Consensus of Open Multi-agent Systems over Dynamic Undirected Graphs with Preserved Connectivity and Collision Avoidance

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Abstract— **We address the consensus problem with collision avoidance for multi-agent systems under limited sensing ranges, in the case where new interconnections and agents may be added at any time. The graph topology is represented by a dynamic undirected graph, assumed to be connected only at an initial time, and the open multi-agent system is modeled via a multidimensional impulsive switched representation. We propose a barrier-Lyapunov-function-based consensus control law that guarantees inter-agent collision-avoidance and connectivity maintenance and, relying on the edge-agreement framework, we establish almost-everywhere asymptotic stability of the consensus manifold. The obtained results are also readily applicable to closed multi-agent systems with edge addition. A numerical simulation illustrates the effectiveness of the proposed approach.**

I. Introduction

Open multi-agent systems are a particular class of networked systems whose number of nodes and/or edges is not fixed. They appear naturally in applications involving, e.g., sensor-based approaches, in which case the network's topology varies depending on the sensing range of each agent. Then, the set of agents may be time-varying, as agents may join or leave the network at any time and new edges may be created as new neighbors "appear" within range. Recent studies of open multi-agent systems include, e.g., [1], where the problem of distributed maximum computation (a.k.a MAX-consensus) is addressed and [2], which focuses on open multi-agent systems interconnected over strongly connected directed graphs. In [3] a stability-analysis framework for multidimensional switched systems is used to address consensus of open multi-agent systems. See also [4], in which some stability results that do not rely on the size of the network are presented.

Both, the addition and removal of nodes and edges modify the graph's topology, so they have an effect on consensus. For instance, as it is well-known, the graph being connected is a necessary condition for consensus. Hence, it is natural to impose that the controller guarantees the graph's connectivity. In addition, in order to ensure the safety of the system, autonomous agents moving "freely" in a workspace must avoid collisions among them. To guarantee connectivity maintenance and the satisfaction of other inter-agent constraints, [5]–[8] rely on the control of the algebraic connectivity of the graph. Furthermore, such approaches implicitly deal with the addition/removal of edges, since the set of edges may change as long as the algebraic connectivity remains positive. However, the latter is a global parameter, so additional distributed-estimation approaches must be implemented. From a local perspective, barrier-function- and navigation-function-based approaches have been proposed, e.g., in [9]–[11], to address the case in which new edges are created as an agent enters the sensing zone of another one. In those references, both consensus and connectivity maintenance are established. In [12] the edge-addition problem for directed graphs is studied, but inter-agent constraints are not considered. A consensus control strategy for nonlinear passive systems that may deal with inter-agent constraints and time-varying topologies is proposed in [13]. None of the above, however, consider open multi-agent systems since the number of agents is considered fixed.

In this letter we address the problem of consensus-based control of an open multi-agent system where the agents do not communicate with each other, but can only measure their relative state with respect to their neighbors, if within a limited sensing zone. The controller is designed such that the initial existing edges as well as all the newly created ones are always maintained. Moreover, all-to-all collision avoidance is guaranteed by imposing a minimal-inter-agent-distance requirement that is satisfied by the proposed control law. As in [14], our control design and analysis rely on the edge-agreement framework introduced in [15], which allows to recast the consensus problem as one of stabilization of the origin and constitutes a more natural setting to consider inter-agent constraints.

Relative to the existing literature, we contribute by proposing a control law that ensures consensus-based formation of an open multi-agent system connected over a dynamic undirected graph with guaranteed connectivity-maintenance and all-to-all collision avoidance. Our results are restricted to the case of first-order systems, but we establish (almost-everywhere) *asymptotic stability* of the consensus manifold, via Lyapunov's direct method and based on the edge-agreement framework. This differs from the existing works in the literature where only *convergence* to consensus is guaranteed. Establishing asymptotic stability and disposing of a Lyapunov function, even for simple linear systems, is an important step towards extending our results to net-

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works of complex nonlinear systems. With respect to the references mentioned above, the novelty of our contribution lies in dealing with consensus under inter-agent constraints (local connectivity maintenance and collision avoidance) and establishing strong stability results for open multi-agent systems, with direct extension to closed switched multi-agent systems with edge addition. This also goes beyond the results presented in [14] applied to first-order systems.

The remainder of this letter is organized as follows. In Section II are presented the model and the problem statement. The main results are stated in Section III and are illustrated via numerical simulations in Section IV. Finally, some concluding remarks are given in Section V.

II. Model and problem description

As mentioned in the Introduction, in numerous practical scenarios the topology is determined by the sensing capabilities of the agents. Thus, in such cases, a new interconnection is created whenever an agent enters the sensing range of a neighbor. Moreover, in an open multi-agent system, new agents may be added to the system as they are detected. This naturally leads to study the multi-agent system using a switched-system representation.

Let $\sigma(t)$, $\sigma : \mathbb{R}_{\geq 0} \to \mathcal{P}$ be the switching signal corresponding to a topology change, where $\mathcal{P} := \{1, 2, \ldots, s\}$ is the set of s possible switching modes. Let $\phi \in \mathcal{P}$ denote a mode of the system, that is, $\phi = \sigma(\tau)$, where $\tau \in [t_l, t_{l+1})$ and t_l, t_{l+1} are switching instants, defined further below.

Next, for the switching signal $\sigma(t)$ we present the concept of transition-dependent average dwell time.

