Bifurcations and Instabilies in Fluid Mechanics
July 11-14, 2017, The Woodlands, TX – USA

Fluid/elastic modes of a flexible plate clamped on a cylinder

Jean-Lou PFISTER

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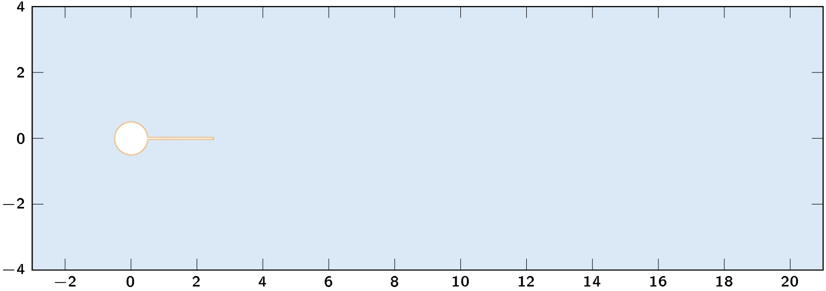
Advisor : Olivier MARQUET (ONERA)

Problem statement



Direct non-linear fluid/elastic simulation : streamwise velocity and plate deformation

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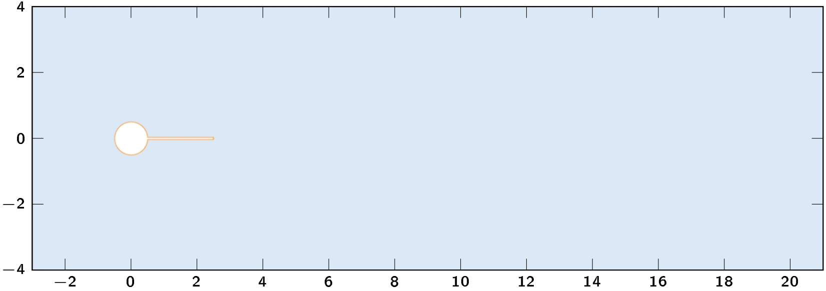

Direct non-linear fluid/elastic simulation : streamwise velocity and plate deformation

Spontaneous plate deviation → non-zero lift force.

- How to predict the instabilities onset ?
- What are the underlying mechanisms ?

Literature review – spontaneous deviations in fluid/solid systems

Motivation : flow control, bio-inspired systems...

1. Lacis, et *al*, Nature, 2014
2. Cimbala, J. and Garg, S., AIAA, 1991
3. Xu, J.C. et *al*., Phys. Fluids, 1990
4. Bagheri, S. et *al*., Phys. Rev. Lett., 2012

Literature review – spontaneous deviations in fluid/solid systems

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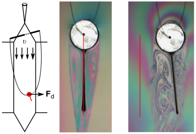
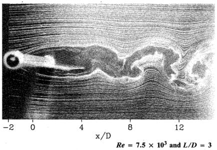
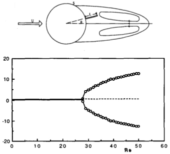
Low-Reynods flow, freely rotatable plate [1]



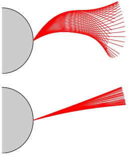
Higher-Reynods flow, freely rotatable plate [2]



Rigid rotatable plate (numeric study) [3]



Flexible attached plate (numeric study) [4]



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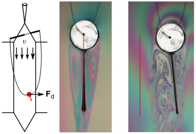

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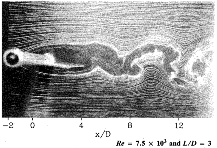

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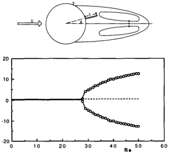

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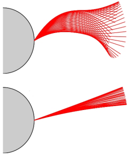

- We propose an approach based on linear stability analysis.

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Outline

Fluid-solid interaction model problem [5]

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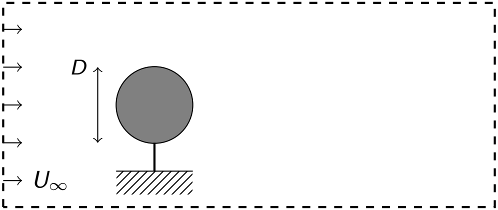

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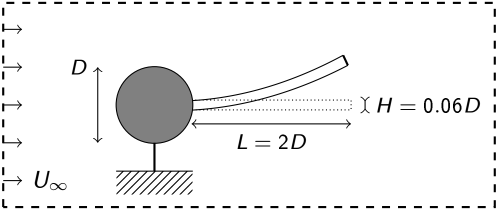
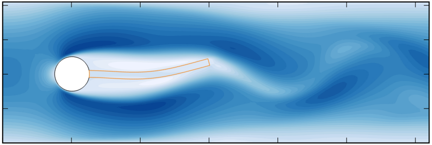


- Parameters :
  - $\rho_{\text{solid}}/\rho_{\text{fluid}} = 1$,
  - Poisson coefficient $\nu_s = 0.4$, variable **rigidity** coefficient $\mathcal{K}_{\text{B}}$,
  - Variable **Reynolds number** $\mathcal{R}e = U_\infty D/\nu$.
- Physics modeling :
  - Incompressible laminar flow,
  - Non-linear elastic solid (Saint-Venant Kirchhoff).

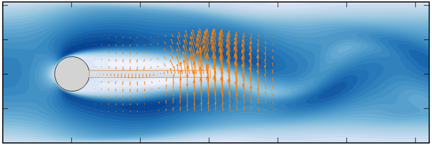
5. Lee, J. and You, D., Phys. Fluids, 2013

How to handle the fluid-structure coupling

To handle the moving domain : Arbitrary Lagrangian Eulerian formulation [6]



actual domain Ω_t (time-dependant)



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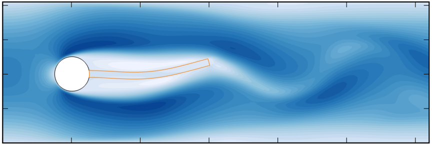

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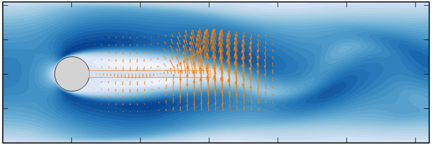

reference domain Ω_0 (fixed)

The coupled problem is then written as

$$\mathcal{B}\frac{\partial \mathbf{q}}{\partial t} - \mathcal{R}(\mathbf{q}) = 0 \quad \text{in a } \textbf{fixed} \text{ domain } \Omega_0$$

$$\mathbf{q} = \begin{cases} \text{velocity (solid/fluid)} \\ \text{displacement (solid/fluid)} \\ \text{pressure (fluid)} \end{cases} \qquad \mathcal{B}, \mathcal{R} = \begin{cases} \text{Navier-Stokes equations (fluid)} \\ \text{Elasticity equations (solid)} \\ \text{Stress/velocity continuity (interface)} \\ \text{Fluid mesh equation} \end{cases}$$

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Steady symmetric solution

Case $\mathcal{R}e = 80$, $\mathcal{K}_B = 0.015$ (same as in the video). Let us solve[7]

$$\mathcal{R}(\mathbf{q}_b) = 0$$

7. Newton method implemented in the finite element software FreeFem++ (www.freefem.org)

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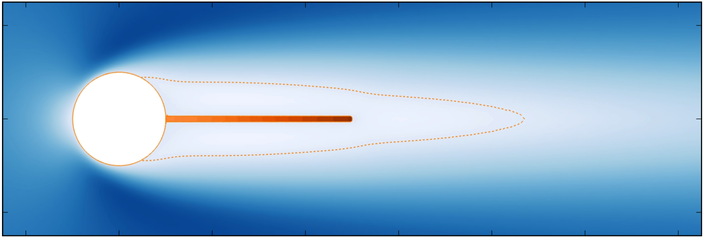
Fluid velocity :



