Exponential Bipartite Containment Tracking over Multi-leader Coopetition Networks

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Abstract— **This paper addresses the distributed bipartite containment tracking-control problem for autonomous vehicles steered by multiple leaders. Some leaders are cooperative and others are competitive, so the vehicles form a so-called coopetition network; in which the interaction links may be negative or positive. The presence of cooperative and antagonistic leaders does not enable the system to achieve consensus. Instead, the followers' states converge to a residual compact set, not predefined, but depending only on the leaders' states. We establish global exponential stability for this so-called bipartite containment set, and we compute the exact equilibria to which all agents converge inside of it. Our proofs are constructive, that is, we provide strict Lyapunov functions, which also allow us to establish robustness with respect to external disturbances. Numerical simulations illustrate our theoretical findings.**

I. Introduction

Coordination of multi-agent networks has received considerable attention due to multiple potential applications in engineering and social sciences [1]. A large number of consensus problems have been extensively studied, e.g., for first-order, second-order, and for linear high-order dynamics [2]. In particular, when the network contains a leader, all followers converge to the leader's states and achieve consensus. On the other hand, when the network contains more than one leader, it is impossible to achieve classical consensus and it appears more appropriate to speak of *containment* control [3]. The latter consists in making all followers' states converge to the convex hull determined only by leaders' initial conditions.

There are various studies on distributed containment control, e.g., for social networks [4] or for networks of single-integrators [5], double-integrators [6], and general linear autonomous systems [7]. Yet, most of the current research on the consensus or containment problems for multi-agent systems focus on cooperative networks, i.e., the interactions between nodes are characterised only by non-negative edge weights, although there are many scenarii in which agents may compete. These may appear, e.g., in robotics, in the context of herding control [8], in aerospace applications involving control of multiple satellites that must face debris represented as non-cooperative agents, or in the context of social networks that include trust/distrust relationship between agents [9]. The latter reference extends the notion of consensus to networks containing antagonistic interactions, and provides a general graph-theory-based framework to deal with *signed* networks, also called *coopetition* networks [10]. In this case, the edge weights may be either positive or negative and at least two consensus equilibria appear—we speak of *bipartite consensus* [9]. In the case of networks with multiple cooperative and antagonistic leaders, the overall behavior is even more complex [11] and more than two consensus equilibria may appear [12].

To analyse the complex behavior of multi-leader coopetition networks, in [13] the notion of containment control is extended to *bipartite containment tracking-control*. The latter consists in making all followers converge to the geometric space spanned by all the leaders' trajectories and their symmetric counterparts. Bipartite containment has been also studied in [14], [15], where the followers converge asymptotically into the convex hull, determined by leaders' initial conditions, but only in [13] limit points for the followers' states are given explicitly. In [14], only cooperative leaders are considered, so they do not apply to scenarii involving, for example, obstacle avoidance, where an obstacle may be considered as a competitive leader.

In this paper, we study the bipartite containment tracking problem presented in [13] over structurally balanced multi-leader coopetition networks described by first-order systems, and we provide the exact equilibrium points for the followers. Unlike [13], we assume that the leaders do not interact. Our main contribution is to establish exponential stability of the containment set, and ensure robustness with respect to additive perturbations. On the other hand, in contrast to all references mentioned previously, our proofs are constructive; we provide a strict Lyapunov function. Exponential stability is a stronger property than the convergence to the interior of a containment set and provides a basis to establish input-to-state stability (ISS).

Our contributions are based on the framework introduced in [16]. We recast the bipartite containment problem into a problem of stability of a set of the appropriately defined errors. Then, we extend the main results of [17] which provides a Lyapunov characterisation for matrices admitting one zero eigenvalue and others having negative real part. We extend this result to the case of matrices admitting multiple zero eigenvalues and by that we establish global exponential stability of the bipartite containment set. Furthermore, we give the explicit limit values of followers by constructing a matrix determined by all eigenvectors associated to the zero eigenvalues.

II. Problem Formulation

Consider a group of n dynamical systems modeled by

$$\dot{x}_i = u_i, \quad x_i, u_i \in \mathbb{R}, \quad i \in \mathcal{I}_N \tag{1}$$

where $\mathcal{I}_N := \{1, 2, \ldots, n\}$, but all the contents of this paper hold if $x_i, u_i \in \mathbb{R}^N$ with $N > 1$. It is well-known–see e.g.