Definition 1: ([3, Definition 2]) On a given interval $[t_0, t_f)$, $t_f > t_0 \geq 0$ and for any $\hat{\phi}, \phi \in \mathcal{P}$, where $\hat{\phi}$ precedes ϕ, denote the total number of switchings from $\hat{\phi}$ to ϕ by $N_{\hat{\phi},\phi}(t_0, t_f)$ and denote the total active duration of mode ϕ by $T_{\hat{\phi},\phi}(t_0, t_f)$. Then, for any given scalar $\hat{N}_{\hat{\phi},\phi}$, the constant $\tau_{\hat{\phi},\phi} > 0$ that satisfies

$$N_{\hat{\phi},\phi}(t_0, t_f) \leq \hat{N}_{\hat{\phi},\phi} + \frac{T_{\hat{\phi},\phi}(t_0, t_f)}{\tau_{\hat{\phi},\phi}}, \tag{1}$$

is called the transition-dependent average dwell time of the switching signal $\sigma(t)$.

At each mode, the open multi-agent system is composed of N_ϕ agents described by the single integrator system

$$\dot{x}_i = u_i \tag{2}$$

where $x_i \in \mathbb{R}$, $i \leq N_\phi$, denotes the state of each agent[1], $u_i \in \mathbb{R}$ is the control input. The agents interact over a dynamic topology described by an *undirected* graph $\mathcal{G}_\phi(\mathcal{V}_\phi, \mathcal{E}_\phi)$ where the set of nodes $\mathcal{V}_\phi := \{1, 2, \ldots, N_\phi\}$ corresponds to the labels of the agents and the set of edges, $\mathcal{E}_\phi \subseteq \mathcal{V}_\phi^2$, of cardinality M_ϕ, represents the communication between a pair of nodes, that is, an edge $e_k := (i, j) \in \mathcal{E}_\phi$, $k \leq M_\phi$, is an unordered pair indicating that agent j has access to information from node i and vice-versa. Additionally, for every instant of time, we define two interaction sets for each agent i. The first one corresponds to the agents that are within the sensing range of agent i during mode ϕ and is defined as

$$\mathcal{N}_{\phi,i} = \{j \in \mathcal{V}_\phi,\ j \neq i \ : \ |x_i(t) - x_j(t)| \leq \Delta_i\}, \tag{3}$$

where Δ_i is the radius of the sensing zone of agent i. The second one is a subset of (3) and is defined as

$$\mathcal{N}^{\varepsilon}_{\phi,i} = \{j \in \mathcal{V}_\phi,\ j \neq i : |x_i(t) - x_j(t)| \leq \Delta_i - \varepsilon\} \tag{4}$$

where $\varepsilon > 0$ is a small positive scalar.

Remark 1: Without loss of generality, for any edge e_k we set $\Delta_i = \Delta_j =: \Delta_k$. Therefore, since the graph is undirected, $j \in \mathcal{N}^{\varepsilon}_{\phi,i}$ implies $i \in \mathcal{N}^{\varepsilon}_{\phi,j}$. •

Based on the sets in (3) and (4), we define the initial topology as the graph formed by the set of edges in the initial mode $\phi_0 = \sigma(\tau)$, $\tau \in [t_0, t_1)$, with t_1 being the first topology-switch instant. That is, $\mathcal{E}_{\phi_0} = \{i, j \in \mathcal{V}_{\phi_0} \ : \ j \in \mathcal{N}_{\phi_0,i}\}$ and we assume that the following holds.

Assumption 1: The initial undirected graph $\mathcal{G}_{\phi_0}$ is connected.

A switching instant t_l is defined as the time when new agents and/or edges are added to the graph, that is, whenever an agent or a group of agents enters the sensing zone of any other agent. Hence, for any mode $\phi = \sigma(\tau) \in \mathcal{P}$, $\tau \in [t_l, t_{l+1})$, with its previous mode denoted $\hat{\phi} = \sigma(\hat{\tau}) \in \mathcal{P}$, $\hat{\tau} \in [t_{l-1}, t_l)$, the set of edges is obtained by the rule

$$\mathcal{E}_\phi = \mathcal{E}_{\hat{\phi}} \cup \mathcal{E}^*_{\sigma(t_l^-)} \tag{5}$$

where

$$\mathcal{E}^*_{\sigma(t_l^-)} = \Big\{ i, j \in \mathcal{V}_\phi : \Big[(i, j) \notin \mathcal{E}_{\hat{\phi}}\Big] \wedge \Big[j \in \mathcal{N}^{\varepsilon}_{\sigma(t_l^-),i}\Big] \Big\}. \tag{6}$$

The rule (5) means that whenever an agent j that did not form an edge with agent i during the mode $\hat{\phi}$ enters the set $\mathcal{N}^{\varepsilon}_{\hat{\phi},i}$, at some time instant t_l, a new edge is created and the system switches from mode $\hat{\phi}$ to mode ϕ. Now, since in this paper the objective is to preserve all the initial and added edges at each mode, while only a finite number of new agents can be added to the system, the system necessarily stops switching as soon as no new edges are created, that is, when the graph is complete. Hence, we pose the following.

Assumption 2: The total number of possible switching modes is finite, that is, $|\mathcal{P}| < \infty$, where $|\mathcal{P}|$ denotes the cardinality of $\mathcal{P}$.