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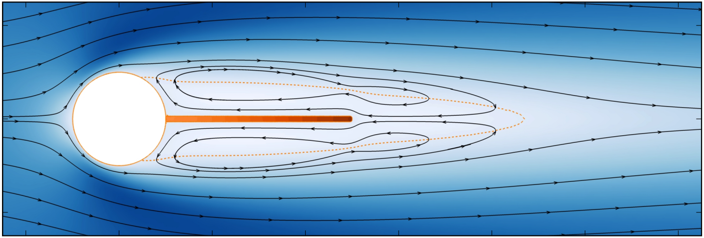

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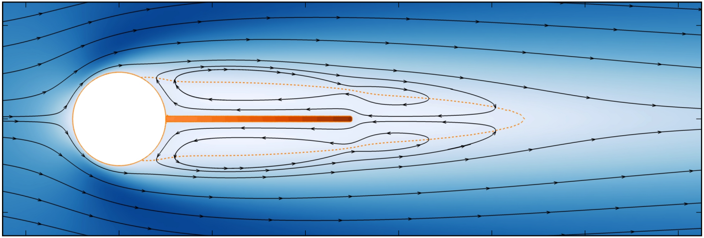
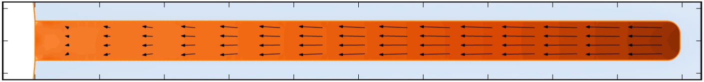


Solid displ. :



- Small compression ($\simeq 0.01\%$ of the length) in the solid due to viscous shear

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Linear stability problem

Non-linear problem

$$\mathcal{B}\frac{\partial \mathbf{q}}{\partial t}(\mathbf{x}) - \mathcal{R}(\mathbf{q})(\mathbf{x}) = 0 \quad \mathbf{x} \in \Omega_0$$

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Long-term asymptotic stability

$$\mathbf{q}(\mathbf{x}, t) = \mathbf{q}_b(\mathbf{x}) + \varepsilon\, \mathbf{q}'(\mathbf{x}, t) \qquad\qquad \varepsilon \ll 1$$

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Linear stability problem $\rightarrow$ look for *unstables* modes $(\lambda > 0, \omega, \hat{\mathbf{q}})$

$$(\lambda + \mathbf{i}\omega)\, \mathcal{B}\, \hat{\mathbf{q}} - \left.\frac{\partial \mathcal{R}}{\partial \mathbf{q}}\right|_{\mathbf{q}_b} \hat{\mathbf{q}} = 0$$

Eigenvalue problem

Monolithic fluid-solid coupling :

- $\hat{\mathbf{q}}_\mathrm{f}$: fluid velocity/pressure,
- $\hat{\mathbf{q}}_\mathrm{s}$: displacement.

Full coupled problem :

$$(\lambda + \mathrm{i}\omega)\begin{bmatrix} \mathcal{B}_\mathrm{ff} & 0 \\ 0 & \mathcal{B}_\mathrm{ss} \end{bmatrix}\begin{bmatrix} \hat{\mathbf{q}}_\mathrm{f} \\ \hat{\mathbf{q}}_\mathrm{s} \end{bmatrix} - \begin{bmatrix} \mathcal{A}_\mathrm{ff} & \mathcal{A}_\mathrm{fs} \\ \mathcal{A}_\mathrm{sf} & \mathcal{A}_\mathrm{ss} \end{bmatrix}\begin{bmatrix} \hat{\mathbf{q}}_\mathrm{f} \\ \hat{\mathbf{q}}_\mathrm{s} \end{bmatrix} = 0$$

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Couplings :

$$(\lambda + \mathbf{i}\omega) \begin{bmatrix} \mathcal{B}_{ff} & 0 \\ 0 & \mathcal{B}_{ss} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{q}}_f \\ \hat{\mathbf{q}}_s \end{bmatrix} - \begin{bmatrix} \mathcal{A}_{ff} & \mathcal{A}_{fs} \\ \mathcal{A}_{sf} & \mathcal{A}_{ss} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{q}}_f \\ \hat{\mathbf{q}}_s \end{bmatrix} = 0$$

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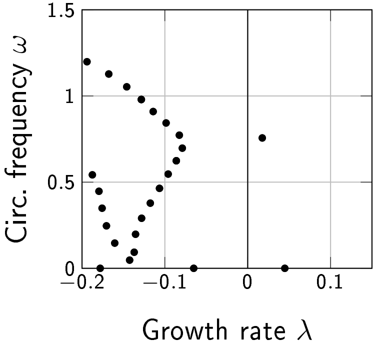
Large eigenvalue problem, solved in `FreeFem++`/ARPACK [8] with finite-elements and an exact Jacobian matrix.

8. www.caam.rice.edu/software/ARPACK/

Fluid-solid linear stability analysis of the symmetric solution

Spectrum ($\mathcal{R}e = 80, \mathcal{K}_B = 0.015$) *:*



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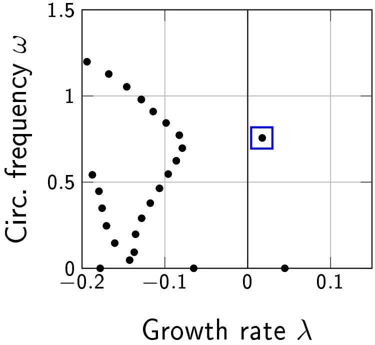
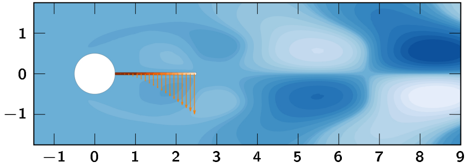


Modes (u velocity and solid displ.) :



Unsteady mode $\sigma_u = 0.017 + 0.76\,\mathrm{i}$

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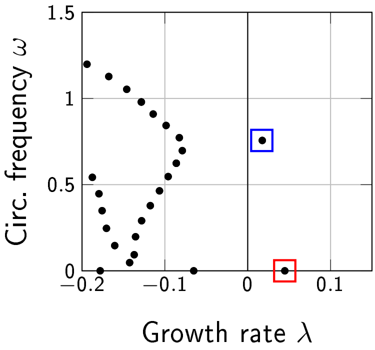

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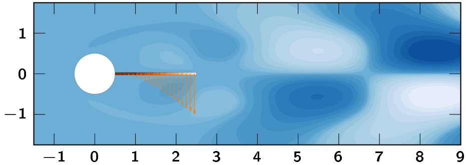

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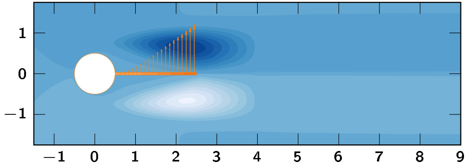
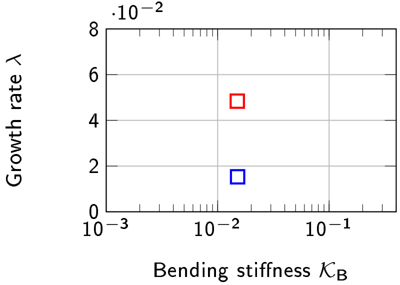
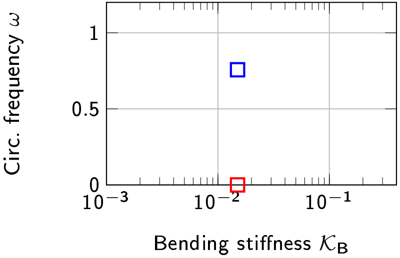