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[3]– that under the distributed control law

$$u_i = -\sum_{j=1}^{n} a_{ij}(x_i - x_j), \tag{2}$$

where $a_{ij} \in \mathbb{R}_{\geq 0}$ is the adjacency weight between the nodes i and j, the consensus problem, that is,

$$\lim_{t\to\infty}[x_j(t) - x_i(t)] = 0 \quad \forall\, i, j \leq n, \tag{3}$$

is solved if and only if the underlying graph contains a directed spanning tree. More precisely, $a_{ij} > 0$ if there is a directed interconnection from the ith node to the jth node, $a_{ij} = 0$ if there is not, and there exists at least one node from which any other node may be reached. Moreover, *the* consensus equilibrium may be computed explicitly. Indeed, if there exists a directed spanning tree, the resulting Laplacian matrix, $L := [\ell_{ij}] \in \mathbb{R}^{N\times N}$, where

$$\ell_{ij} = \begin{cases} \sum\limits_{k\in\mathcal{I}_{\mathcal{N}}} a_{ik} & i = j \\ -a_{ij} & i \neq j, \end{cases} \tag{4}$$

has exactly one zero eigenvalue [3]. Thus, the consensus equilibrium x_m is uniquely calculated by the left eigenvector v_l associated to that eigenvalue, $x_m := v_l^\top x(0)$. Furthermore, a strict Lyapunov function can be constructed to establish exponential stability of the origin in the space of the synchronisation errors $e := x - v_r x_m$ [17], where v_r is the right eigenvector associated to the zero eigenvalue.

In the case of a network containing nodes with interactions that can be either *cooperative*, such that $a_{ij} > 0$ for *some* i, $j \leq n$ or *competitive*, such that, $a_{ij} < 0$ for *some* $i, j \leq n$, the distributed consensus control law (2) becomes

$$u_i = -\sum_{j=1}^{n} |a_{ij}|(x_i - \text{sgn}(a_{ij})x_j), \tag{5}$$

and the elements of the associated Laplacian matrix are

$$\ell_{ij} = \begin{cases} \sum\limits_{k\in\mathcal{I}_N} |a_{ik}| & i = j \\ -a_{ij} & i \neq j. \end{cases} \tag{6}$$

Agents on a directed coopetition network with a leader or a directed spanning tree achieve *bipartite consensus*, under the distributed control law (5), if and only if the underlying graph is structurally balanced [9]. A signed graph is *structurally balanced* if it may be split into two disjoint sets of vertices $\mathcal{V}_1$ and $\mathcal{V}_2$, where $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}, \mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ such that for every $i, j \in \mathcal{V}_p, p \in \{1, 2\}$ if $a_{ij} \geq 0$ while for every $i \in \mathcal{V}_p, j \in \mathcal{V}_q$, with $p, q \in \{1, 2\}, p \neq q$ if $a_{ij} \leq 0$. It is *structurally unbalanced*, otherwise. The Laplacian matrix of a structurally balanced graph has a unique zero eigenvalue [9] and the associated right eigenvector has all entries equal to ± 1, thereby agents converge to the same state in modulus but different in signs. In such networks, moreover, several antagonistic agents, described as *competitive leaders*, that inject disinformation into the network may appear. In this particular case, agents can no longer achieve bipartite consensus. Instead, they can achieve *bipartite containment* [13], where followers converge to the convex hull spanned by all cooperative leaders' trajectories and competitive leaders' symmetric trajectories. More precisely, the achievable objective is *bipartite containment tracking*, that is,

$$\lim_{t\to\infty}[|x_j(t)| - \max_{i\in\mathcal{L}}|x_i(t)|] \leq 0, \quad j \in \mathcal{F}, \tag{7}$$

which is the problem solved, e.g., in [13]– [15], where $\mathcal{L}$ and $\mathcal{F}$ are sets of leader and follower nodes respectively.

In this paper we analyse the behavior of the networked systems (1) in closed loop with the distributed control law (5) and under the assumption that multiple leaders (cooperative and competitive) interfere. Beyond the inequality in (7), commonly found in the literature—cf. [13], we give the explicit limit values of the followers' states depending only on the initial conditions of the leaders. For clarity we stress that we consider a leader node to be one that has no incoming edges and we assume them to be static. To that end, we pose the following

Standing Assumption:

1) the signed graph is structurally balanced;
2) there are m leaders such that $1 \leq m \leq n$;
3) given each follower ν_j, for all $j \in \mathcal{F}$ with $\mathcal{F} := \{m+1, m+2, \ldots, n\}$, there exists at least one leader ν_i, for all $i \in \mathcal{L}$ with $\mathcal{L} := \{1, 2, \ldots, m\}$, such that there exists at least one path from ν_i to ν_j —cf. [13, Condition 1].

