16th European Turbulence Conference
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Fluid/elastic stationary modes of a flexible plate clamped on a cylinder

Jean-Lou PFISTER[1], Marco CARINI[2], Olivier MARQUET[2]

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[2] ONERA (Meudon, France)

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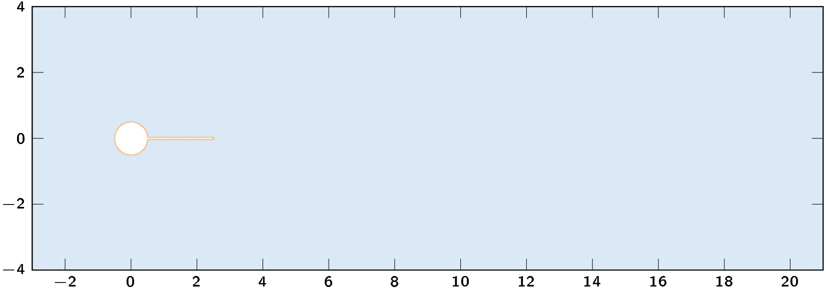
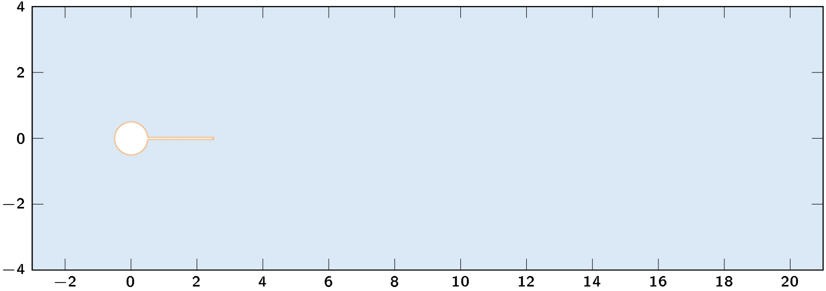
Problem statement



Direct non-linear fluid/elastic simulation : streamwise velocity and plate deformation

Problem statement



Direct non-linear fluid/elastic simulation : streamwise velocity and plate deformation

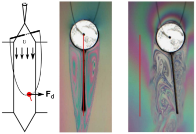
Spontaneous plate deviation → non-zero lift force.

- How to predict the instabilitiy onset ?
- What are the underlying mechanisms ?

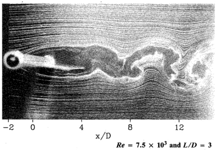
Literature review – spontaneous deviations in fluid/solid systems

Motivation : flow control, bio-inspired systems...

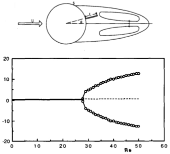
Low-Reynods flow, freely rotatable plate [1]



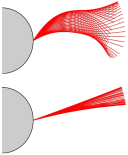
Higher-Reynods flow, freely rotatable plate [2]



Rigid rotatable plate (numeric study) [3]



Flexible attached plate (numeric study) [4]



1. Lacis, et *al*, Nature, 2014
2. Cimbala, J. and Garg, S., AIAA, 1991
3. Xu, J.C. et *al.*, Phys. Fluids, 1990
4. Bagheri, S. et *al.*, Phys. Rev. Lett., 2012

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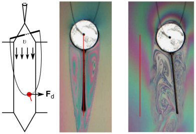

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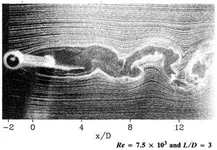

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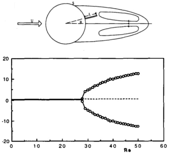

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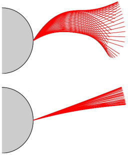

- Here : approach based on linear stability analysis.

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Fluid-solid interaction model problem [5]

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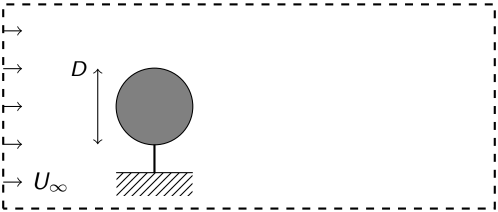

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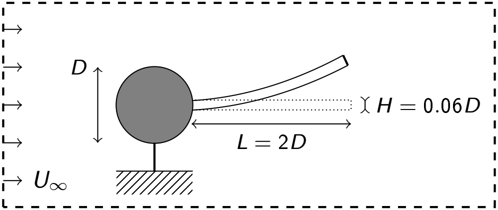
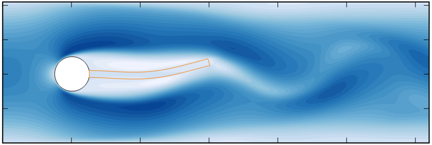


- Parameters :
  - $\rho_{\text{solid}}/\rho_{\text{fluid}} = 1$,
  - Poisson coefficient $\nu_s = 0.4$, variable **rigidity** coefficient $\mathcal{K}_{\text{B}}$,
  - Variable **Reynolds number** $\mathcal{R}e = U_\infty D/\nu$.
- Physics modeling :
  - Incompressible laminar flow,
  - Non-linear elastic solid (Saint-Venant Kirchhoff).

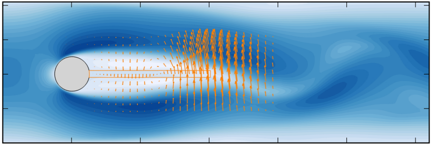
5. Lee, J. and You, D., Phys. Fluids, 2013

How to handle the fluid-structure coupling

To handle the moving domain : Arbitrary Lagrangian Eulerian formulation [6]



actual domain Ω_t (time-dependant)



reference domain Ω_0 (fixed)

6. Donea, J. et al. Enc. Comp. Mech., 2004

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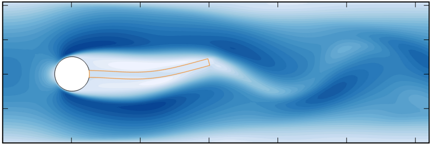

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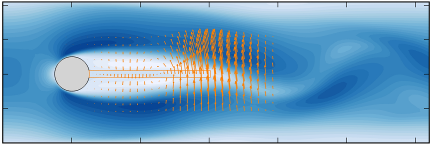

reference domain Ω_0 (fixed)

The coupled problem is then written as

$$\mathcal{B}\frac{\partial \mathbf{q}}{\partial t} - \mathcal{R}(\mathbf{q}) = 0 \quad \text{in a } \textbf{fixed} \text{ domain } \Omega_0$$

$$\mathbf{q} = \begin{cases} \text{velocity (solid/fluid)} \\ \text{displacement (solid/fluid)} \\ \text{pressure (fluid)} \end{cases} \qquad \mathcal{B},\mathcal{R} = \begin{cases} \text{Navier-Stokes equations (fluid)} \\ \text{Elasticity equations (solid)} \\ \text{Stress/velocity continuity (interface)} \\ \text{Fluid mesh equation} \end{cases}$$