Steady mode $\sigma_s = 0.044 + 0\,\mathbf{i}$

Effect of the bending stiffness $\mathcal{K}_\mathrm{B}$





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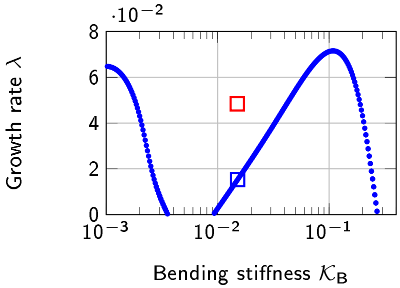

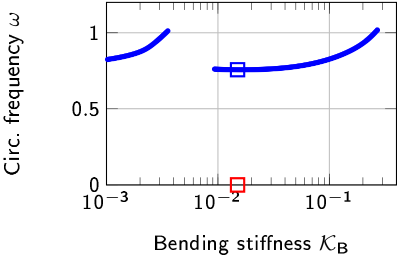

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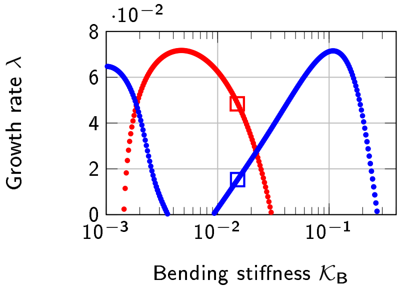

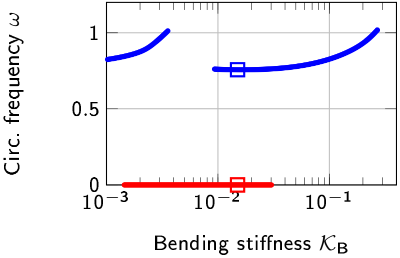

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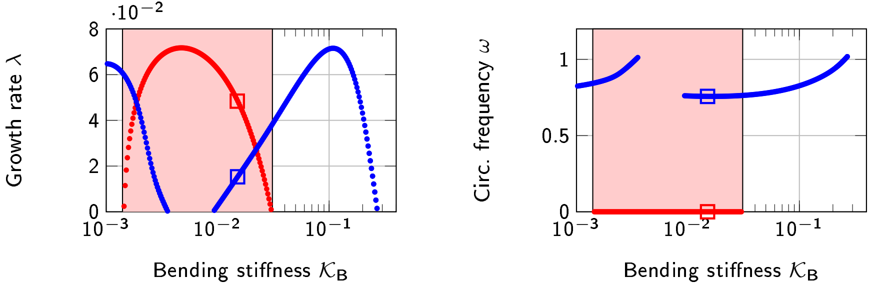
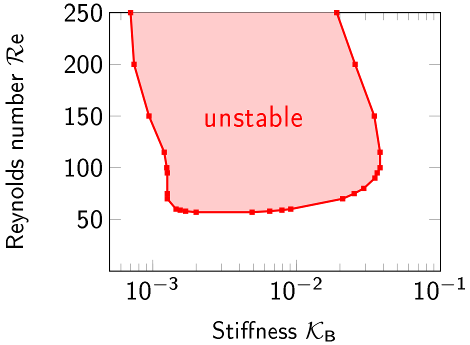


- The steady mode is unstable over a finite range of rigidities,
- In what follows we focus on this range.

Neutral curve for the steady mode



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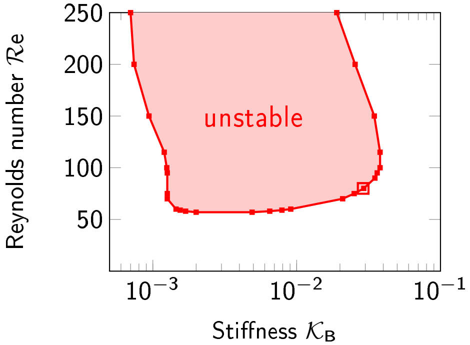
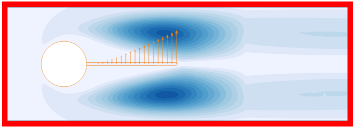


Steady mode shape, $\mathcal{R}e = 80$

$\mathcal{K}_B = 0.0290$



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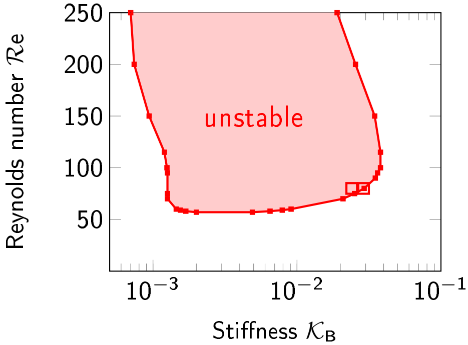

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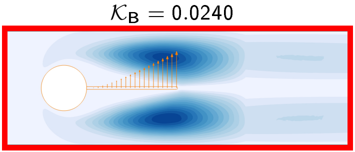

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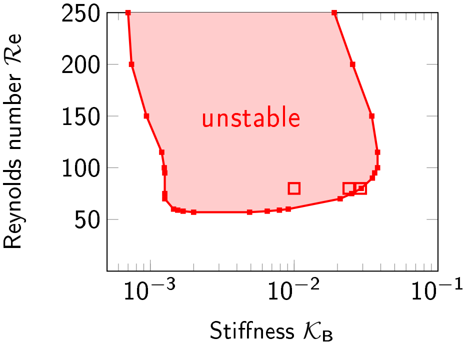
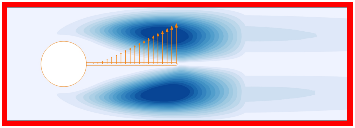


Steady mode shape, $\mathcal{R}e = 80$

$\mathcal{K}_B = 0.0150$



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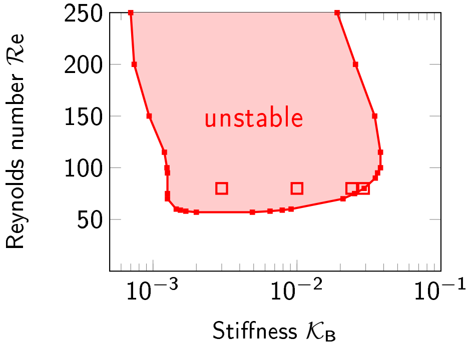
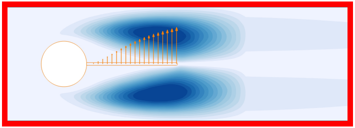


Steady mode shape, $\mathcal{R}e = 80$

$\mathcal{K}_{\mathrm{B}} = 0.0030$



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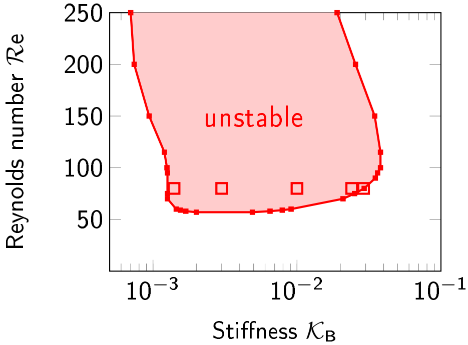