In the case of a network containing only one leader, the Standing Assumption boils down to the necessary condition for consensus that requires the existence of a directed spanning tree. As the networks considered here contain, *a priori*, more than one leader, the resulting Laplacian matrix has as many zero eigenvalues and associated eigenvectors as the number of leaders [18]. This also results in multiple convergence points for the agents. Therefore, in contrast to the consensus equilibrium $x_m = v_l^\top x(0)$ for networks with one leader, the final states of the agents, for multi-leader networks, are determined by all eigenvectors associated to the zero eigenvalues. One of this paper's contributions is to show that under the control law (5) and the Standing Assumption, the limit-values of the agents are given by

$$x_m := \mathbb{V}x, \tag{8}$$

in which $\mathbb{V}$ is a matrix determined by all the eigenvectors associated to the m zero eigenvalues of the Laplacian matrix. More precisely, the matrix $\mathbb{V}$ is given by

$$\mathbb{V} := \sum_{i=1}^{m} \begin{bmatrix} v_{r_{i,1}} \\ v_{r_{i,2}} \\ \vdots \\ v_{r_{i,n}} \end{bmatrix} \begin{bmatrix} v_{l_{i,1}} & v_{l_{i,2}} & \ldots & v_{l_{i,n}} \end{bmatrix}, \tag{9}$$

where for each $j \in \mathcal{I}_{\mathcal{N}}$, $v_{r_{i,j}}$ and $v_{l_{i,j}}$ denote, respectively, the jth element of the ith right and left eigenvector of the Laplacian matrix corresponding to the ith 0 eigenvalue. We will demonstrate further below, the properties of the terms of the right and left eigenvectors. Considering these, we establish bipartite containment of the system and, more significantly, that $x \to x_m$ exponentially.

III. Analysis Approach

Our main results are based on original technical statements for networks having an associated Laplacian matrix with multiple null eigenvalues. These are of two kinds. First, we follow the framework brought in [16], and we show how to construct the matrix $\mathbb{V}$ in (8), which defines the average states of the agents. Then, we extend the method of [17] on constructing strict Lyapunov functions for linear systems with a simple zero eigenvalue, to the case of multiple zero eigenvalues. To that end, we recall the following definition from [18] to introduce some useful sets of vertices in a graph.

Definition 1: A reachable set $\mathcal{R}_j$ is the set containing vertex j and all vertices i belonging to the directed path from j to i. A set $\mathcal{R}$ of vertices in a graph is called a *reach* if it is a maximal reachable set that consists in a leader and its followers. For each reach $\mathcal{R}_i$ of a graph, we define the *exclusive part* of $\mathcal{R}_i$ to be the set $\mathcal{H}_i = \mathcal{R}_i \backslash \bigcup_{j \neq i} \mathcal{R}_j$, that is, the set of followers influenced only by the leader i, and the *common part* of $\mathcal{R}_i$ to be the set $\mathcal{C}_i = \mathcal{R}_i \backslash \mathcal{H}_i$, that is, the set of followers influenced by other leaders than the ith one.

The following statement, which is an original contribution of this paper, extends Corollary 4.2 of [18] to the case of structurally balanced signed networks and leads to the construction of the matrix $\mathbb{V}$.

Lemma 1: Let $\mathcal{G}$ denote a structurally balanced directed signed graph, and let L denote the associated Laplacian matrix. Suppose G has n vertices and m leaders. Then the algebraic and geometric multiplicity of the eigenvalue 0 is equal to m. Furthermore, the associated eigenspace in $\mathbb{R}^n$ has as basis $\{v_{r_1}, v_{r_2}, \ldots, v_{r_m}\}$, where

1) $v_{r_{i,j}} = 0$ for $j \notin \mathcal{R}_i$,
2) $v_{r_{i,j}} = \begin{cases} 1, & \text{if } (\nu_j, \nu_i) \in \mathcal{V}_1 \\ -1, & \text{if } \nu_j \in \mathcal{V}_1, \nu_i \in \mathcal{V}_2 \end{cases}$ for $j \in \mathcal{H}_i$,
3) $v_{r_{i,j}} \in \begin{cases} (0,1), & \text{if } (\nu_j, \nu_i) \in \mathcal{V}_1 \\ (-1,0), & \text{if } \nu_j \in \mathcal{V}_1, \nu_i \in \mathcal{V}_2 \end{cases}$ for $j \in \mathcal{C}_i$,
4) $\sum_j |v_{r_i}| = \mathbf{1}_n$,

$\mathcal{V}_1$ and $\mathcal{V}_2$ are the two disjoint sets of vertices, $i \in \mathcal{I}_{\mathcal{M}} := \{1, 2, \ldots, m\}$, $j \in \mathcal{I}_{\mathcal{N}}$, and $v_{r_{i,j}}$ denotes the jth element of v_{r_i}.