6. Donea, J. et al. Enc. Comp. Mech., 2004

Steady symmetric solution

Case $\mathcal{R}\mathrm{e} = 80$, $\mathcal{K}_\mathrm{B} = 0.015$ (same as in the video). Let us solve [7]

$$\mathcal{R}(\mathbf{q}_b) = 0$$

7. Newton method implemented in the finite element software `FreeFem++` (www.freefem.org)

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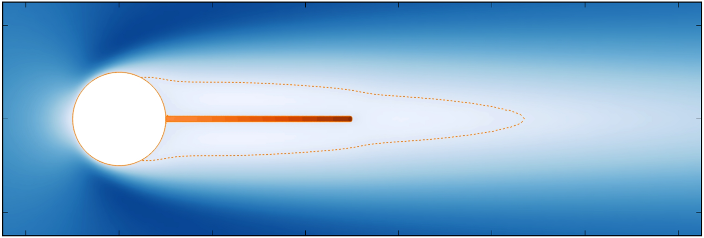
Fluid velocity :



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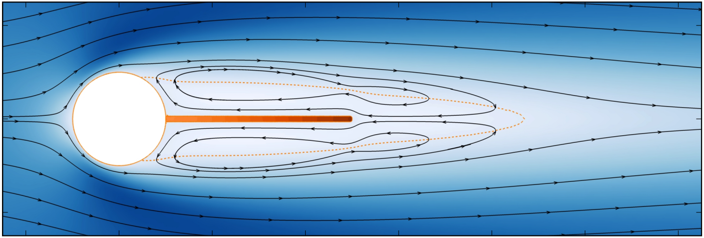

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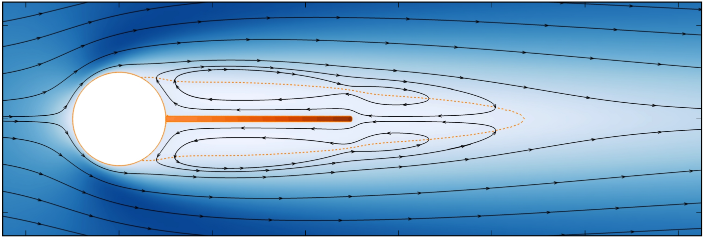
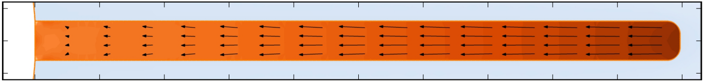


Solid displ. :



- Small compression ($\simeq 0.01\%$ of the length) in the solid due to viscous shear

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Linear stability problem

Non-linear problem

$$\mathcal{B}\frac{\partial \mathbf{q}}{\partial t}(\mathbf{x}) - \mathcal{R}(\mathbf{q})(\mathbf{x}) = 0 \quad \mathbf{x} \in \Omega_0$$

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Long-term asymptotic stability

$$\mathbf{q}(\mathbf{x}, t) = \mathbf{q}_b(\mathbf{x}) + \varepsilon\, \mathbf{q}'(\mathbf{x}, t) \qquad \varepsilon \ll 1$$

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$$\mathbf{q}(\mathbf{x}, t) = \mathbf{q}_b(\mathbf{x}) + \varepsilon\,\hat{\mathbf{q}}(\mathbf{x})\exp(\sigma t) \qquad \varepsilon \ll 1$$

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Long-term asymptotic stability

$$\mathbf{q}(\mathbf{x}, t) = \mathbf{q}_b(\mathbf{x}) + \varepsilon\, \hat{\mathbf{q}}(\mathbf{x}) \exp(\sigma t) \qquad \varepsilon \ll 1$$

Linear stability problem $\rightarrow$ look for *unstables* modes $(\mathrm{Re}(\sigma > 0), \hat{\mathbf{q}})$

$$\sigma\, \mathcal{B}\, \hat{\mathbf{q}} - \left.\frac{\partial \mathcal{R}}{\partial \mathbf{q}}\right|_{\mathbf{q}_b} \hat{\mathbf{q}} = 0$$

Eigenvalue problem

Monolithic fluid-solid coupling :

- $\hat{\mathbf{q}}_\mathsf{f}$: fluid velocity/pressure,
- $\hat{\mathbf{q}}_\mathsf{s}^a$: displacement & displacement velocity

Full coupled problem :

$$\sigma \begin{bmatrix} \mathcal{B}_\mathsf{ff} & 0 \\ 0 & \mathcal{B}_\mathsf{ss}^a \end{bmatrix} \begin{bmatrix} \hat{\mathbf{q}}_\mathsf{f} \\ \hat{\mathbf{q}}_\mathsf{s}^a \end{bmatrix} - \begin{bmatrix} \mathcal{A}_\mathsf{ff} & \mathcal{A}_\mathsf{fs} \\ \mathcal{A}_\mathsf{sf}^a & \mathcal{A}_\mathsf{ss}^a \end{bmatrix} \begin{bmatrix} \hat{\mathbf{q}}_\mathsf{f} \\ \hat{\mathbf{q}}_\mathsf{s}^a \end{bmatrix} = 0$$

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Fluid-only subproblem :

$$\sigma \begin{bmatrix} \mathcal{B}_{ff} & 0 \\ 0 & \mathcal{B}_{ss}^a \end{bmatrix} \begin{bmatrix} \hat{\mathbf{q}}_f \\ \hat{\mathbf{q}}_s^a \end{bmatrix} - \begin{bmatrix} \mathcal{A}_{ff} & \mathcal{A}_{fs} \\ \mathcal{A}_{sf}^a & \mathcal{A}_{ss}^a \end{bmatrix} \begin{bmatrix} \hat{\mathbf{q}}_f \\ \hat{\mathbf{q}}_s^a \end{bmatrix} = 0$$

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Solid-only subproblem (**a**ugmented → linear eigenvalue problem) :

$$\sigma \begin{bmatrix} \mathcal{B}_{ff} & 0 \\ 0 & \mathcal{B}_{ss}^a \end{bmatrix} \begin{bmatrix} \hat{\mathbf{q}}_f \\ \hat{\mathbf{q}}_s^a \end{bmatrix} - \begin{bmatrix} \mathcal{A}_{ff} & \mathcal{A}_{fs} \\ \mathcal{A}_{sf}^a & \mathcal{A}_{ss}^a \end{bmatrix} \begin{bmatrix} \hat{\mathbf{q}}_f \\ \hat{\mathbf{q}}_s^a \end{bmatrix} = 0$$

