller without guarantee of connectivity. Dashed lines: distance constraints.

The simulation results of our proposed control law (35) are depicted in Figs. 2 and 3. During the first 15s the perturbation $d(t)$ stymies the achievement of consensus; the systems stabilise with a steady-state error. After 15s, the perturbation vanishes, so the trajectories move from their previous steady state towards the consensus equilibrium. Moreover, the distance constraints (dashed lines) for all initially existing edges are always preserved as can be seen in Fig. 2, implying that the initially connected graph remains so.

For comparison, a second scenario was studied taking the same initial conditions satisfying $z(0) \in \mathcal{J}$, and the same disturbances acting on the system. For this comparison the controller is an edge-based linear consensus protocol without connectivity maintenance, as proposed in Mukherjee and Zelazo (2019). As can be seen in Fig. 4, a linear consensus protocol does not guarantee the respect of the range constraints, thus preventing the multi-agent system from reaching consensus.

5 Conclusions

The edge-based representation of graphs opens new perspectives for consensus control as it allows to rely on Lyapunov theory. We established uniform asymptotic stability and input-to-state stability of the consensus manifold for first and second-order multi-agent systems subject to proximity constraints by means of the construction of strict Lyapunov functions. Our results, however, apply to specific topologies; the extension to arbitrary directed connected graphs remains an open problem under study. Other application-driven problems under investigation involve consensus control under additional inter-agent constraints such as collision avoidance, input saturation, *etc.*, as well as formation tracking control.

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