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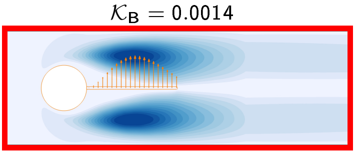
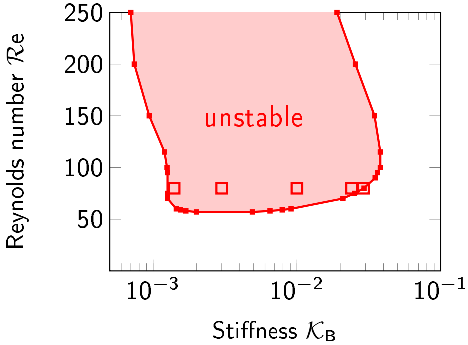
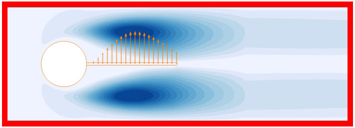


Neutral curve for the steady mode



Steady mode shape, $\mathcal{R}e = 80$

$\mathcal{K}_B = 0.0014$



Can we describe the steady mode neutral curve with a simple model ?

Reduced model

Coupled system :

$$(\lambda + \mathbf{i}\omega)\begin{bmatrix}\mathcal{B}_{\mathsf{ff}} & 0 \\ 0 & \mathcal{B}_{\mathsf{ss}}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{q}}_{\mathsf{f}} \\ \hat{\mathbf{q}}_{\mathsf{s}}\end{bmatrix} - \begin{bmatrix}\mathcal{A}_{\mathsf{ff}} & \mathcal{A}_{\mathsf{fs}} \\ \mathcal{A}_{\mathsf{sf}} & \mathcal{A}_{\mathsf{ss}}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{q}}_{\mathsf{f}} \\ \hat{\mathbf{q}}_{\mathsf{s}}\end{bmatrix} = 0$$

Reduced model

Coupled system :

$$
\begin{aligned}
(\lambda + \mathbf{i}\omega)\, \mathcal{B}_{\mathsf{ff}}\, \hat{\mathbf{q}}_{\mathsf{f}} - \mathcal{A}_{\mathsf{ff}}\, \hat{\mathbf{q}}_{\mathsf{f}} - \mathcal{A}_{\mathsf{fs}}\, \hat{\mathbf{q}}_{\mathsf{s}} &= 0 \\
(\lambda + \mathbf{i}\omega)\, \mathcal{B}_{\mathsf{ss}}\, \hat{\mathbf{q}}_{\mathsf{s}} - \mathcal{A}_{\mathsf{sf}}\, \hat{\mathbf{q}}_{\mathsf{f}} - \mathcal{A}_{\mathsf{ss}}\, \hat{\mathbf{q}}_{\mathsf{s}} &= 0
\end{aligned}
$$

Reduced model

Coupled system : stationary modes

$$(\lambda + \cancel{i\omega})\,\mathcal{B}_{\mathrm{ff}}\,\hat{\mathbf{q}}_{\mathrm{f}} - \mathcal{A}_{\mathrm{ff}}\,\hat{\mathbf{q}}_{\mathrm{f}} - \mathcal{A}_{\mathrm{fs}}\,\hat{\mathbf{q}}_{\mathrm{s}} = 0$$
$$(\lambda + \cancel{i\omega})\,\mathcal{B}_{\mathrm{ss}}\,\hat{\mathbf{q}}_{\mathrm{s}} - \mathcal{A}_{\mathrm{sf}}\,\hat{\mathbf{q}}_{\mathrm{f}} - \mathcal{A}_{\mathrm{ss}}\,\hat{\mathbf{q}}_{\mathrm{s}} = 0$$

Reduced model

Coupled system : stationary modes

$$\lambda\,\mathcal{B}_{\mathrm{ff}}\,\hat{\mathbf{q}}_{\mathrm{f}} - \mathcal{A}_{\mathrm{ff}}\,\hat{\mathbf{q}}_{\mathrm{f}} - \mathcal{A}_{\mathrm{fs}}\,\hat{\mathbf{q}}_{\mathrm{s}} = 0$$
$$\lambda\,\mathcal{B}_{\mathrm{ss}}\,\hat{\mathbf{q}}_{\mathrm{s}} - \mathcal{A}_{\mathrm{sf}}\,\hat{\mathbf{q}}_{\mathrm{f}} - \mathcal{A}_{\mathrm{ss}}\,\hat{\mathbf{q}}_{\mathrm{s}} = 0$$

Reduced model

Coupled system : stationary modes

$$\lambda\,\mathcal{B}_{\mathsf{ss}}\,\hat{\mathbf{q}}_{\mathsf{s}} - \mathcal{A}_{\mathsf{ss}}\,\hat{\mathbf{q}}_{\mathsf{s}} = \mathcal{A}_{\mathsf{sf}}\,(\lambda\,\mathcal{B}_{\mathsf{ff}} - \mathcal{A}_{\mathsf{ff}})^{-1}\mathcal{A}_{\mathsf{fs}}\,\hat{\mathbf{q}}_{\mathsf{s}} \qquad (a)$$

Reduced model

Coupled system : stationary modes

$$\underbrace{\lambda\,\mathcal{B}_{\mathsf{ss}}\,\hat{\mathbf{q}}_{\mathsf{s}} - \mathcal{A}_{\mathsf{ss}}\,\hat{\mathbf{q}}_{\mathsf{s}}}_{\text{linearized solid equation}} = \underbrace{\mathcal{A}_{\mathsf{sf}}\,(\lambda\,\mathcal{B}_{\mathsf{ff}} - \mathcal{A}_{\mathsf{ff}})^{-1}\mathcal{A}_{\mathsf{fs}}\,\hat{\mathbf{q}}_{\mathsf{s}}}_{\text{linearized fluid load}} \qquad (a)$$

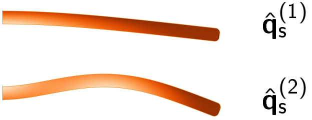
Reduced model

Coupled system : stationary modes

$$\underbrace{\lambda \mathcal{B}_{ss}\, \hat{\mathbf{q}}_s - \mathcal{A}_{ss}\, \hat{\mathbf{q}}_s}_{\text{linearized solid equation}} = \underbrace{\mathcal{A}_{sf}\,(\lambda \mathcal{B}_{ff} - \mathcal{A}_{ff})^{-1} \mathcal{A}_{fs}\, \hat{\mathbf{q}}_s}_{\text{linearized fluid load}} \qquad (a)$$

Free solid modes are orthogonal, solution of

$$\lambda \mathcal{B}_{ss}\, \hat{\mathbf{q}}_s^{(j)} - \mathcal{A}_{ss}\, \hat{\mathbf{q}}_s^{(j)} = 0$$



Reduced model

Coupled system : stationary modes

$$\underbrace{\lambda \mathcal{B}_{ss}\,\hat{\mathbf{q}}_s - \mathcal{A}_{ss}\,\hat{\mathbf{q}}_s}_{\text{linearized solid equation}} = \underbrace{\mathcal{A}_{sf}\,(\lambda \mathcal{B}_{ff} - \mathcal{A}_{ff})^{-1}\mathcal{A}_{fs}\,\hat{\mathbf{q}}_s}_{\text{linearized fluid load}} \qquad (a)$$