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Coupling operators :

$$\sigma \begin{bmatrix} \mathcal{B}_\mathrm{ff} & 0 \\ 0 & \mathcal{B}_\mathrm{ss}^a \end{bmatrix} \begin{bmatrix} \hat{\mathbf{q}}_\mathrm{f} \\ \hat{\mathbf{q}}_\mathrm{s}^a \end{bmatrix} - \begin{bmatrix} \mathcal{A}_\mathrm{ff} & \mathcal{A}_\mathrm{fs} \\ \mathcal{A}_\mathrm{sf}^a & \mathcal{A}_\mathrm{ss}^a \end{bmatrix} \begin{bmatrix} \hat{\mathbf{q}}_\mathrm{f} \\ \hat{\mathbf{q}}_\mathrm{s}^a \end{bmatrix} = 0$$

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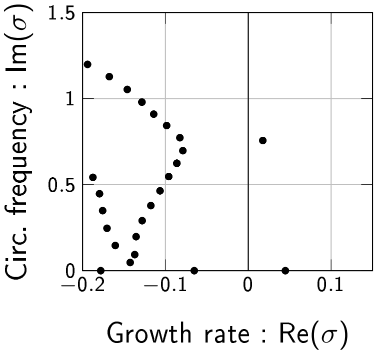
Large eigenvalue problem, solved in `FreeFem++`/ARPACK [8] with finite-elements and an exact Jacobian matrix.

8. www.caam.rice.edu/software/ARPACK/

Fluid-solid linear stability analysis of the steady solution

Spectrum ($\mathcal{R}e = 80$, $\mathcal{K}_B = 0.015$) *:*



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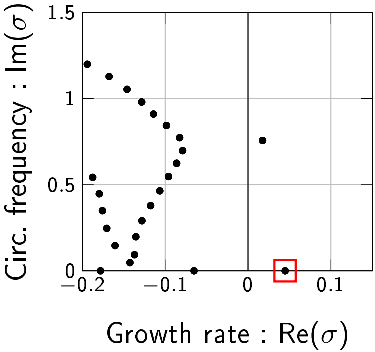
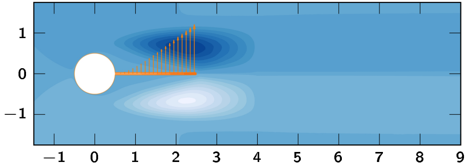
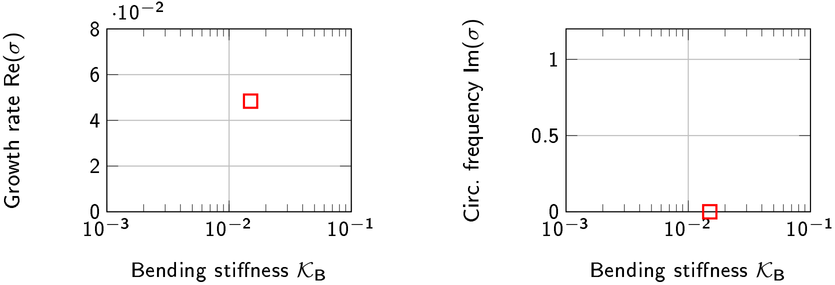


Mode (u velocity and solid displ.) :



Steady mode $\sigma = 0.044 + 0\,\mathbf{i}$

Effect of the bending stiffness $\mathcal{K}_B$ on the eigenvalue



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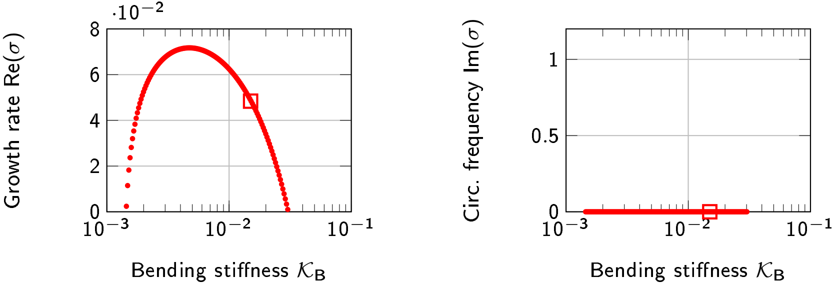

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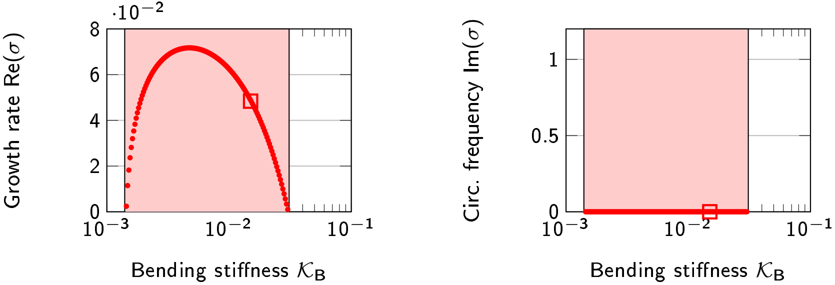

- The steady mode is unstable over a finite range of rigidities,
- In what follows we focus on this range.

An useful tool : solid modal decomposition

Consider the solid sub-problem only (linear, augmented) :

$$\sigma \begin{bmatrix} \mathcal{B}_{\mathrm{ff}} & 0 \\ 0 & \mathcal{B}^{a}_{\mathrm{ss}} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{q}}_{\mathrm{f}} \\ \hat{\mathbf{q}}^{a}_{\mathrm{s}} \end{bmatrix} - \begin{bmatrix} \mathcal{A}_{\mathrm{ff}} & \mathcal{A}_{\mathrm{fs}} \\ \mathcal{A}^{a}_{\mathrm{sf}} & \mathcal{A}^{a}_{\mathrm{ss}} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{q}}_{\mathrm{f}} \\ \hat{\mathbf{q}}^{a}_{\mathrm{s}} \end{bmatrix} = 0$$

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Consider the solid sub-problem only (linear, augmented) :

$$\mathbf{i}\omega_{s,j}\,\mathcal{B}^{a}_{\mathsf{ss}}\,\hat{\mathbf{q}}_{\mathsf{s}}^{a,(j)} - \mathcal{A}^{a}_{\mathsf{ss}}\,\hat{\mathbf{q}}_{\mathsf{s}}^{a,(j)} = 0$$

An useful tool : solid modal decomposition

Consider the solid sub-problem only (rewritten in a quadratic form) :

$$(\mathbf{i}\omega_{s,j})^2 \mathcal{B}_{\mathsf{ss}}\, \hat{\mathbf{q}}_{\mathsf{s}}^{(j)} - \mathcal{A}_{\mathsf{ss}}\, \hat{\mathbf{q}}_{\mathsf{s}}^{(j)} = 0$$

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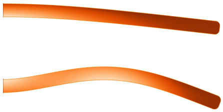
Free vibration modes $\hat{\mathbf{q}}_s^{(j)}$:



$\hat{\mathbf{q}}_s^{(1)}$

$\hat{\mathbf{q}}_s^{(2)}$

... $\hat{\mathbf{q}}_s^{(N)}$

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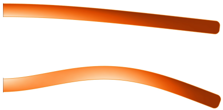


$\hat{\mathbf{q}}_s^{(1)}$

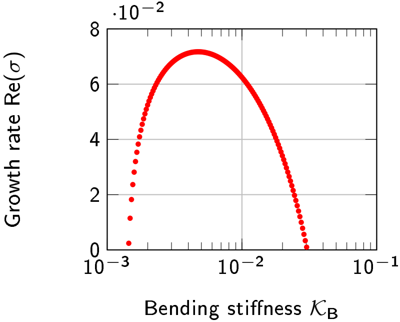
$\hat{\mathbf{q}}_s^{(2)}$

... $\hat{\mathbf{q}}_s^{(N)}$

Free solid modes are orthogonal → **projection basis** (coefficients y_j)

$$\hat{\mathbf{q}}_s = \sum_{j=1}^{N} y_j\, \hat{\mathbf{q}}_s^{(j)}$$

Effect of the bending stiffness $\mathcal{K}_B$ on the mode



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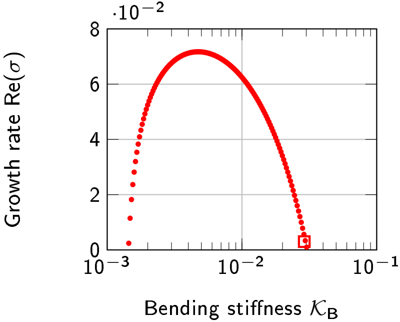
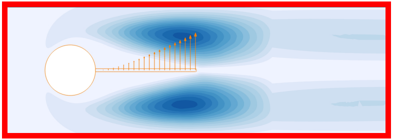


Steady mode shape, $\mathcal{R}e = 80$

$\mathcal{K}_B = 0.0290$



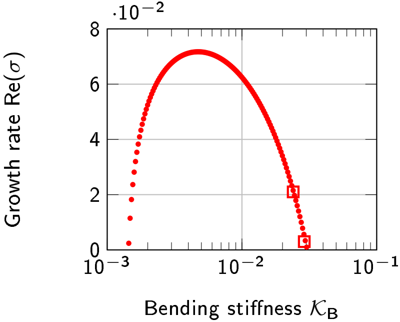
y_1

y_2

Projections :

$$y_1/(y_1 + y_2) = 97\%$$
$$y_2/(y_1 + y_2) = 3\%$$

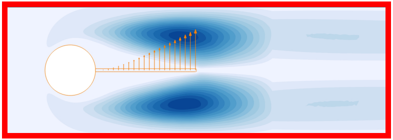
Effect of the bending stiffness $\mathcal{K}_B$ on the mode



y_1 y_2

Steady mode shape, $\mathcal{R}e = 80$

$\mathcal{K}_B = 0.0240$

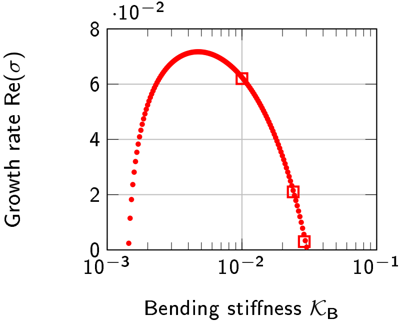


Projections :

$y_1/(y_1 + y_2) = 96\%$

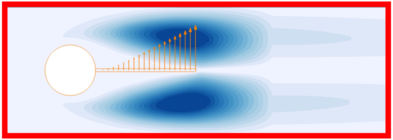
$y_2/(y_1 + y_2) = 4\%$

Effect of the bending stiffness $\mathcal{K}_B$ on the mode



Steady mode shape, $\mathcal{R}e = 80$

$\mathcal{K}_B = 0.0150$



y_1

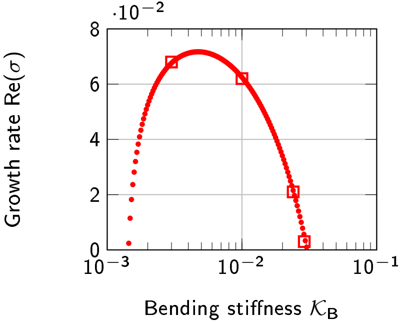
y_2

Projections :

$y_1/(y_1 + y_2) = 94\%$

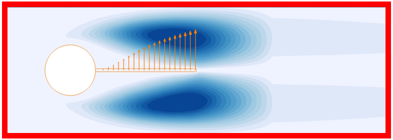
$y_2/(y_1 + y_2) = 6\%$

Effect of the bending stiffness $\mathcal{K}_B$ on the mode



Steady mode shape, $\mathcal{R}e = 80$

$\mathcal{K}_B = 0.0030$



y_1

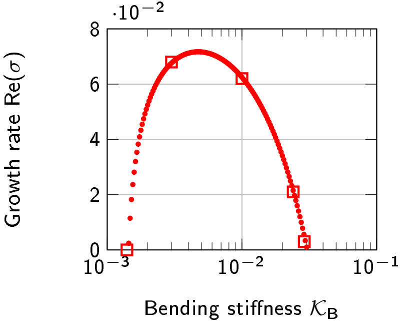
y_2

Projections :

$y_1/(y_1 + y_2) = 75\%$

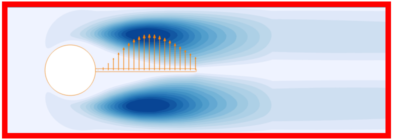
$y_2/(y_1 + y_2) = 25\%$

Effect of the bending stiffness $\mathcal{K}_B$ on the mode



Steady mode shape, $\mathcal{R}e = 80$

$\mathcal{K}_B = 0.0014$



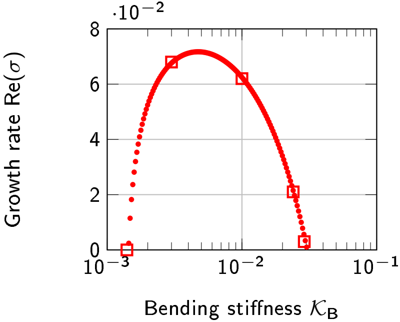
y_1

y_2

Projections :

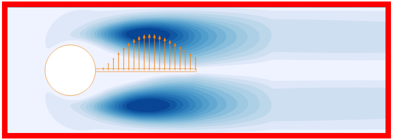
$$y_1/(y_1 + y_2) = 66\%$$
$$y_2/(y_1 + y_2) = 34\%$$