Now, for each mode $\phi \in \mathcal{P}$, we define an edge state as

$$z_{\phi,k} := x_i - x_j \quad \forall k \leq M_\phi, \quad e_k \in \mathcal{E}_\phi \tag{7}$$

and, for each $k \leq M_\phi$, the set of inter-agent constraints as

$$\mathcal{D}_{\phi,k} := \big\{ z_{\phi,k} \in \mathbb{R} : \delta < |z_{\phi,k}| < \Delta_k, \big\}, \tag{8}$$

where $\delta > 0$ is the minimum distance between any pair of agents that guarantees collision avoidance.

The control goal is for the agents to achieve a consensus-based formation, under the inter-agent constraints given by the sets $\mathcal{D}_{\phi,k}$ in (8) and the edges' update rule in (5). Mathematically, the formation problem translates into

[1]For simplicity, we consider $x_i \in \mathbb{R}$, but all the statements hold, and may be rewritten, for $x_i \in \mathbb{R}^n$ under appropriate modifications involving the use of the Kronecker product.

making $x_i - x_j \to z^d_{\phi,k}$, or equivalently, $z_{\phi,k} \to z^d_{\phi,k}$ in the relative coordinates, where $z^d_{\phi,k} \in \mathbb{R}$ denotes the desired relative state between a pair of neighboring agents i and j for each mode. Hence, we address this problem using the edge-based representation method of [14]. This has the advantage of recasting the consensus-based problem as the stabilization of the origin in error coordinates.

For each mode, let us denote the so-called incidence matrix of the graph $E_\phi \in \mathbb{R}^{N_\phi \times M_\phi}$ with rows indexed by the nodes and columns indexed by the edges. Its (i,k)th entry is defined as follows: $[E_\phi]_{ik} := -1$ if i is the terminal node of edge e_k, $[E_\phi]_{ik} := 1$ if i is the initial node of edge e_k, and $[E_\phi]_{ik} := 0$ otherwise. Then, the edge states in (7) satisfy

$$z_\phi := E_\phi^\top x_\phi, \tag{9}$$

where $x_\phi \in \mathbb{R}^{N_\phi}$ and $z_\phi \in \mathbb{R}^{M_\phi}$ are defined, respectively, as $x_\phi^\top = \begin{bmatrix} x_1^\top \cdots x_{N_\phi}^\top \end{bmatrix}$ and $z_\phi^\top := \begin{bmatrix} z_{\phi,1}^\top \ldots z_{\phi,M_\phi}^\top \end{bmatrix}$, so the consensus-based formation objective may be defined in terms of the error

$$\tilde{z}_\phi = E_\phi^\top x_\phi - z_\phi^d, \tag{10}$$

where $z_\phi^{d\top} = [z_{\phi,1}^{d\top} \cdots z_{\phi,M_\phi}^{d\top}] \in \mathbb{R}^{M_\phi}$, as

$$\lim_{t\to\infty} |\tilde{z}_\phi(t)| = 0. \tag{11}$$

To address this problem we rewrite the system's dynamics in terms of $\tilde{z}_\phi$ by collecting the inputs of the multiple agents into the vector $u_\phi^\top = \begin{bmatrix} u_1^\top \cdots u_{N_\phi}^\top \end{bmatrix} \in \mathbb{R}^{N_\phi}$. Then, taking the time derivative of (10) and using (2), we obtain

$$\dot{\tilde{z}}_\phi = E_\phi^\top u_\phi. \tag{12}$$

In these coordinates, the formation objective as defined in (11) is achieved if the origin is asymptotically stabilized for the edge system (12).

One of the advantages of considering the edge states rather than the node's is that it is possible to obtain an equivalent reduced system that is easier to analyze using stability theory. Indeed, as observed in [15], using an appropriate labeling of the edges, the incidence matrix is expressed as $E_\phi = [E_{\mathcal{T}\phi} \quad E_{\mathcal{C}\phi}]$, where $E_{\mathcal{T}\phi} \in \mathbb{R}^{N_\phi \times (N_\phi - 1)}$ denotes the full-column-rank incidence matrix corresponding to an arbitrary spanning tree $\mathcal{G}_{\mathcal{T}\phi} \subset \mathcal{G}_\phi$ and $E_{\mathcal{C}\phi} \in \mathbb{R}^{N_\phi \times (M_\phi - N_\phi + 1)}$ represents the incidence matrix corresponding to the remaining edges not contained in $\mathcal{G}_{\mathcal{T}\phi}$. Moreover,

$$E_\phi = E_{\mathcal{T}\phi} R_\phi, \tag{13}$$

with

$$R_\phi := \begin{bmatrix} I_{N_\phi - 1} & T_\phi \end{bmatrix}, \quad T_\phi := \left(E_{\mathcal{T}\phi}^\top E_{\mathcal{T}\phi} \right)^{-1} E_{\mathcal{T}\phi}^\top E_{\mathcal{C}\phi}. \tag{14}$$