Sketch of Proof: From Definition 1 and under the Standing Assumption, the number of leaders is equal to the number of reaches. The statement follows by applying a *gauge transformation*, which consists in a change of coordinates performed by the matrix $D = diag(\sigma)$, where $\sigma = [\sigma_1, ..., \sigma_n], \sigma_j \in \{1, -1\}, j \in \mathcal{I}_{\mathcal{N}}$ [9], as the considered signed network is structurally balanced, to transform it into an unsigned graph and following along the lines of the proof in [18, Corollary 4.2]. ■

As the m leaders have no incoming edges, the Laplacian matrix has all entries equal to 0 for its first m rows. Then, we obtain the following form for the m left eigenvectors associated to the zero eigenvalues:

$$v_{l_{i,j}} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad \forall i \in \mathcal{I}_{\mathcal{M}}, \forall j \in \mathcal{I}_{\mathcal{N}}. \tag{10}$$

Hence, we can split $\mathbb{V}$ in four block as follows:

$$\mathbb{V} = \sum_{i=1}^{m} v_{r_i} v_{l_i}^\top = \left[\begin{array}{c|c} V_l & V_{0_1} \\ \hline V_f & V_{0_2} \end{array}\right], \tag{11}$$

where $V_l \in \mathbb{R}^{m \times m}$ represents the leaders' interactions, $V_f \in \mathbb{R}^{(n-m) \times m}$ represents the leader-follower interactions and $V_{0_1} \in \mathbb{R}^{m \times (n-m)}$ and $V_{0_2} \in \mathbb{R}^{(n-m) \times (n-m)}$ are null. More precisely, from (10), we have

$$V_l = \begin{bmatrix} v_{r_{1,1}} v_{l_{1,1}} & & \\ & \ddots & \\ & & v_{r_{m,m}} v_{l_{m,m}} \end{bmatrix} = I_{m \times m}, \tag{12a}$$

$$V_f = \begin{bmatrix} v_{r_{1,m+1}} & \cdots & v_{r_{m,m+1}} \\ \vdots & \vdots & \vdots \\ v_{r_{1,n}} & \cdots & v_{r_{m,n}} \end{bmatrix}, \tag{12b}$$

so $\mathbb{V}$ has the following particular form

$$\mathbb{V} = \left[\begin{array}{c|c} I_{m \times m} & 0_{m \times (n-m)} \\ \hline V_f & 0_{(n-m) \times (n-m)} \end{array}\right]. \tag{13}$$

Notice that in view of (12b) the elements of V_f have the same properties as the basis defined in Lemma 1. This is significant because V_f is the matrix that defines the limit points of the followers as $x \to x_m$, where x_m is defined by (8). Moreover, the followers' states may be influenced by other followers' states during their trajectories depending on the network's topology, but it follows from (8) that the final states of the followers are defined only by the leaders' states.

Now, similarly to the case of networks with one leader, where the error is defined as $e := x - v_r x_m$, with $x_m := v_l^\top x$, for multi-leader coopetition networks, we define the consensus errors as

$$e := [I - \mathbb{V}]x. \tag{14}$$

Then, to establish that $x \to x_m$ and, consequently, the bipartite containment objective defined by (7), we will prove the stronger property of global exponential stability of the set $\{e = 0\}$. For that, we shall show how to construct strict—in the space of e— Lyapunov functions, based on the following proposition, which is another original contribution of this paper and extends Proposition 1 of [17] to the case of signed networks with multiple leaders.

Proposition 1: Let $\mathcal{G}$ be a structurally balanced directed signed network containing multiple leaders. Then, the following are equivalent:

(i) the graph has m leaders, and given each follower $\nu_j, \forall j \in \mathcal{F}$, there exists at least one leader $\nu_i, \forall i \in \mathcal{L}$ such that there exists at least one path from ν_i to ν_j,

(ii) for any $Q \in \mathbb{R}^{N \times N}, Q = Q^\top > 0$ and for any $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ with $\alpha_i > 0$, there exists a matrix $P \in \mathbb{R}^{N \times N}$, $P = P^\top > 0$ such that

$$PL + L^\top P = Q - \sum_{i=1}^{m} \alpha_i (P v_{ri} v_{li}^\top + v_{li} v_{ri}^\top P), \tag{15}$$

where $v_{ri}, v_{li} \in \mathbb{R}$ are the right and left eigenvectors of L associated with the ith 0 eigenvalue.