Effect of the bending stiffness $\mathcal{K}_B$ on the mode

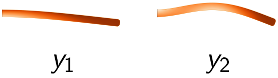


Steady mode shape, $\mathcal{R}e = 80$

$\mathcal{K}_B = 0.0014$



Projections :



$$y_1/(y_1 + y_2) = 66\%$$
$$y_2/(y_1 + y_2) = 34\%$$

- When $\mathcal{K}_B$ is decreased, in the solid the contribution of the second mode increases

Reduced model

Coupled system :

$$\sigma \begin{bmatrix} \mathcal{B}_{\mathrm{ff}} & 0 \\ 0 & \sigma \mathcal{B}_{\mathrm{ss}} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{q}}_{\mathrm{f}} \\ \hat{\mathbf{q}}_{\mathrm{s}} \end{bmatrix} - \begin{bmatrix} \mathcal{A}_{\mathrm{ff}} & \mathcal{A}_{\mathrm{fs}} \\ \mathcal{A}_{\mathrm{sf}} & \mathcal{A}_{\mathrm{ss}} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{q}}_{\mathrm{f}} \\ \hat{\mathbf{q}}_{\mathrm{s}} \end{bmatrix} = 0$$

Reduced model

Coupled system :

$$\sigma\, \mathcal{B}_{\mathrm{ff}}\, \hat{\mathbf{q}}_{\mathrm{f}} - \mathcal{A}_{\mathrm{ff}}\, \hat{\mathbf{q}}_{\mathrm{f}} - \mathcal{A}_{\mathrm{fs}}\, \hat{\mathbf{q}}_{\mathrm{s}} = 0$$
$$\sigma^2\, \mathcal{B}_{\mathrm{ss}}\, \hat{\mathbf{q}}_{\mathrm{s}} - \mathcal{A}_{\mathrm{sf}}\, \hat{\mathbf{q}}_{\mathrm{f}} - \mathcal{A}_{\mathrm{ss}}\, \hat{\mathbf{q}}_{\mathrm{s}} = 0$$

Reduced model

Coupled system :

$$\sigma^2 \, \mathcal{B}_{\mathsf{ss}} \, \hat{\mathbf{q}}_{\mathsf{s}} - \mathcal{A}_{\mathsf{ss}} \, \hat{\mathbf{q}}_{\mathsf{s}} = \mathcal{A}_{\mathsf{sf}} \left(\sigma \, \mathcal{B}_{\mathsf{ff}} - \mathcal{A}_{\mathsf{ff}} \right)^{-1} \mathcal{A}_{\mathsf{fs}} \, \hat{\mathbf{q}}_{\mathsf{s}} \qquad (a)$$

Reduced model

Coupled system :

$$\underbrace{\sigma^2 \mathcal{B}_{\mathrm{ss}} \hat{\mathbf{q}}_{\mathrm{s}} - \mathcal{A}_{\mathrm{ss}} \hat{\mathbf{q}}_{\mathrm{s}}}_{\text{linearized solid equation}} = \underbrace{\mathcal{A}_{\mathrm{sf}} \left(\sigma \mathcal{B}_{\mathrm{ff}} - \mathcal{A}_{\mathrm{ff}}\right)^{-1} \mathcal{A}_{\mathrm{fs}} \hat{\mathbf{q}}_{\mathrm{s}}}_{\text{linearized fluid load}} \qquad (a)$$

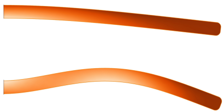
Reduced model

Coupled system :

$$\underbrace{\sigma^2 \mathcal{B}_{ss}\,\hat{\mathbf{q}}_s - \mathcal{A}_{ss}\,\hat{\mathbf{q}}_s}_{\text{linearized solid equation}} = \underbrace{\mathcal{A}_{sf}\,(\sigma\,\mathcal{B}_{ff} - \mathcal{A}_{ff})^{-1}\mathcal{A}_{fs}\,\hat{\mathbf{q}}_s}_{\text{linearized fluid load}} \qquad (a)$$

Free solid modes :

$$(\mathrm{i}\omega_{s,j})^2\,\mathcal{B}_{ss}\,\hat{\mathbf{q}}_s^{(j)} - \mathcal{A}_{ss}\,\hat{\mathbf{q}}_s^{(j)} = 0$$



$\hat{\mathbf{q}}_s^{(1)}$

$\hat{\mathbf{q}}_s^{(2)}$

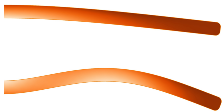
Reduced model

Coupled system :

$$\underbrace{\sigma^2 \mathcal{B}_{ss}\,\hat{\mathbf{q}}_s - \mathcal{A}_{ss}\,\hat{\mathbf{q}}_s}_{\text{linearized solid equation}} = \underbrace{\mathcal{A}_{sf}\,(\sigma\,\mathcal{B}_{ff} - \mathcal{A}_{ff})^{-1}\mathcal{A}_{fs}\,\hat{\mathbf{q}}_s}_{\text{linearized fluid load}} \qquad (a)$$

Free solid modes :

$$(\mathrm{i}\omega_{s,j})^2\,\mathcal{B}_{ss}\,\hat{\mathbf{q}}_s^{(j)} - \mathcal{A}_{ss}\,\hat{\mathbf{q}}_s^{(j)} = 0$$



$\hat{\mathbf{q}}_s^{(1)}$

$\hat{\mathbf{q}}_s^{(2)}$

Modal basis

$$\hat{\mathbf{q}}_s = \sum_j \hat{\mathbf{q}}_s^{(j)}\,y_j$$

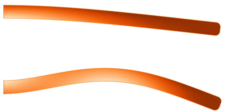
Reduced model

Coupled system :

$$\underbrace{\sigma^2\,\mathcal{B}_{\mathrm{ss}}\,\hat{\mathbf{q}}_{\mathrm{s}} - \mathcal{A}_{\mathrm{ss}}\,\hat{\mathbf{q}}_{\mathrm{s}}}_{\text{linearized solid equation}} = \underbrace{\mathcal{A}_{\mathrm{sf}}\,(\sigma\,\mathcal{B}_{\mathrm{ff}} - \mathcal{A}_{\mathrm{ff}})^{-1}\mathcal{A}_{\mathrm{fs}}\,\hat{\mathbf{q}}_{\mathrm{s}}}_{\text{linearized fluid load}} \qquad (a)$$

Free solid modes :

$$(\mathbf{i}\omega_{s,j})^2\,\mathcal{B}_{\mathrm{ss}}\,\hat{\mathbf{q}}_{\mathrm{s}}^{(j)} - \mathcal{A}_{\mathrm{ss}}\,\hat{\mathbf{q}}_{\mathrm{s}}^{(j)} = 0$$