The identity (13) is useful to derive a reduced-order model—cf. [15]. Indeed, the edges' states are also split as

$$\tilde{z}_\phi = \begin{bmatrix} \tilde{z}_{\mathcal{T}\phi}^\top & \tilde{z}_{\mathcal{C}\phi}^\top \end{bmatrix}^\top, \tag{15}$$

where $\tilde{z}_{\mathcal{T}\phi} \in \mathbb{R}^{N_\phi - 1}$ are the states of the edges of an arbitrary spanning tree $\mathcal{G}_{\mathcal{T}\phi}$ and $\tilde{z}_{\mathcal{C}\phi} \in \mathbb{R}^{M_\phi - N_\phi + 1}$ are the states of the remaining edges $e_k \in \mathcal{G}_\phi \backslash \mathcal{G}_{\mathcal{T}\phi}$. Thus, after (9), (11), and (15), and denoting $z_{\mathcal{T}\phi}^d \in \mathbb{R}^{N_\phi - 1}$ as the vector of desired relative displacements corresponding to $\mathcal{G}_{\mathcal{T}\phi}$, we obtain

$$\tilde{z}_\phi = R_\phi^\top \tilde{z}_{\mathcal{T}\phi}. \tag{16}$$

Using the identity (16), the edge system (12) can be transformed into the reduced-order system

$$\dot{\tilde{z}}_{\mathcal{T}\phi} = E_{\mathcal{T}\phi}^\top u_\phi \tag{17}$$

Then, asymptotic stability for the origin of (17) implies asymptotic stability for the origin of (12). Therefore, the formation objective defined in (11) is achieved if the origin is asymptotically stabilized for the reduced-order system (17).

III. Main result

A. Control approach

To address the formation problem with inter-agent constraints formulated above, we propose a control law based on the gradient of a *barrier Lyapunov function* (BLF). These are positive definite, but their domain of definition is restricted by design to open subsets of the Euclidean space, and they grow unbounded as the state approaches the boundary of their domain. We define a BLF as follows—cf. [16].

Definition 2 (Barrier Lyapunov function): Consider the system $\dot{x} = f(x)$ and let $\mathcal{J}$ be an open set with boundary $\partial\mathcal{J}$ and containing the origin. A BLF is a positive definite function $V : \mathcal{J} \to \mathbb{R}_{\geq 0}$, $x \mapsto V(x)$, that is $\mathcal{C}^1$, satisfies $\dot{V}(x) \leq 0$ and has the property $V(x) \to \infty$ and $|\nabla V(x)| \to \infty$ as $x \to \partial\mathcal{J}$, where $\nabla V(x) = \partial V(x)/\partial x$.

Now, akin to (8), for each mode and for each $k \leq M_\phi$, the inter-agent constraints in terms of the error coordinates are given by the set

$$\tilde{\mathcal{D}}_{\phi,k} := \{\tilde{z}_{\phi,k} \in \mathbb{R} \, : \, \delta < |\tilde{z}_{\phi,k} + z^d_{\phi,k}| < \Delta_k\}. \tag{18}$$

Then, for each $k \leq M_\phi$, we define a candidate BLF $W_{\phi,k} : \tilde{\mathcal{D}}_{\phi,k} \to \mathbb{R}_{\geq 0}$, as

$$W_{\phi,k}(\tilde{z}_{\phi,k}) = \frac{1}{2}\left[\tilde{z}^2_{\phi,k} + B_{\phi,k}(\tilde{z}_{\phi,k} + z^d_{\phi,k}) \right], \tag{19}$$

where $B_{\phi,k}(\tilde{z}_{\phi,k} + z^d_{\phi,k})$ is non-negative and satisfies $B_{\phi,k}(z^d_{\phi,k}) = 0$, $B_{\phi,k}(\tilde{z}_{\phi,k} + z^d_{\phi,k}) \to \infty$ as $|\tilde{z}_{\phi,k} + z^d_{\phi,k}| \to \Delta_k$ or as $|\tilde{z}_{\phi,k} + z^d_{\phi,k}| \to \delta_k$, and $\nabla B_{\phi,k}(z^d_{\phi,k}) = 0$, where

$$\nabla B_{\phi,k}(z^d_{\phi,k}) = \left. \frac{\partial B_{\phi,k}(\tilde{z}_{\phi,k} + z^d_{\phi,k})}{\partial \tilde{z}_{\phi,k}} \right|_{\tilde{z}_{\phi,k}=0}.$$

Therefore, $W_{\phi,k}(\tilde{z}_{\phi,k}) \to \infty$ as $|\tilde{z}_{\phi,k} + z^d_{\phi,k}| \to \Delta_k$ or as $|\tilde{z}_{\phi,k} + z^d_{\phi,k}| \to \delta_k$.[2]

Then, the formation control law for each agent is set to

$$u_i := -c \sum_{k \leq M_\phi} [E_\phi]_{ik} \nabla W_{\phi,k}(\tilde{z}_{\phi,k}), \quad \forall i \leq N_\phi \tag{20}$$

and, introducing the candidate BLF for the overall system,

$$V_\phi(\tilde{z}_\phi) = \sum_{k \leq M_\phi} W_{\phi,k}(\tilde{z}_{\phi,k}) \tag{21}$$

[2]For agents evolving in $\mathbb{R}^n$, in (18) and (19) , consider $|\cdot|$ to be the Euclidean norm and replace $(\cdot)^2$ with $|\cdot|^2$.

we write the control law may in compact form as

$$u_\phi := -c_1 E_\phi \nabla V_\phi(\tilde{z}_\phi). \tag{22}$$

Remark 2: The functions defined in (19) are reminiscent of scalar potential functions in constrained environments [17] and, as for the latter, the appearance of multiple critical points is inevitable. Indeed, the gradient of the BLF (19), $\nabla W_{\phi,k}(\tilde{z}_{\phi,k})$, vanishes at the origin and at an isolated saddle point separated from the origin —see [18]. Therefore, when using the gradient of (19) for the control, the closed-loop system has multiple equilibria. •