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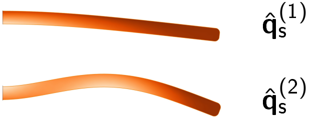

Modal basis

$$\hat{\mathbf{q}}_s = \sum_j y_j\,\hat{\mathbf{q}}_s^{(j)}$$

Reduced model

Coupled system : stationary modes

$$\underbrace{\lambda\,\mathcal{B}_{ss}\,\hat{\mathbf{q}}_s - \mathcal{A}_{ss}\,\hat{\mathbf{q}}_s}_{\text{linearized solid equation}} = \underbrace{\mathcal{A}_{sf}\,(\lambda\,\mathcal{B}_{ff} - \mathcal{A}_{ff})^{-1}\mathcal{A}_{fs}\,\hat{\mathbf{q}}_s}_{\text{linearized fluid load}} \qquad (a)$$

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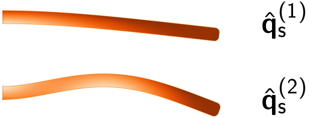


Modal basis $\rightarrow$ projection of (a)

$$y_i\left[\hat{\mathbf{q}}_s^{(i)}\cdot(\lambda\,\mathcal{B}_{ss} - \mathcal{A}_{ss})\,\hat{\mathbf{q}}_s^{(i)}\right] = \sum_j y_j\left[\hat{\mathbf{q}}_s^{(i)}\cdot\mathcal{A}_{sf}\,(\lambda\mathcal{B}_{ff} - \mathcal{A}_{ff})^{-1}\mathcal{A}_{fs}\,\hat{\mathbf{q}}_s^{(j)}\right] \quad \forall i$$

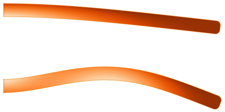
Reduced model

Coupled system : stationary modes

$$\underbrace{\lambda\,\mathcal{B}_{ss}\,\hat{\mathbf{q}}_s - \mathcal{A}_{ss}\,\hat{\mathbf{q}}_s}_{\text{linearized solid equation}} = \underbrace{\mathcal{A}_{sf}\,(\lambda\,\mathcal{B}_{ff} - \mathcal{A}_{ff})^{-1}\mathcal{A}_{fs}\,\hat{\mathbf{q}}_s}_{\text{linearized fluid load}} \qquad (a)$$

Free solid modes are orthogonal, solution of

$$\lambda\,\mathcal{B}_{ss}\,\hat{\mathbf{q}}_s^{(j)} - \mathcal{A}_{ss}\,\hat{\mathbf{q}}_s^{(j)} = 0$$



$\hat{\mathbf{q}}_s^{(1)}$

$\hat{\mathbf{q}}_s^{(2)}$

Modal basis $\rightarrow$ projection of (a) $\rightarrow$ marginality

$$y_i\left[\hat{\mathbf{q}}_s^{(i)}\cdot(\cancel{\lambda\mathcal{B}_{ss}} - \mathcal{A}_{ss})\,\hat{\mathbf{q}}_s^{(i)}\right] = \sum_j y_j\left[\hat{\mathbf{q}}_s^{(i)}\cdot\mathcal{A}_{sf}\,(\cancel{\lambda\mathcal{B}_{ff}} - \mathcal{A}_{ff})^{-1}\mathcal{A}_{fs}\,\hat{\mathbf{q}}_s^{(j)}\right] \quad \forall i$$

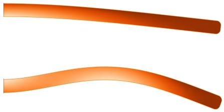
Reduced model

Coupled system : stationary modes

$$\underbrace{\lambda\,\mathcal{B}_{\mathrm{ss}}\,\hat{\mathbf{q}}_{\mathrm{s}} - \mathcal{A}_{\mathrm{ss}}\,\hat{\mathbf{q}}_{\mathrm{s}}}_{\text{linearized solid equation}} = \underbrace{\mathcal{A}_{\mathrm{sf}}\,(\lambda\,\mathcal{B}_{\mathrm{ff}} - \mathcal{A}_{\mathrm{ff}})^{-1}\mathcal{A}_{\mathrm{fs}}\,\hat{\mathbf{q}}_{\mathrm{s}}}_{\text{linearized fluid load}} \qquad (a)$$

Free solid modes are orthogonal, solution of

$$\lambda\,\mathcal{B}_{\mathrm{ss}}\,\hat{\mathbf{q}}_{\mathrm{s}}^{(j)} - \mathcal{A}_{\mathrm{ss}}\,\hat{\mathbf{q}}_{\mathrm{s}}^{(j)} = 0$$



$\hat{\mathbf{q}}_{\mathrm{s}}^{(1)}$

$\hat{\mathbf{q}}_{\mathrm{s}}^{(2)}$

Modal basis → projection of (a) → marginality → modal equilibrium

$$y_i\ \underbrace{\left[\hat{\mathbf{q}}_{\mathrm{s}}^{(i)}\cdot\mathcal{A}_{\mathrm{ss}}\,\hat{\mathbf{q}}_{\mathrm{s}}^{(i)}\right]}_{\text{modal stifness } ki(\mathcal{K}_{\mathrm{B}})} = \sum_j y_j\ \underbrace{\left[\hat{\mathbf{q}}_{\mathrm{s}}^{(i)}\cdot\mathcal{A}_{\mathrm{sf}}\,\mathcal{A}_{\mathrm{ff}}^{-1}\mathcal{A}_{\mathrm{fs}}\,\hat{\mathbf{q}}_{\mathrm{s}}^{(j)}\right]}_{\text{fluid linearized load projection}} \qquad \forall\, i$$

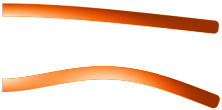
Reduced model

Coupled system : stationary modes

$$\underbrace{\lambda\,\mathcal{B}_{ss}\,\hat{\mathbf{q}}_s - \mathcal{A}_{ss}\,\hat{\mathbf{q}}_s}_{\text{linearized solid equation}} = \underbrace{\mathcal{A}_{sf}\,(\lambda\,\mathcal{B}_{ff} - \mathcal{A}_{ff})^{-1}\mathcal{A}_{fs}\,\hat{\mathbf{q}}_s}_{\text{linearized fluid load}} \qquad (a)$$

Free solid modes are orthogonal, solution of

$$\lambda\,\mathcal{B}_{ss}\,\hat{\mathbf{q}}_s^{(j)} - \mathcal{A}_{ss}\,\hat{\mathbf{q}}_s^{(j)} = 0$$



$\hat{\mathbf{q}}_s^{(1)}$

$\hat{\mathbf{q}}_s^{(2)}$

Modal basis $\rightarrow$ projection of (a) $\rightarrow$ marginality $\rightarrow$ modal equilibrium

$$k_i(\mathcal{K}_B)\,y_i = \sum_j y_j \underbrace{\left[\hat{\mathbf{q}}_s^{(i)} \cdot \mathcal{A}_{sf}\,\mathcal{A}_{ff}^{-1}\mathcal{A}_{fs}\,\hat{\mathbf{q}}_s^{(j)}\right]}_{\text{fluid linearized load projection}} \qquad \forall\, i$$