$\hat{\mathbf{q}}_{\mathrm{s}}^{(1)}$

$\hat{\mathbf{q}}_{\mathrm{s}}^{(2)}$

Modal basis $\rightarrow$ projection of (a)

$$\left[\hat{\mathbf{q}}_{\mathrm{s}}^{(i)} \cdot (\sigma^2\,\mathcal{B}_{\mathrm{ss}} - \mathcal{A}_{\mathrm{ss}})\,\hat{\mathbf{q}}_{\mathrm{s}}^{(i)}\right] y_i = \sum_j \left[\hat{\mathbf{q}}_{\mathrm{s}}^{(i)} \cdot \mathcal{A}_{\mathrm{sf}}\,(\sigma\mathcal{B}_{\mathrm{ff}} - \mathcal{A}_{\mathrm{ff}})^{-1}\mathcal{A}_{\mathrm{fs}}\,\hat{\mathbf{q}}_{\mathrm{s}}^{(j)}\right] y_j \quad \forall i$$

Reduced model

Coupled system :

$$\underbrace{\sigma^2\,\mathcal{B}_{ss}\,\hat{\mathbf{q}}_s - \mathcal{A}_{ss}\,\hat{\mathbf{q}}_s}_{\text{linearized solid equation}} = \underbrace{\mathcal{A}_{sf}\,(\sigma\,\mathcal{B}_{ff} - \mathcal{A}_{ff})^{-1}\mathcal{A}_{fs}\,\hat{\mathbf{q}}_s}_{\text{linearized fluid load}} \qquad (a)$$

Free solid modes :

$$(\mathrm{i}\omega_{s,j})^2\,\mathcal{B}_{ss}\,\hat{\mathbf{q}}_s^{(j)} - \mathcal{A}_{ss}\,\hat{\mathbf{q}}_s^{(j)} = 0$$

$\hat{\mathbf{q}}_s^{(1)}$

$\hat{\mathbf{q}}_s^{(2)}$

Modal basis → projection of (a)

$$(\sigma^2 + \omega_{s,j}^2)\,y_i = \sum_j \frac{\hat{\mathbf{q}}_s^{(i)}\cdot\mathcal{A}_{sf}\,(\sigma\mathcal{B}_{ff} - \mathcal{A}_{ff})^{-1}\mathcal{A}_{fs}\,\hat{\mathbf{q}}_s^{(j)}}{\hat{\mathbf{q}}_s^{(i)}\cdot\mathcal{B}_{ss}\,\hat{\mathbf{q}}_s^{(i)}}\,y_j \qquad \forall i$$

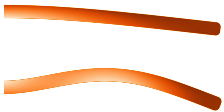
Reduced model

Coupled system :

$$\underbrace{\sigma^2\,\mathcal{B}_{\mathrm{ss}}\,\hat{\mathbf{q}}_{\mathrm{s}} - \mathcal{A}_{\mathrm{ss}}\,\hat{\mathbf{q}}_{\mathrm{s}}}_{\text{linearized solid equation}} = \underbrace{\mathcal{A}_{\mathrm{sf}}\,(\sigma\,\mathcal{B}_{\mathrm{ff}} - \mathcal{A}_{\mathrm{ff}})^{-1}\mathcal{A}_{\mathrm{fs}}\,\hat{\mathbf{q}}_{\mathrm{s}}}_{\text{linearized fluid load}} \qquad (a)$$

Free solid modes :

$$(\mathrm{i}\omega_{s,j})^2\,\mathcal{B}_{\mathrm{ss}}\,\hat{\mathbf{q}}_{\mathrm{s}}^{(j)} - \mathcal{A}_{\mathrm{ss}}\,\hat{\mathbf{q}}_{\mathrm{s}}^{(j)} = 0$$



$\hat{\mathbf{q}}_{\mathrm{s}}^{(1)}$

$\hat{\mathbf{q}}_{\mathrm{s}}^{(2)}$

Modal basis $\rightarrow$ projection of (a)

$$\underbrace{\left(\sigma^2 + \omega_{s,j}^2\right) y_i}_{\text{modal solid eqn.}} = \underbrace{\sum_j K_{ij}(\sigma)\,y_j}_{\text{modal fluid loads}} \quad \forall i \qquad (a')$$

Example : one-mode approximation

Modal basis of one mode :

$$\left(\sigma^2 + \omega_{s,j}^2\right) y_i = \sum_j K_{ij}(\sigma)\, y_j \quad i, j = 1$$

Example : one-mode approximation

Modal basis of one mode :

$$\left(\sigma^2 + \omega_{s,1}^2\right) y_1 = K_{11}(\sigma)\, y_1 \qquad (a')$$

Example : one-mode approximation

Modal basis of one mode :

$$\left(\sigma^2 + \omega_{s,1}^2\right) y_1 = K_{11}(\sigma)\, y_1 \qquad (a')$$

As we consider $\sigma \simeq 0$,

$$K_{11}(\sigma) \simeq K_{11,0} + K_{11,1}\,\sigma + K_{11,2}\,\sigma^2 + \ldots$$

Example : one-mode approximation

Modal basis of one mode :

$$\left(\sigma^2 + \omega_{s,1}^2\right) y_1 = K_{11}(\sigma)\, y_1 \qquad (a')$$

As we consider $\sigma \simeq 0$,

$$K_{11}(\sigma) = \underbrace{K_{11,0}}_{\simeq 3.7} + \underbrace{K_{11,1}}_{10^{-4}}\, \sigma + \underbrace{K_{11,2}}_{10^{-7}}\, \sigma^2 + \dots$$

Example : one-mode approximation

Modal basis of one mode :

$$\left(\sigma^2 + \omega_{s,1}^2\right) y_1 = K_{11}(\sigma)\, y_1 \qquad (a')$$

As we consider $\sigma \simeq 0$,

$$K_{11}(\sigma) \simeq K_{11,0} = 3.71$$

Example : one-mode approximation

Modal basis of one mode :

$$\left(\sigma^2 + \omega_{s,1}^2\right) y_1 = K_{11}(\sigma)\, y_1 \qquad (a')$$

As we consider $\sigma \simeq 0$,

$$K_{11}(\sigma) \simeq K_{11,0} = 3.71$$

Then in (a') : equation for the mode

$$\frac{\mathrm{d}^2 Y_1}{\mathrm{d}t^2} + \left(\omega_{s,1}^2 - K_{11,0}\right) \; Y_1 = 0 \qquad \mid \; Y_1 = y_1 e^{\sigma t} \qquad (a'')$$

Example : one-mode approximation

Modal basis of one mode :

$$\left(\sigma^2 + \omega_{s,1}^2\right) y_1 = K_{11}(\sigma)\, y_1 \qquad (a')$$