In light of Remark 2, let us denote by $\tilde{z}^*_\phi \in \mathbb{R}^{M\phi}$ the vector containing the saddle points of the BLF for each edge (19). Then, define the disjoint set $\mathcal{W}_\phi := \{0\} \cup \{\tilde{z}^*_\phi\}$, which corresponds to the critical points of $\tilde{z}_\phi \mapsto V_\phi(\tilde{z}_\phi)$ in (21). Then, V_ϕ satisfies $\alpha_1 |\tilde{z}_\phi|^2 \le V_\phi(\tilde{z}_\phi)$, with $\alpha_1 > 0$, so

$$\alpha_1 |\tilde{z}_\phi|^2_{\mathcal{W}_\phi} \le V_\phi(\tilde{z}_\phi), \quad \alpha_1 > 0, \tag{23}$$

since $|\tilde{z}_\phi|_{\mathcal{W}_\phi} \le |\tilde{z}_\phi|$, where $|\tilde{z}_\phi|_{\mathcal{W}_\phi} := \min\left\{|\tilde{z}_\phi|, |\tilde{z}_\phi - \tilde{z}^*_\phi|\right\}$.

Due to the open nature of the system, the dimension of the state is not constant for all switching modes $\phi \in \mathcal{P}$. In fact, even if the number of agents in the system does not change, the dimension of $\tilde{z}_\phi \in \mathbb{R}^{M_\phi}$ increases each time a new edge is added to the graph. Therefore, the system (17) in closed loop with (22) may be represented as a multi-dimensional impulsive switched system of the form

$$\dot{\tilde{z}}_{\mathcal{T}\phi}(t) = -c_1 E^\top_{\mathcal{T}\phi} E_{\mathcal{T}\phi} \nabla V_\phi(\tilde{z}_{\mathcal{T}\phi}(t)), \quad \text{if} \quad t \in (t_l, t_{l+1}) \tag{24a}$$

$$\tilde{z}_{\mathcal{T}\phi}(t_l^+) = \Xi_{\phi,\hat{\phi}} \tilde{z}_{\mathcal{T}\hat{\phi}}(t_l^-) + \Phi_l, \qquad \text{if} \quad t = t_l \tag{24b}$$

where t_l are the switching instants, $\Xi_{\phi,\hat{\phi}} \in \mathbb{B}^{M_\phi \times M_{\hat{\phi}}}$ is a 0-1 matrix that indicates the dimension variation of the state at t_l, and $\Phi_l \in \mathbb{R}^{M_\phi}$ is a real vector indicating the value changes of $\tilde{z}_\phi$ at any t_l. Note also that in the right-hand side of (24a) we used (13) and $R^\top_\phi \nabla V_\phi(\tilde{z}_\phi) = \nabla V_\phi(\tilde{z}_{\mathcal{T}\phi})$, where

$$\nabla V_\phi(\tilde{z}_{\mathcal{T}\phi}) := \frac{\partial V_\phi\left(R^\top_\phi \tilde{z}_{\mathcal{T}\phi}\right)}{\partial \tilde{z}_{\mathcal{T}\phi}}.$$

B. Main statement

Proposition 1: Consider the open multi-agent system (2) under Assumption 1 and in closed loop with the switching control law (22). Then,

1) for every mode $\phi \in \mathcal{P}$ the constraints set $\mathcal{D}_\phi := \bigcap_{k \le M_\phi} \mathcal{D}_{\phi,k}$, with $\mathcal{D}_{\phi,k}$ defined in (8) is forward invariant, hence, collisions are avoided and all the initial and added edges are maintained;
2) if the switching signal σ admits an average dwell time, as in Definition 1, satisfying

$$\tau_{\phi,\hat{\phi}} > \frac{\ln(\Omega_{\phi,\hat{\phi}})}{\gamma_\phi}, \tag{25}$$

where $\Omega_{\phi,\hat{\phi}}$ and γ_ϕ are positive constants defined further below, the origin of the closed-loop system (24) is asymptotically practically stable for all initial conditions such that $z_{\phi,k}(t_0) \in \mathcal{D}_{\phi,k}$, $\forall k \le M_\phi$, except for a zero-measure set of unstable equilibria;
3) under Assumption 2, the origin of the closed-loop system (24) is asymptotically stable for all initial conditions as in 2) above. □

Proof: We start by establishing the second claim. For each mode $\phi \in \mathcal{P}$ and for all $\tilde{z}_{\phi,k} \in \tilde{\mathcal{D}}_{\phi,k}$, $k \le M_\phi$, the candidate Lyapunov functions in (21) satisfy

$$\alpha_1 |\tilde{z}_{\mathcal{T}\phi}|^2 \le V_\phi(\tilde{z}_{\mathcal{T}\phi}) \le \alpha_2 |\nabla V_\phi(\tilde{z}_{\mathcal{T}\phi})|^2, \tag{26}$$

where $\alpha_1, \alpha_2 > 0$ and, with an abuse of notation, we write V_ϕ as a function of $\tilde{z}_{\mathcal{T}\phi}$ since $V_\phi(\tilde{z}_\phi) = V_\phi(R^\top \tilde{z}_{\mathcal{T}\phi})$. Furthermore, between switching times, that is for all $\tau \in (t_l, t_{l+1})$ with $l \in \{0, \ldots, |\mathcal{P}| - 1\}$, the derivative of V_ϕ satisfies