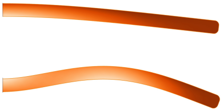
Reduced model

Coupled system : stationary modes

$$\underbrace{\lambda\,\mathcal{B}_{\mathrm{ss}}\,\hat{\mathbf{q}}_{\mathrm{s}} - \mathcal{A}_{\mathrm{ss}}\,\hat{\mathbf{q}}_{\mathrm{s}}}_{\text{linearized solid equation}} = \underbrace{\mathcal{A}_{\mathrm{sf}}\,(\lambda\,\mathcal{B}_{\mathrm{ff}} - \mathcal{A}_{\mathrm{ff}})^{-1}\mathcal{A}_{\mathrm{fs}}\,\hat{\mathbf{q}}_{\mathrm{s}}}_{\text{linearized fluid load}} \qquad (a)$$

Free solid modes are orthogonal, solution of

$$\lambda\,\mathcal{B}_{\mathrm{ss}}\,\hat{\mathbf{q}}_{\mathrm{s}}^{(j)} - \mathcal{A}_{\mathrm{ss}}\,\hat{\mathbf{q}}_{\mathrm{s}}^{(j)} = 0$$



$\hat{\mathbf{q}}_{\mathrm{s}}^{(1)}$

$\hat{\mathbf{q}}_{\mathrm{s}}^{(2)}$

Modal basis → projection of (a) → marginality → modal equilibrium

$$k_i(\mathcal{K}_{\mathrm{B}})\,y_i = \sum_j \frac{\partial F_{(i)}^{\mathrm{f}\to\mathrm{s}}}{\partial \hat{\mathbf{q}}_{\mathrm{s}}^{(j)}} y_j \qquad \forall\, i \qquad (a')$$

Example : one-mode approximation

Equilibrium between the solid stiffness and a fluid *added stiffness*[9]

$$\left(-k_1(\mathcal{K}_\mathrm{B}) + \frac{\partial F^{\mathrm{f}\to\mathrm{s}}_{(1)}}{\partial \hat{\mathbf{q}}^{(1)}_\mathrm{s}}\right) y_1 = 0$$

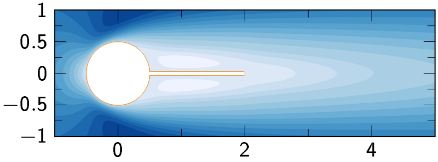
9. Dowell, et *al*, Springer, 2004

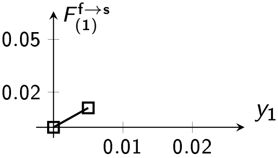
Example : one-mode approximation

Equilibrium between the solid stiffness and a fluid *added stiffness*[9]

$$\left(-k_1(\mathcal{K}_{\mathrm{B}}) + \frac{\partial F_{(1)}^{\mathrm{f}\rightarrow\mathrm{s}}}{\partial \hat{\mathbf{q}}_{\mathrm{s}}^{(1)}}\right) y_1 = 0$$

Approximation of the linearized fluid load





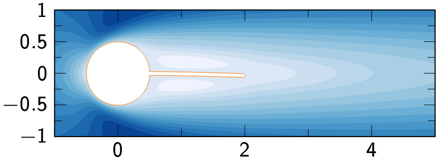
9. Dowell, et *al*, Springer, 2004

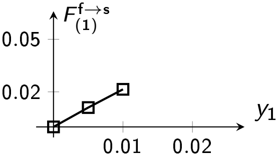
Example : one-mode approximation

Equilibrium between the solid stiffness and a fluid *added stiffness* [9]

$$\left(-k_1(\mathcal{K}_B) + \frac{\partial F_{(1)}^{f\to s}}{\partial \hat{\mathbf{q}}_s^{(1)}} \right) y_1 = 0$$

Approximation of the linearized fluid load





9. Dowell, et *al*, Springer, 2004

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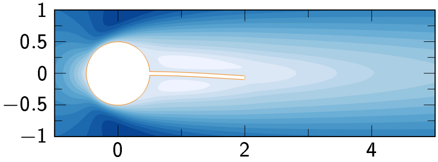
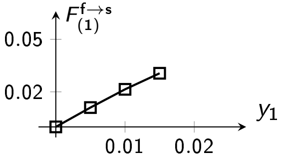




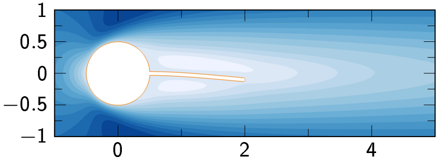
9. Dowell, et *al*, Springer, 2004

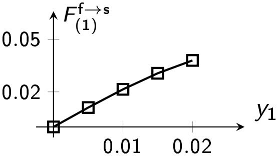
Example : one-mode approximation

Equilibrium between the solid stiffness and a fluid *added stiffness*[9]

$$\left(-k_1(\mathcal{K}_B) + \frac{\partial F^{f\to s}_{(1)}}{\partial \hat{\mathbf{q}}^{(1)}_s}\right) y_1 = 0$$

Approximation of the linearized fluid load





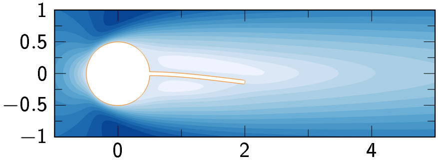
9. Dowell, et *al*, Springer, 2004

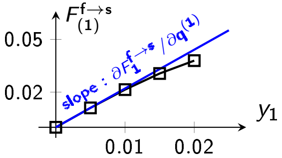
Example : one-mode approximation

Equilibrium between the solid stiffness and a fluid *added stiffness*[9]

$$\left(-k_1(\mathcal{K}_B) + \frac{\partial F_{(1)}^{f\to s}}{\partial \hat{\mathbf{q}}_s^{(1)}}\right) y_1 = 0$$

Approximation of the linearized fluid load





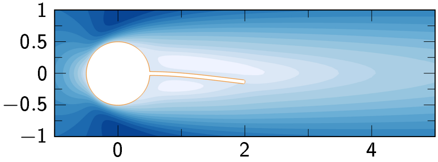
9. Dowell, et *al*, Springer, 2004

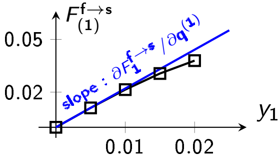
Example : one-mode approximation

Equilibrium between the solid stiffness and a fluid *added stiffness*[9]

$$\left(-k_1(\mathcal{K}_\mathrm{B}) + \frac{\partial F_{(1)}^{\mathrm{f}\to\mathrm{s}}}{\partial \hat{\mathbf{q}}_\mathrm{s}^{(1)}}\right) y_1 = 0$$

Approximation of the linearized fluid load





Non-zero (=deviated) solutions if $y_1 \neq 0 \Rightarrow$

$$-k_1(\mathcal{K}_\mathrm{B}) + \frac{\partial F_{(1)}^{\mathrm{f}\to\mathrm{s}}}{\partial \hat{\mathbf{q}}_\mathrm{s}^{(1)}}(\mathcal{R}\mathrm{e}) = 0 \qquad \text{(solid stiffness cancelled by the fluid added stiffness)}$$

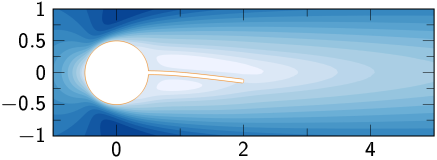
9. Dowell, et *al*, Springer, 2004

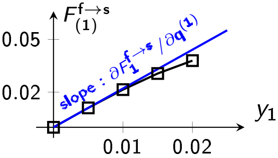
Example : one-mode approximation

Equilibrium between the solid stiffness and a fluid *added stiffness* [9]

$$\left(-k_1(\mathcal{K}_\mathrm{B}) + \frac{\partial F^{\mathrm{f}\rightarrow\mathrm{s}}_{(1)}}{\partial \hat{\mathbf{q}}^{(1)}_\mathrm{s}} \right) y_1 = 0$$