Proof: (i) $\Rightarrow$ (ii): By assumption, the graph $\mathcal{G}$ has m leaders. Then, from Lemma 1, it follows that L has m zero eigenvalues: $0 = \lambda_1 = \cdots = \lambda_m < \Re e(\lambda_{m+1}) \leq \cdots \leq \Re e(\lambda_n)$. Following the lines the proof of Lemma 2 of [17], we write the Jordan decomposition of L as $L = U\Lambda U^{-1} = \sum_{i=1}^{m} \lambda_i(L) v_{ri} v_{li}^\top + U_1 \Lambda_1 U_1^\dagger$ with $\Lambda_1 \in \mathbb{C}^{n-m \times n-m}$, $U = \begin{bmatrix} v_{r_1} & \dots & v_{r_m} & U_1 \end{bmatrix} \in \mathbb{C}^{n\times n}$, and $U^{-1} = \begin{bmatrix} v_{l_1}^\top & \dots & v_{l_m}^\top & U_1^\dagger \end{bmatrix}^\top \in \mathbb{C}^{n\times n}$. For any $\alpha_i > 0$ define $R(\alpha_i) = L + \sum_{i=1}^{m} \alpha_i v_{ri} v_{li}^\top$. From this decomposition and the properties of Λ_1, $\Re e\{\lambda_j(R)\} > 0$ for all $j \leq n$. $-R(\alpha_i)$ is Hurwitz, therefore for any $Q = Q^\top > 0$ and $\alpha_i > 0, i \leq m$, there exists $P = P^\top > 0$ such that

$$- P(L + \sum_{i=1}^{m} \alpha_i v_{ri} v_{li}^\top) - (L + \sum_{i=1}^{m} \alpha_i v_{ri} v_{li}^\top)^\top P = -Q.$$

Then, we obtain the equation in (15).

(ii) $\Rightarrow$ (i): Let statement (ii) hold and assume that the Laplacian matrix has $m+1$ zero eigenvalues. In view of Lemma 1, the assumption that the system has m leaders does not hold. Now, the Jordan decomposition of L has the form $L = \sum_{i=1}^{m+1} \lambda_i(L) v_{ri} v_{li}^\top + U_1 \Lambda_1 U_1^\dagger$ with $U = \begin{bmatrix} v_{r_1} & \dots & v_{r_{m+1}} & U_1 \end{bmatrix}$ and $U^{-1} = \begin{bmatrix} v_{l_1}^\top & \dots & v_{l_{m+1}}^\top & U_1^\dagger \end{bmatrix}^\top$. Next let us consider $R(\alpha_i) = L + \sum_{i=1}^{m} \alpha_i v_{ri} v_{li}^\top$ which admits the Jordan decomposition $R := U \Lambda_R U^{-1}$, where

$$\Lambda_R := \begin{bmatrix} \alpha_1 & & & & \\ & \ddots & & & \\ & & \alpha_m & & \\ & & & 0 & \\ & & & & \Lambda_1 \end{bmatrix}.$$

Clearly, R is not positive definite beause one of its eigenvalues is equal to zero. Then, there exists a matrix $Q = Q^\top$ for which there does not exist a matrix $P = P^\top$ such that $-PR - R^\top P = -Q$, which contradicts statement (ii). ∎

IV. Main Results

In this section, we will present our main results on first-order systems and we will establish robustness of the bipartite containment tracking in the sense of ISS, with respect to external bounded perturbations.

A. Exponential Stability

Consider the system (1), interconnected with the bipartite containment control law (5). We analyse the dynamics of the errors (14). Differentiating the latter on both sides, to obtain

$$\dot{e} = [I - \mathbb{V}]\dot{x} \tag{16}$$

and using (1) and (5), we obtain the closed-loop dynamical equations

$$\dot{e} = -Le. \tag{17}$$

The bipartite containment problem is now recast as a problem of stability analysis of the dynamical system (17). Thus, relying on Proposition 1, our next statement provides sufficient conditions to achieve global exponential stability of the set $\{e = 0\}$, which is equivalent to the bipartite containment tracking objective (7).

Proposition 2: Consider the system (1) with the bipartite containment control law (5). Under the Standing Assumption, for any $Q = Q^\top > 0$ there exists $P = P^\top > 0$ such that

$$V(e) = e^\top P e, \quad \dot{V}(e) = -e^\top Q e. \tag{18}$$

Then, the consensus set $\{e = 0\}$ is exponentially stable for all initial states $x(0) \in \mathbb{R}^n$.