$$\begin{aligned}\dot{V}_\phi(\tilde{z}_{\mathcal{T}\phi}(\tau)) &= -c_1 \nabla V_\phi(\tilde{z}_{\mathcal{T}\phi}(\tau))^\top E^\top_{\mathcal{T}\phi} E_{\mathcal{T}\phi} \nabla V_\phi(\tilde{z}_{\mathcal{T}\phi}(\tau)) \\ &\le -c_1 \lambda_{min}(E^\top_{\mathcal{T}\phi} E_{\mathcal{T}\phi}) |\nabla V_\phi(\tilde{z}_{\mathcal{T}\phi}(\tau))|^2 \\ &\le -\gamma_\phi V_\phi(\tilde{z}_{\mathcal{T}\phi}(\tau)),\end{aligned} \tag{27}$$

where $\gamma_\phi := c_1 \lambda_{min}(E^\top_{\mathcal{T}\phi} E_{\mathcal{T}\phi})/\alpha_2$ is a positive constant since $E^\top_{\mathcal{T}\phi} E_{\mathcal{T}\phi}$ is positive definite—cf. [15].

Now, let $\hat{\phi}, \phi \in \mathcal{P}$ denote two consecutive modes where $\hat{\phi}$ precedes ϕ. Assume, without loss of generality, that the added edges at each switching instant t_l are labeled $z_{\hat{\phi},k}$ with $M_{\hat{\phi}} < k \le M_{\hat{\phi}} + r =: M_\phi$, for r new edges. Then, from (21), we have

$$\begin{aligned}V_\phi(\tilde{z}_{\mathcal{T}\phi}(t_l^+)) &= \sum_{k=1}^{M_{\hat{\phi}}} W_{\phi,k}(\tilde{z}_{\phi,k}(t_l^+)) + \sum_{k=M_{\hat{\phi}}+1}^{M_\phi} W_{\phi,k}(\tilde{z}_{\phi,k}(t_l^+)) \\ &= V_{\hat{\phi}}(\tilde{z}_{\mathcal{T}\hat{\phi}}(t_l^-)) + \sum_{k=M_{\hat{\phi}}+1}^{M_\phi} W_{\phi,k}(\tilde{z}_{\phi,k}(t_l^+))\end{aligned} \tag{28}$$

Recall that according to the update law (5) the topology switches whenever an agent j is in sufficient proximity of the ith agent, i.e., if $j \in \mathcal{N}^\varepsilon_{\hat{\phi},i}$. Hence, right after at the instant t_l, we have $|\tilde{z}_{\phi,k}(t_l^+)| \le \Delta_k - \varepsilon$, with $M_{\hat{\phi}} < k \le M_\phi$. Therefore, using (26), we obtain

$$\begin{aligned}\sum_{k=M_{\hat{\phi}}+1}^{M_\phi} W_{\phi,k}(\tilde{z}_{\phi,k}(t_l^+)) &\le \alpha_2 |\nabla V_\phi(\tilde{z}_{\mathcal{T}\phi}(t_l^+))|^2 \\ &\le \alpha_2 \beta |\tilde{z}_{\mathcal{T}\phi}(t_l^+)|^2_{\mathcal{W}_\phi}\end{aligned} \tag{29}$$

with $\beta > 0$. Then, from (24b), (29) becomes

$$\sum_{k=M_{\hat{\phi}}+1}^{M_\phi} W_{\phi,k}(\tilde{z}_{\hat{\phi},k}(t_l^+)) \le \alpha_2 \beta \Big(|\Xi_{\phi,\hat{\phi}}|^2 |\tilde{z}_{\mathcal{T}\phi}(t_l^-)|^2_{\mathcal{W}_\phi} + |\Phi_l|^2_{\mathcal{W}_\phi}\Big). \tag{30}$$

Note that $|\Xi_{\phi,\hat{\phi}}| \equiv 1$ and $|\Phi_l|_{\mathcal{W}_\phi} \le \bar{\Phi}$, where the latter holds from the fact that at a switching instant $|\tilde{z}_{\phi,k}(t_l)| \le \Delta_k - \varepsilon$, $M_{\hat{\phi}} < k \le M_\phi$. Therefore, $\bar{\Phi} := (\Delta_k - \varepsilon) r$, where r is the number of newly added edges. Hence, replacing (30) and (26) into (28), we obtain

$$V_\phi(\tilde{z}_{\mathcal{T}\phi}(t_l^+)) \le \Omega_{\phi,\hat{\phi}} V_{\hat{\phi}}(\tilde{z}_{\mathcal{T}\hat{\phi},t}(t_l^-)) + \Theta, \tag{31}$$

with $\Omega_{\phi,\hat{\phi}} := (1 + \alpha_2 \beta / \alpha_1)$ and $\Theta := \alpha_2 \beta \bar{\Phi}^2$.

From (26)-(27) and (31), and invoking [3, Theorem 1], it follows that the origin for the reduced-order system (24) is almost-everywhere asymptotically practically stable (that is, except for a zero-measure set of initial conditions corresponding to the unstable equilibria—see Remark 2) if the switching σ admits an average dwell-time satisfying (25).

Next, we establish the third item. In view of Assumption 2, there is a finite number of new agents (nodes) that may be added into the system. Hence, after the last new node or set of nodes have joined the system, only new edges can be added to the graph. Moreover, from (15) and (16), we know that any new edge in mode ϕ can be represented as a linear combination of the edges of the spanning tree in the precedent mode $\hat{\phi}$. Therefore, if only new edges appear, (24b) holds with $\Phi_l = 0$ and $\Xi_{\phi,\hat{\phi}} = I$, with I the identity matrix of appropriate dimension. Consequently, (31) holds with $\Theta = 0$ and, by [3, Theorem 1], the origin for the reduced-order system (24) is almost-everywhere asymptotically stable if the average-dwell time satisfies (25). Note that the latter necessarily holds under Assumption 2. Indeed, since only a finite number of new *nodes* may be added (from practical considerations), for any (large) $\tau_{\phi,\hat{\phi}}$ there necessarily exists t_f, and consequently $T_{\phi,\hat{\phi}}$, so that (1) in Definition 1 holds.