As we consider $\sigma \simeq 0$,

$$K_{11}(\sigma) \simeq K_{11,0} = 3.71$$

Then in (a') : equation for the mode

$$\frac{\mathrm{d}^2 Y_1}{\mathrm{d}t^2} + \underbrace{\left(\omega_{s,1}^2 - K_{11,0}\right)}_{\text{can be } < 0\,!} Y_1 = 0 \qquad \mid\ Y_1 = y_1 e^{\sigma t} \qquad (a'')$$

Example : one-mode approximation

Modal basis of one mode :

$$\left(\sigma^2 + \omega_{s,1}^2\right) y_1 = K_{11}(\sigma)\, y_1 \qquad (a')$$

As we consider $\sigma \simeq 0$,

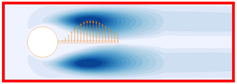
$$K_{11}(\sigma) \simeq K_{11,0} = 3.71$$

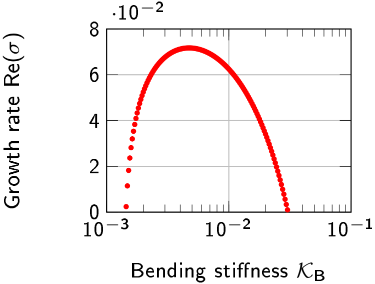
Then in (a') : equation for the mode

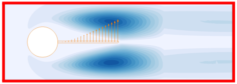
$$\frac{\mathrm{d}^2 Y_1}{\mathrm{d}t^2} + \underbrace{\left(\omega_{s,1}^2 - K_{11,0}\right)}_{\text{can be } < 0\,!} Y_1 = 0 \qquad \mid Y_1 = y_1 e^{\sigma t} \qquad (a'')$$

The fluid load (factor $K_{11,0}$) acts as a *negative added stiffness*
$\rightarrow$ possible *divergence* instability.

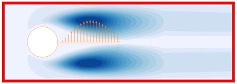
Reduced model results

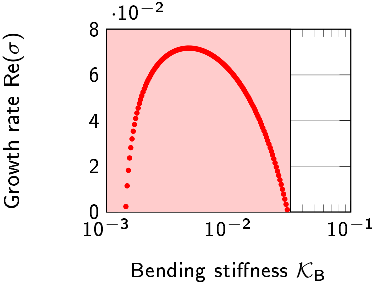


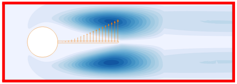




Reduced model results



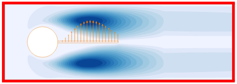


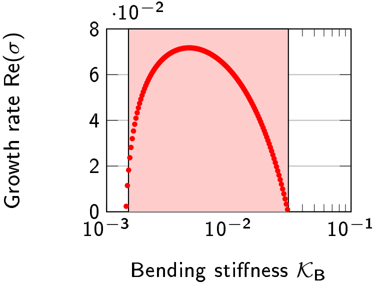


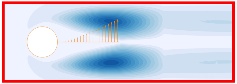
Unstable region, reduced model with mode 1

$$\omega_{s,1}^2 - K_{11,0} < 0$$

Reduced model results



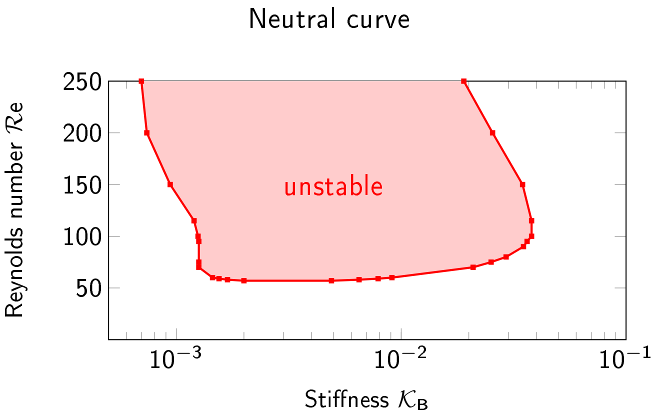




Unstable region, reduced model with mode 1 and mode 2

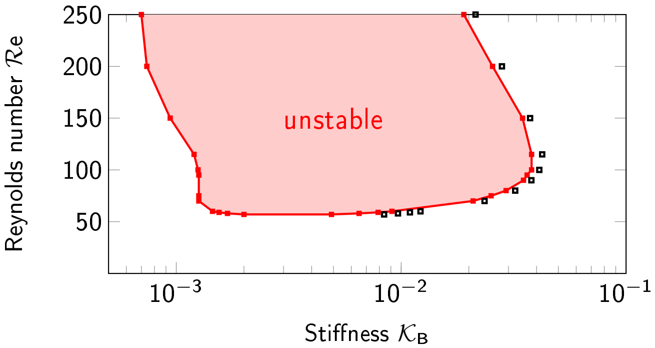
$$\left(\omega_{s,1}^2 - K_{11,0}\right)\left(\omega_{s,2}^2 - K_{22,0}\right) - K_{12,0}\,K_{21,0} < 0$$

Neutral curve in $(\mathcal{K}_B, \mathcal{R}e)$ plane



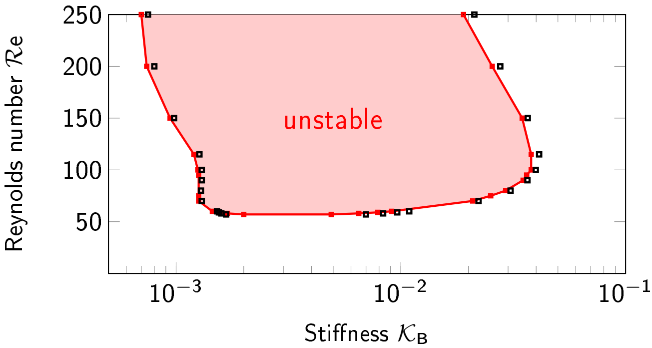
Neutral curve in $(\mathcal{K}_B, \mathcal{R}e)$ plane

Neutral curve *vs.* one-mode approximation



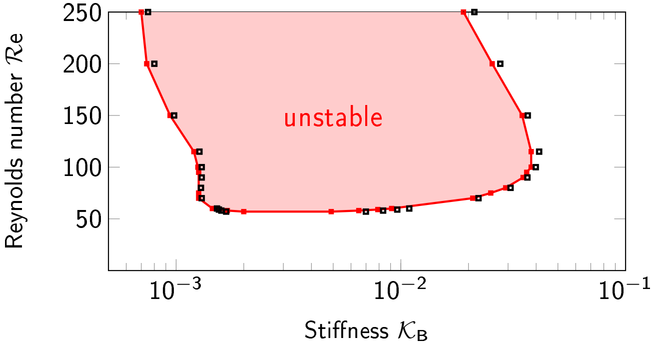
Neutral curve in $(\mathcal{K}_B, \mathcal{R}e)$ plane

Neutral curve *vs.* two-modes approximation



Neutral curve in $(\mathcal{K}_B, \mathcal{R}e)$ plane

Neutral curve *vs.* two-modes approximation



Good agreement : the hypothesis is validated.