Proof: Let $Q = Q^\top > 0$ and $\alpha > 0$ be arbitrarily fixed. Since by the Standing Assumption, by Proposition 1, $\exists P = P^\top > 0$ such that (15) holds. Then, consider the Lyapunov function candidate $V(e) := e^\top P e$. The total time derivative of V along the trajectories yields

$$\dot{V}(e) = -e^\top Q e + e^\top \sum_{i=1}^{m} \alpha_i (P v_{ri} v_{li}^\top + v_{li} v_{ri}^\top P) e.$$

On the other hand, replacing (14) we obtain

$$\begin{aligned} \sum_{i=1}^{m} \alpha_i P v_{ri} v_{li}^\top e &= \sum_{i=1}^{m} \alpha_i P v_{ri} v_{li}^\top [I - \sum_{i=1}^{m} v_{ri} v_{li}^\top] x \\ &= \sum_{i=1}^{m} \alpha_i (P v_{ri} v_{li}^\top - P v_{ri} v_{li}^\top) x = 0 \end{aligned}$$

for which we used the identity $v_{li}^\top v_{ri} = 1, i \leq m$. Similarly, we obtain $e^\top \sum_{i=1}^{m} \alpha_i v_{li} v_{ri}^\top P = 0$. In consequence,

$$\dot{V}(e) = -e^\top Q e \leq -q_m |e|^2, \tag{19}$$

where $q_m > 0$ is the smallest eigenvalue of Q, so the statement of the proposition follows. ∎

The following statement provides explicit expressions for the limit values of the followers' states.

Proposition 3: Consider the system (1) with the bipartite containment control law (5). Under the Standing Assumption, the bipartite containment objective is achieved, that is the inequality (7) holds. Furthermore, if the leaders are static (i.e., $\dot{x}_l = 0$), the final states of the followers satisfy

$$\lim_{t\to\infty} x_f(t) = V_f x_l, \tag{20}$$

where x_l and x_f are the leaders' and the followers' states respectively and V_f is given in (12b).

Proof: Differentiating the weighted average of the system (8), we obtain the dynamical equation below

$$\dot{x}_m = \mathbb{V}\dot{x} = -\mathbb{V}Lx = 0, \tag{21}$$

with $v_{l_i}^\top L = 0$ for each $i \leq m$. Its solution gives $x_m(t) = x_m(0)$. From Proposition 2, we have $\lim_{t\to\infty} e(t) = 0$, which gives $\lim_{t\to\infty} x(t) = x_m(t) = x_m(0)$. Then, using (13), we obtain the relation in (20). Under the Standing Assumption and from Item 4 of Lemma 1, we have

$$\begin{aligned} \lim_{t\to\infty} |x_{f_j}(t)| = |\sum_{i=1}^{m} v_{r_{i,m+j}} x_{l_i}| &\leq \sum_{i=1}^{m} |v_{r_{i,m+j}}| \max_{1\leq i\leq m} |x_{l_i}| \\ &\leq \max_{1\leq i\leq m} |x_{l_i}| \end{aligned}$$

Then, the bipartite containment objective in (7) is achieved. ∎

B. Robustness Analysis

Consider the perturbed first-order systems

$$\dot{x}_i = u_i + d_i(t), \tag{22}$$

where the disturbances $d_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ are assumed to be locally integrable functions. Under the control law (5), the system (22) becomes

$$\dot{x} = -Lx + d(t). \tag{23}$$

Differentiating the errors in (16) on both sides, and using (23) we obtain

$$\dot{e} = -Le + [I - \mathbb{V}]d(t). \tag{24}$$

Then, we have the following.

Proposition 4: The closed-loop system (24), under the Standing Assumption, is ISS with respect to an essentially bounded, locally integrable external disturbance.

Proof: Consider the Lyapunov function candidate in (18). Its derivative gives

$$\dot{V}(e) = \frac{\partial V}{\partial e}(-Le) + \frac{\partial V}{\partial e}[I - \mathbb{V}]d.$$

From (19), we have

$$\begin{aligned}\dot{V}(e) &\leq -e^\top Qe + \frac{\partial V}{\partial e}[I - \mathbb{V}]d \\ &\leq -q_m|e|^2 + 2\overline{\lambda}_P|e||[I - \mathbb{V}]||d|.\end{aligned}$$

We know that $0 \leq |[I - \mathbb{V}]| \leq |I| + |\mathbb{V}| \leq 2$, because all eigenvalues of I are equal to 1 and all eigenvalues of $|\mathbb{V}|$ are either 1 or 0. Let $\delta > 0$ be such that $c := q_m - \frac{2\overline{\lambda}_P}{\delta} > 0$. Then,

$$\dot{V}(e) \leq -c|e|^2 + 2\delta|d|^2.$$

The statement follows. ■

V. Simulation Results

To illustrate our theoretical findings we present a numerical example on a system of multi-wheeled mobile robots modeled as unicycles. Let $\begin{bmatrix} r_{x_i} & r_{y_i} \end{bmatrix}^\top \in \mathbb{R}^2$ be the position of the center of the ith robot, $\theta_i \in \mathbb{R}$ the orientation of the ith robot, and $v_i \in \mathbb{R}$ and $\omega_i \in \mathbb{R}$ the linear and angular velocities of the ith robot. Then, the dynamics of the wheeled mobile robots can be modeled as [19]

$$\dot{r}_{x_i} = v_i \cos(\theta_i), \quad \dot{r}_{y_i} = v_i \sin(\theta_i), \quad \dot{\theta}_i = \omega_i. \tag{25}$$