Now we establish invariance of the set $\mathcal{D}_\phi$ (first item). To that end, given a $\phi \in \mathcal{P}$, $\tilde{z}_{\phi,k} \in \tilde{\mathcal{D}}_{\phi,k}$, $k \leq M_\phi$, we show that $\tilde{\mathcal{D}}_{\phi,k}$ is forward invariant. We proceed by contradiction. We conclude from (27) that for any continuous mode $\phi = \sigma(\tau)$, $\tau \in (t_l, t_{l+1})$, $V_\phi(\tilde{z}_{\mathcal{T}\phi}(\tau))$ is bounded. Moreover, from (31) at every switching instant t_l, $V_\phi(\tilde{z}_{\mathcal{T}\phi}(t_l^+))$ remains bounded. Now, assume that there exists $T > 0$ such that for all $t \in [t_0, T)$, $\tilde{z}_{\sigma(t),k}(t) \in \tilde{\mathcal{D}}_{\sigma(t),k}$ for all $k \leq M_{\sigma(t)}$, and $z_{\sigma(t),k}(T) \notin \tilde{\mathcal{D}}_{\sigma(T),k}$. More precisely, we have $|\tilde{z}_{\sigma(t),k}(t)| \to \Delta_k$ as $t \to T$ for at least one $k \leq M_{\sigma(t)}$. From the definition of barrier Lyapunov function, this implies that $V_{\sigma(t)}(\tilde{z}_{\mathcal{T}\sigma(t)}(t)) \to \infty$ as $t \to T$ which is a contradiction since $V_{\sigma(t)}(\tilde{z}_{\mathcal{T}\sigma(t)}(t))$ is bounded for all $t \geq t_0$. Forward invariance of $\tilde{\mathcal{D}}_{\sigma(t),k}$, hence of $\mathcal{D}_{\sigma(t)}$, follows.

It is left to show that the set $\mathcal{D}_{\sigma(t)}$ corresponds to the domain of attraction for the closed-loop system. This follows by showing that all solutions of (24) starting in $\mathcal{D}_{\sigma(t)}$ converge to the origin. To that end, for any $\epsilon \in (0, \Delta_k)$, consider a subset $\mathcal{D}_{\sigma(t),\epsilon} \subset \mathcal{D}_{\sigma(t)}$ defined as $\mathcal{D}_{\sigma(t),\epsilon} := \{z_{\sigma(t)} \in \mathbb{R}^{N_\sigma(t)} : \delta + \epsilon < |z_{\sigma(t),k}| < \Delta_k - \epsilon, \ \forall\, k \leq M_{\sigma(t)}\}$ and let $\bar{\mathcal{D}}_{\sigma(t),\epsilon}$, denote the closure of $\mathcal{D}_{\sigma(t),\epsilon}$. From Definition 2 and (26) it follows that $V_{\sigma(t)}(\tilde{z}_{\mathcal{T}\sigma(t)})$ is positive definite on $\bar{\mathcal{D}}_{\sigma(t),\epsilon}$ and it satisfies the bounds in (26) and (23), where $|\nabla V_\phi(s)|$ is defined and strictly increasing everywhere in $\bar{\mathcal{D}}_{\sigma(t),\epsilon}$, $|\nabla V_\phi(s)| > 0$ for all $|s| > 0$, and $|\nabla V_\phi(0)| = 0$. This means that $V_{\sigma(t)}(\tilde{z}_{\mathcal{T}\sigma(t)}) \to 0$ as $\tilde{z}_{\mathcal{T}\sigma(t)} \to 0$. Therefore, from (27), (31), and standard Lyapunov theory it follows that all trajectories of (24) starting in $\mathcal{D}_{\sigma(t)}$, except for a zero-measure set of initial conditions, converge to the origin. The previous arguments hold for any $\epsilon \to 0$, so the origin is attractive for all trajectories $z_{\sigma(t)}(t)$ starting in $\mathcal{D}_{\sigma(t)}$, except for a zero-measure set of initial conditions. This completes the proofs of statements 2) and 3). ■

Remark 3: The impulsive and multi-dimensional nature of the open multi-agent system in the edge-based representation (24) stymies the construction of a common Lyapunov function [19]. Indeed, unlike the scenario of edge-addition problem for *closed* multi-agent systems in the nodes' representation—see e.g. [9], the candidate BLF (21) is a sum over all the edges of the graph at each mode. But, since each time an edge is added to the graph the state changes dimension, as denoted by Eq. (24b), the stability analysis relies on the concept of multiple Lyapunov functions [20]. •

IV. Numerical example

We illustrate the performance of the controller (22) via a numerical example. The simulation consists in the consensus-based formation of a network composed initially of six agents subject to connectivity and collision-avoidance restrictions and with an initial topology modeled by the connected undirected graph in Fig. 1. Two new agents, $i = 7$ and $i = 8$ are added to the network at $t = 0.5$s and $t = 3$s, respectively. Moreover, as the nodes get close to each other, new edges are also added. This can be appreciated by the appearance of new trajectories in Figs. 3 and 4, after 1s and 3s.