Conclusion

- Linearized fluid/solid stability analysis can predict well the instability thresholds,
- The static fluid/solid bifurcation mechanism is a solid stiffness cancellation due to negative fluid added stiffness.

Thank you – Questions

Appendix – Parameters

Non-dimensional parameters :

$$\mathcal{R}\text{e} = \frac{DU_\infty}{\nu}, \quad \mathcal{K}_\text{B} = \frac{EH^3}{12\rho_f U_\infty^2 L^3}, \quad \mathcal{M}_\rho = \rho_s/\rho_f$$

In this study :

- $H/L \sim 1\%$
- $\mathcal{R}\text{e} \sim 100$
- $\mathcal{K}_\text{B} \sim 10^{-3} - 1$
- $\mathcal{M}_\rho = 1$

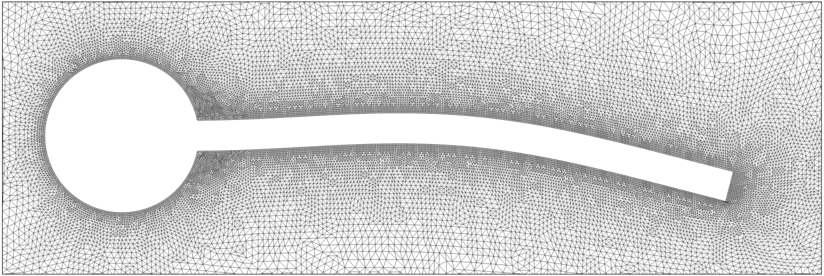
Corresponding values in the real world (e.g. in a water chanel) :

- $D \sim 1$ mm, $U_\infty \sim 0.1$ m/s, $E \sim 1 - 100$ MPa (rubber)

$\rightarrow$ Haemodynamics, small water-swimmers,...

Appendix – Spatial discretization

- Domain $x/D \in [-20, 50]$, $y/D \in [-20, 20]$
- Finite elements P2 for velocity/displacement, P1 for the pressure
- Conformal mesh with $\sim$ 30k elements $\rightarrow$ 200k d.o.f. in the stability problem



Appendix – Non-linear ALE system

Local equations[9]

$$J\frac{\partial \mathbf{u}}{\partial t} + \nabla \mathbf{u}\boldsymbol{\Phi}\left(\mathbf{u} - \frac{\partial \boldsymbol{\xi}}{\partial t}\right) - \nabla \cdot \boldsymbol{\Sigma}^f = 0, \qquad \text{fluid momentum eq.}$$

$$\nabla \mathbf{u} : \boldsymbol{\Phi}^{\mathsf{T}} = 0, \qquad \text{fluid continuity eq.}$$

$$\nabla^2 \boldsymbol{\xi} = 0, \qquad \text{extension eq.}$$

$$\mathcal{M}_\rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} - \nabla \cdot \boldsymbol{\Sigma}^s = 0 \qquad \text{solid momentum eq.}$$

Constitutive relations

$$\boldsymbol{\Sigma}^f = \left(-p\mathbf{I} + \frac{1}{\mathcal{R}e}\frac{1}{J}\left(\nabla \mathbf{u}\boldsymbol{\Phi} + \boldsymbol{\Phi}^{\mathsf{T}}\nabla \mathbf{u}^{\mathsf{T}}\right)\right)\boldsymbol{\Phi}^{\mathsf{T}}, \qquad \text{viscous fluid}$$

$$\boldsymbol{\Sigma}^s = \lambda \,\mathrm{tr}\left(\mathbf{F}^{\mathsf{T}}\mathbf{F} - \mathbf{I}\right)\mathbf{I} + 2\mu\left(\mathbf{F}^{\mathsf{T}}\mathbf{F} - \mathbf{I}\right) \qquad \text{Saint-Venant Kirchhoff solid}$$

Interface conditions : stress, displacement and velocity continuity.

9. $\boldsymbol{\Phi} = J\,\mathbf{F}^{-1}$, $\mathbf{F} = \mathbf{I} + \nabla\boldsymbol{\xi}$, $J = \det \mathbf{F}$ (geometric transformation operators)

Appendix – Stationary solver

Problem : Solve the stationary *non-linear* system :

$$\mathcal{R}(\mathbf{q}) = 0.$$

Approach : Newton method $\rightarrow$ solve successively *linear* systems :

$$\left.\frac{\partial \mathcal{R}}{\partial \mathbf{q}}\right|_{\mathbf{q}^k} \left(\mathbf{q}^{k+1} - \mathbf{q}^k\right) = -\mathcal{R}(\mathbf{q}^k).$$

- Monolithic approach : the fluid, solid and extension equation are solved simultaneously.
- *Exact Jacobian* implementation : an analytical derivation of $\partial\mathcal{R}/\partial\mathbf{q}$ is done rather than finite-differences $\rightarrow$ gain in convergence.
- Direct solver `MUMPS` for linear systems.

Appendix – Eigenvalue solver

Problem : Solve the eigenvalue problem :

$$\sigma \mathcal{B}\hat{\mathbf{q}} - \left.\frac{\partial \mathcal{R}}{\partial \mathbf{q}}\right|_{\mathbf{q}_b} \hat{\mathbf{q}} = 0$$

Unknowns (on the whole domain) in $\mathbf{q}$: 2×velocity + 4×displacement + 4×displacement velocity (fluid/solid, overlapping) + 1×pressure + interface Lagrange multipliers → symmetric $\mathcal{B}$ matrix.

Procedures :

- Exact Jacobian implementation,
- Shift-and-invert & ARNOLDI method,
- MUMPS solver for LU matrix decomposition,
- ARPACK interface in `FreeFem++`.

Appendix – DNS

Approach : ALE equations in reference conf. + monolithic weak form [10] :

- \+ No mesh-moving procedure (except for post-treatments)
- \+ No numerical added-mass effect
- – Strongly non-linear

Procedure : At each time-step n the fully implicit problem

$$\left.\frac{\partial \mathbf{q}}{\partial t}\right|^{n+1} + \mathcal{R}(\mathbf{q}^{n+1}, \mathbf{q}^{n}, \ldots) = 0$$

is solved using `FreeFem++`, with

- a (evtl. shifted) Crank-Nicholson time-discretization scheme (2$^{\text{nd}}$ order),
- a Newton method + exact Jacobian expressions,
- parallel matrix assembly & MUMPS direct solver.

Validated by comparison with popular FSI benchmarks (Turek & Hron)

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10. Richter, T. et Wick, T. *Finite elements for fluid-structure interaction in ALE and fully Eulerian coordinates* Comp. Meth. in Appl. Mech. Eng., 2010