To apply the consensus control law (5)—designed for (1)—on this system we apply a preliminary feedback linearizing control. To that end, we rewrite the system's dynamics in terms of the position of a point located at a distance δ off the axis joining the wheels. That is, the point $p_i = \begin{bmatrix} p_{x_i} & p_{y_i} \end{bmatrix}^\top$, where $p_{x_i} = r_{x_i} + \delta_i \cos(\theta_i)$ and $p_{y_i} = r_{y_i} + \delta_i \sin(\theta_i)$. For the purpose of simulation, we use $\delta_i = 0.1$m. Differentiating p_i with respect to time and by letting

$$\begin{bmatrix} v_i \\ \omega_i \end{bmatrix} = \begin{bmatrix} \cos(\theta_i) & \sin(\theta_i) \\ -\frac{1}{\delta_i}\sin(\theta_i) & \frac{1}{\delta_i}\cos(\theta_i) \end{bmatrix} \begin{bmatrix} u_{x_i} \\ u_{y_i} \end{bmatrix}, \tag{26}$$

we get $\begin{bmatrix} \dot{p}_{x_i} & \dot{p}_{y_i} \end{bmatrix}^\top = \begin{bmatrix} u_{x_i} & u_{y_i} \end{bmatrix}^\top$, which is a simplified kinematic equation in the form of first-order dynamics. For the simulations examples, we implemented (26) with u_i as in (5), where $x_i = [p_{x_i}\ p_{y_i}]$.

We consider a coopetition network containing three leaders $x_i, i \leq 3$ and four followers $x_j, 4 \leq j \leq 7$, communicating over a directed graph as the one depicted in Figure 1. The competitive leader x_3 represents an obstacle in the system.

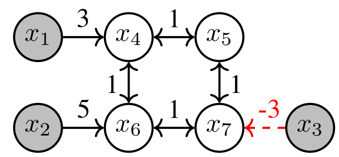

Fig. 1. A network of seven mobile robots with 2 cooperative and 1 competitive leaders.

According to (6) , the Laplacian matrix corresponding to the graph is

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 5 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 & -1 \\ 0 & -5 & 0 & -1 & 0 & 7 & -1 \\ 0 & 0 & 3 & 0 & -1 & -1 & 5 \end{bmatrix}$$

and its eigenvalues are $\lambda_L = \{0, 0, 0, 1.38, 4.80, 7.81, 5\}$. The network may be bipartitioned into two subgroups as $\mathcal{V}_1 = \{x_1, x_2, x_4, x_5, x_6, x_7\}$, $\mathcal{V}_2 = \{x_3\}$ so is structurally balanced. The matrix V_f in (12b) is calculated as below.

$$V_f = \begin{bmatrix} 0.7038 & 0.1923 & -0.1038 \\ 0.4038 & 0.1923 & -0.4038 \\ 0.1154 & 0.7692 & -0.1154 \\ 0.1038 & 0.1923 & -0.7038 \end{bmatrix}.$$

We notice that V_f has the properties stated on Items 1–4 of Lemma 1. Since each follower is influenced by the three leaders, 0 is not an element of V_f (Item 1). Moreover, none of the followers corresponds to the exclusive part of a reach, so V_f does not have an element equal to ± 1 (Item 2). From the structural-balance property, all elements corresponding to leaders x_1 and x_2 (the first two columns) are positive and less than one, whereas the elements corresponding to leader x_3 (on the last column) are negative and greater than -1 (Item 3). We also remark that the sum of the absolute value of the terms on each row is equal to 1 (Item 4).

Let P be generated by (15) with $Q = I_N$ and $\alpha = 20$, then we obtain $\overline{\lambda}_P = 0.6247$. Consider the system (25) and the bipartite containment law (5). The respective initial states of the robots are $r_x(0) = [3.5, 4, -2, -6.5, 5.5, -3.5, 6]^\top$, $r_y(0) = [2, 3.5, -3, -1, -3, -3, -2.5]^\top$, $\theta_i(0) = \frac{\pi}{2}$ for all $i \in \mathcal{I}_N$. Figure 2 depicts the simulation results. The followers converge to the convex hull spanned by cooperative leaders' states and competitive leader's x_3 symmetric state. Using (20) and the coordinate transformation, we obtain the following limit values for the followers' states: $\lim_{t\to\infty} r_{x_f}(t) = \begin{bmatrix} 3.44 & 2.99 & 3.69 & 2.54 \end{bmatrix}^\top$ and $\lim_{t\to\infty} r_{y_f}(t) = \begin{bmatrix} 2.37 & 2.61 & 3.25 & 2.85 \end{bmatrix}^\top$.