Approximation of the linearized fluid load





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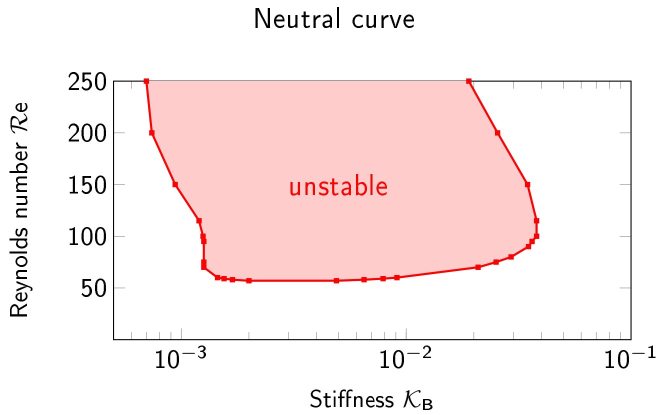
Same mechanism as the *divergence instability* in aeroelasticity

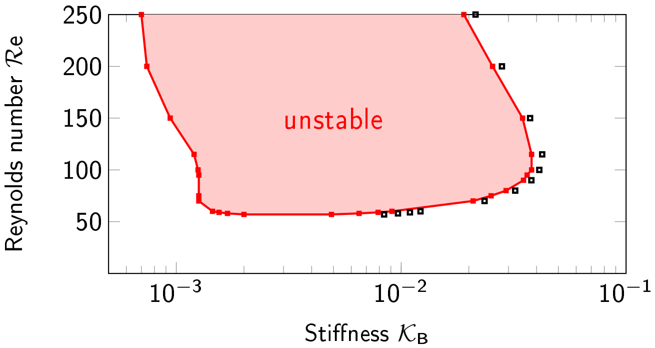
9. Dowell, et *al*, Springer, 2004

Model results



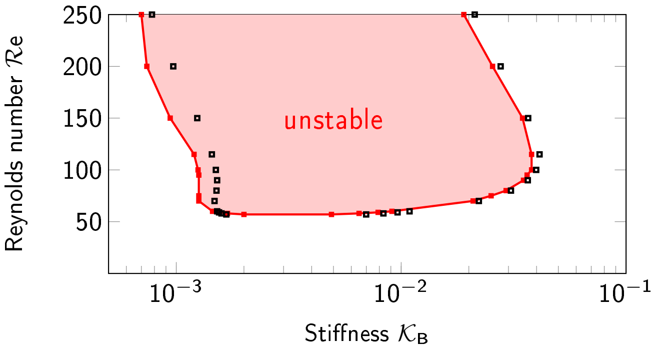
Model results

Neutral curve *vs.* one-mode approximation



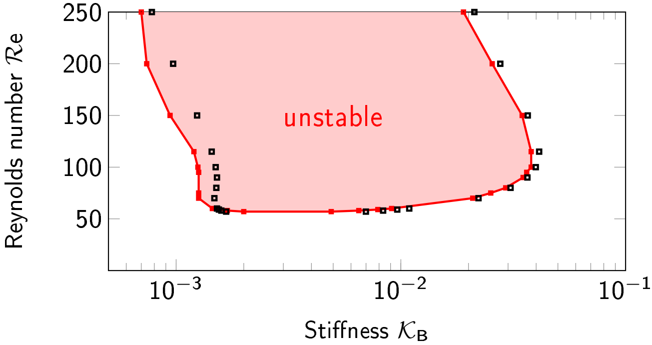
Model results

Neutral curve *vs.* two-modes approximation



Model results

Neutral curve *vs.* two-modes approximation



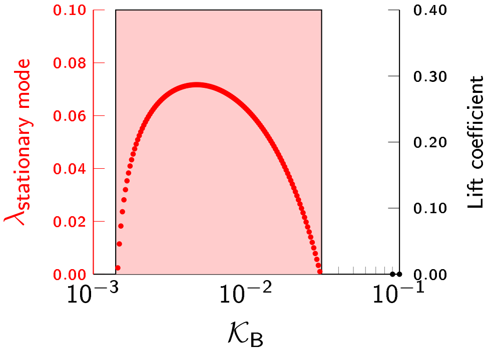
Good agreement : the hypothesis is validated.

Outline

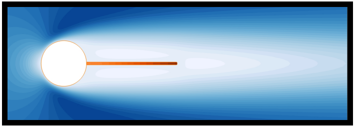
Deviated steady solutions

Starting form $\mathbf{q}_b$, we now follow the deviated stationary solutions

$$\mathcal{R}(\mathbf{q}_{\text{dev}}) = 0$$



$\mathcal{K}_{\text{B}} = 0.1000$



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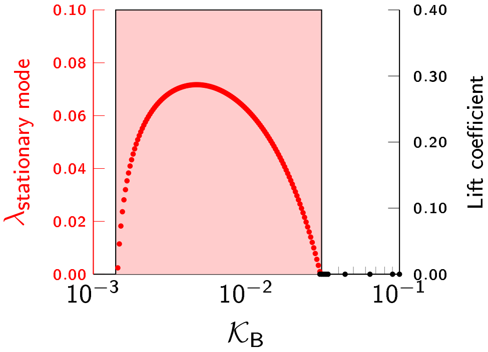

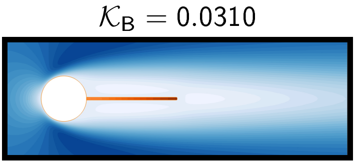

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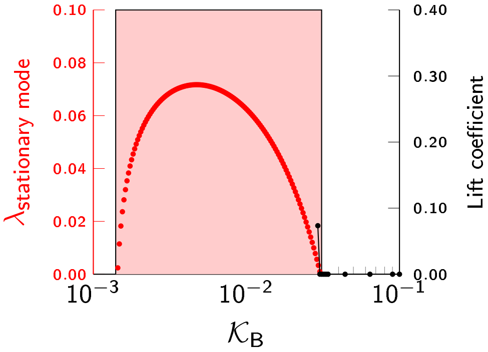
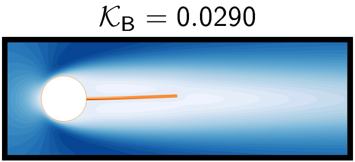