The initial conditions of the six original agents in the simulation, as well as $(x_7(1), y_7(1))$ and $(x_8(3), y_8(3))$ are presented in Table I. The limited range is taken equal for all the agents and is set to $\Delta_k = 4.6$m. The controller gain is set to $c_1 = 1.5$. The barrier function is given by

$$B_{\phi,k}(z_{\phi,k}) = \kappa_{1,k}\left[\ln\left(\frac{\Delta_k^2}{\Delta_k^2 - |z_{\phi,k}|^2}\right) - \ln\left(\frac{\Delta_k^2}{\Delta_k^2 - |z_{\phi,k}^d|^2}\right)\right] + \kappa_{2,k}\left[\ln\left(\frac{|z_{\phi,k}|^2}{|z_{\phi,k}|^2 - \delta^2}\right) - \ln\left(\frac{|z_{\phi,k}^d|^2}{|z_{\phi,k}^d|^2 - \delta^2}\right)\right]$$

where

$$\kappa_{1,k} := \frac{\delta^2}{|z_{\phi,k}^d|^2(|z_{\phi,k}^d|^2 - \delta^2)}, \quad \kappa_{2,k} := \frac{1}{\Delta_k^2 - |z_{\phi,k}^d|^2}.$$

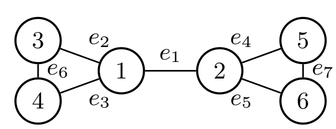

Fig. 1: Initial interaction topology: undirected connected graph.

TABLE I: Initial conditions

Index	1	2	3	4	5	6	7	8
x **[m]**	1.9	-2	5.5	-5.5	5.5	-5.5	0	5
y **[m]**	0	0	2	2	-2	-2	5	8

The paths of each agent up to consensus are shown in Fig. 2. It can be appreciated in Fig. 2 that the systems successfully reach the desired formation after the eighth agent is added to the system. The same can be appreciated by the convergence of edge-error variables to the origin, as shown in Fig. 4. Furthermore, it is clear from Fig. 3 that both the connectivity and collision avoidance constraints (represented by the red dashed lines) are always respected. Indeed, it can be seen that as a new edge is created, it never goes beyond or below the imposed limits, meaning that it is maintained for the rest of the simulation and that collisions between each pair of agents are avoided.

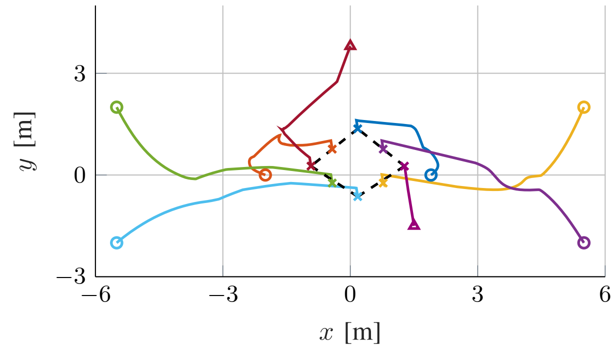

Fig. 2: Paths described by six agents from initial postures, represented by circles, and by two more agents joining the network from "initial" postures, represented by triangles. The crosses denote the final positions. The black dashed lines represent the final desired formation.

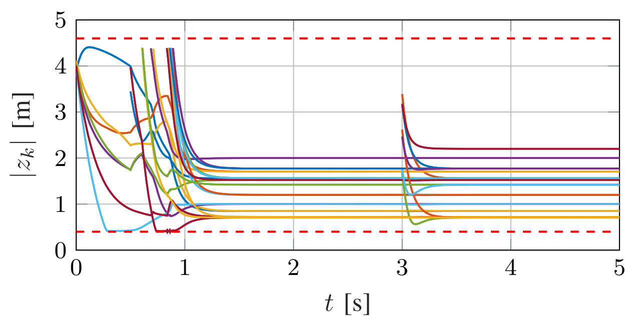

Fig. 3: Inter-agent distances. The dashed red lines denote the connectivity and collision avoidance constraints.

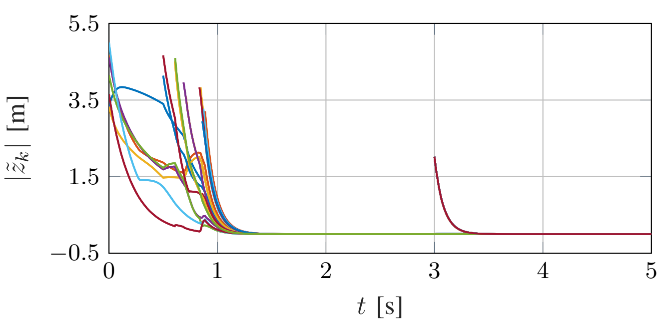

Fig. 4: Norms of the formation errors for each edge.

V. Conclusions

This letter is the first attempt to address consensus-based control of open multi-agent systems under inter-agent constraints. The controller that we propose is designed using a gradient-based law and a barrier Lyapunov function, to guarantee collision-avoidance and connectivity-maintenance constraints. Even though our preliminary results are limited to first-order systems, we establish almost-everywhere asymptotic stability of the consensus manifold for open multi-agent systems. The results obtained also apply to closed multi-agent systems with edge addition, but not to directed-graph topologies or the more challenging case of node/edge removal. These are subjects of future research.

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