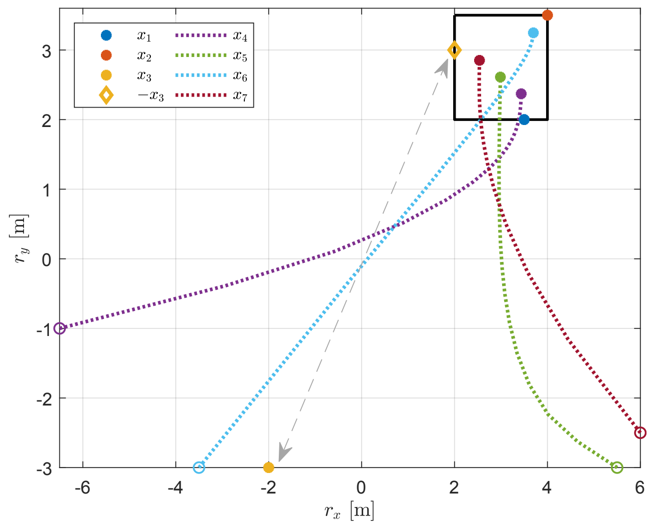

Fig. 2. Bipartite containment tracking of system (25) with control input (26), $[u_{x_i}\ u_{y_i}] =: u_i$ and u_i as in (5). The filled dots are the final states of the mobile robots and the dotted lines represent the trajectory of the four followers. The yellow diamond represents the symmetric state of the antagonistic leader x_3.

We now perform simulations for the system (25) with the bipartite containment law (5), with $d_i(t) = \sigma_i(t)\begin{bmatrix}1 & 1\end{bmatrix}^\top$ where $\sigma_i(t)$ is given as below

$$\sigma_i(t) = \begin{cases} \tanh(t-10) - 1 + \frac{1}{(t+10)} & i \in \{5,6\} \\ -\tanh(t-10) + 1 - \frac{1}{(t+10)} & i = 4 \\ 0 & i \in \{1,2,3,7\}. \end{cases} \tag{27}$$

Figure 3 depicts the simulation results. During the first $10s$, the perturbation $d(t)$ prevents the achievement of bipartite containment tracking and the followers reach a stable state with a steady-state error. However, as the perturbation vahishes, after $10s$, the trajectories of the followers move towards the convex hull, spanned by cooperative leaders' states and antagonistic leader's symmetric state. We obtain the same limit values as before for the followers.

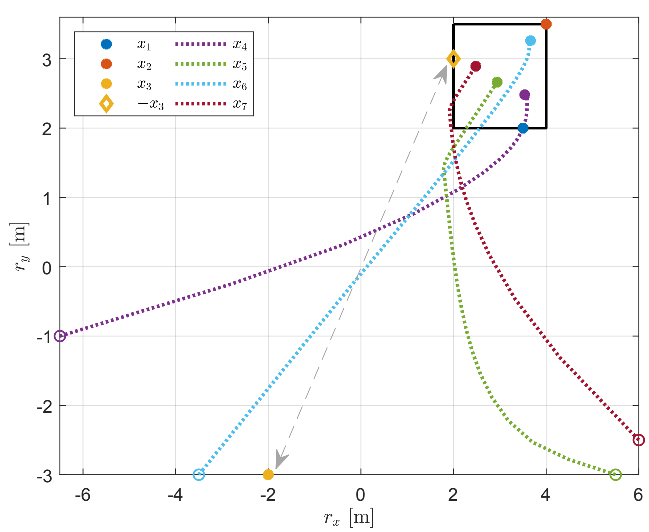

Fig. 3. Bipartite containment tracking of system (25) under the same conditions as in Figure 2 and under the effect of the perturbation in (27).

VI. Conclusions

We presented a Lyapunov approach to analyse the exponential stability of the bipartite containment tracking problem of simple-integrators over structurally balanced multi-leader coopetition networks. Via a change of coordinates, we have shown a bound for the convergence of the followers. Moreover, we have generalised the Lyapunov equation characterisation of the Hurwitz property of a matrix to matrices having more than one zero eigenvalue, which allowed us to construct strict Lyapunov functions. Disposing of strict Lyapunov functions allowed us to establish the robustness of the system with a bounded disturbance. Further research is focused on extending these results to more general classes of dynamical systems and industrial deployment of multi-robot systems.

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