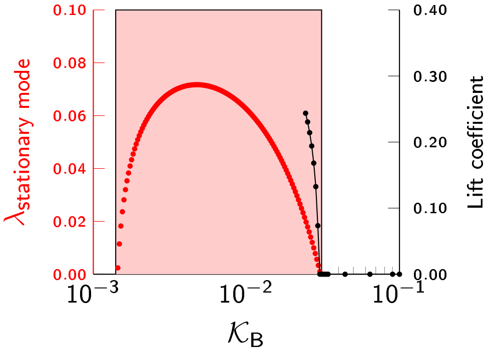


Supercritical pitchfork bifurcation

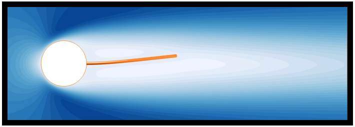
Deviated steady solutions

Starting form $\mathbf{q}_b$, we now follow the deviated stationary solutions

$$\mathcal{R}(\mathbf{q}_{\mathrm{dev}}) = 0$$



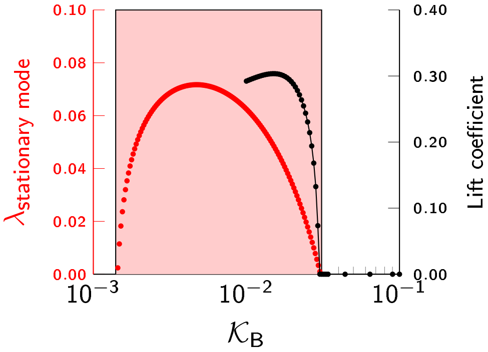
$\mathcal{K}_{\mathrm{B}} = 0.0240$



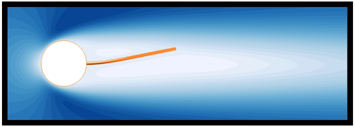
Deviated steady solutions

Starting form $\mathbf{q}_b$, we now follow the deviated stationary solutions

$$\mathcal{R}(\mathbf{q}_{\mathrm{dev}}) = 0$$



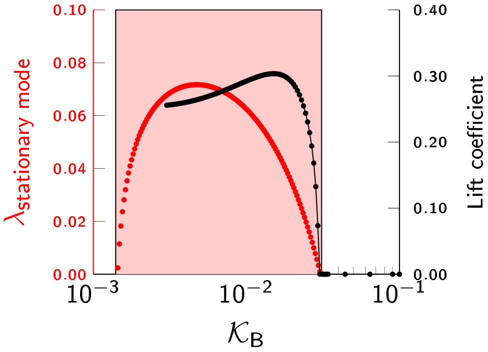
$\mathcal{K}_{\mathrm{B}} = 0.0150$



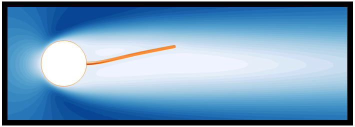
Deviated steady solutions

Starting form $\mathbf{q}_b$, we now follow the deviated stationary solutions

$$\mathcal{R}(\mathbf{q}_{\text{dev}}) = 0$$



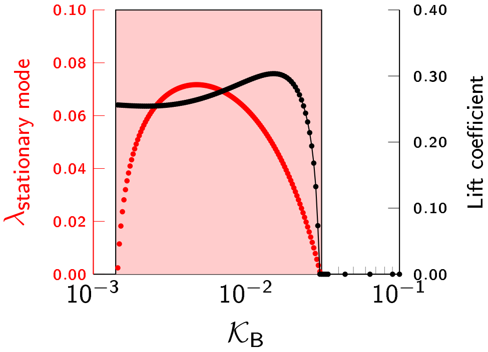
$\mathcal{K}_\text{B} = 0.0030$



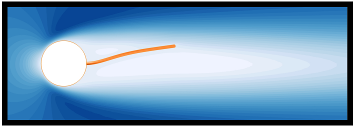
Deviated steady solutions

Starting form $\mathbf{q}_b$, we now follow the deviated stationary solutions

$$\mathcal{R}(\mathbf{q}_{\mathrm{dev}}) = 0$$



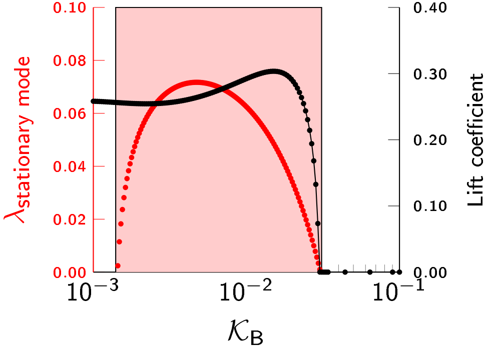
$\mathcal{K}_{\mathrm{B}} = 0.0014$



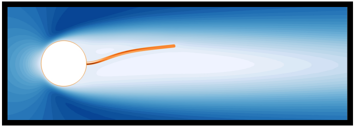
Deviated steady solutions

Starting form $\mathbf{q}_b$, we now follow the deviated stationary solutions

$$\mathcal{R}(\mathbf{q}_{\text{dev}}) = 0$$

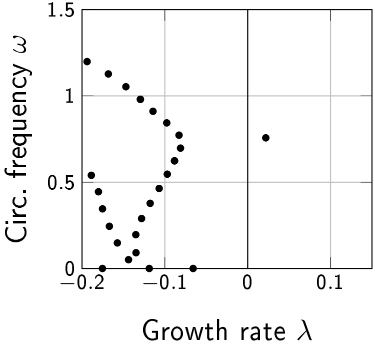


$\mathcal{K}_\text{B} = 0.0010$



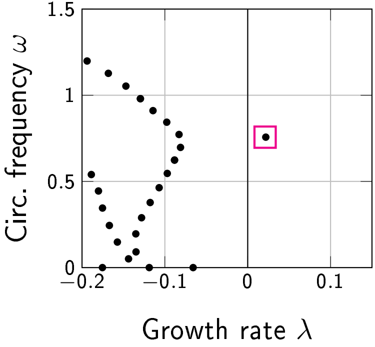
Fluid-solid linear stability analysis of the deviated solution

Spectrum ($\mathcal{R}e = 80$, $\mathcal{K}_B = 0.015$) *:*

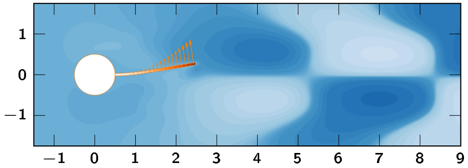


Fluid-solid linear stability analysis of the deviated solution

Spectrum ($\mathcal{R}e = 80$, $\mathcal{K}_B = 0.015$) *:*



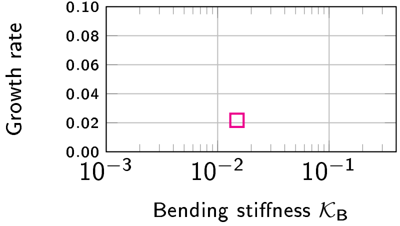
Mode (u velocity and solid displ.) :

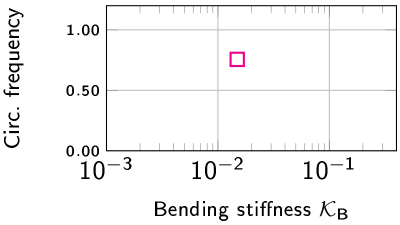


Unsteady mode $\sigma_u^{\text{dev}} = 0.021 + 0.756\,\mathbf{i}$

Stability on the deviated state

Now we compute the linear global modes of the **deviated** state $\mathbf{q}_{\mathrm{dev}}(\mathbf{x})$:





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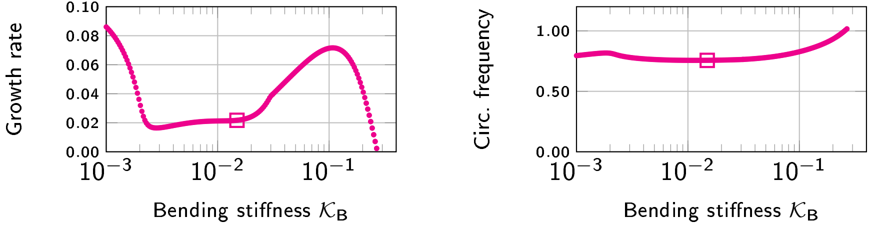

Stability on the deviated state

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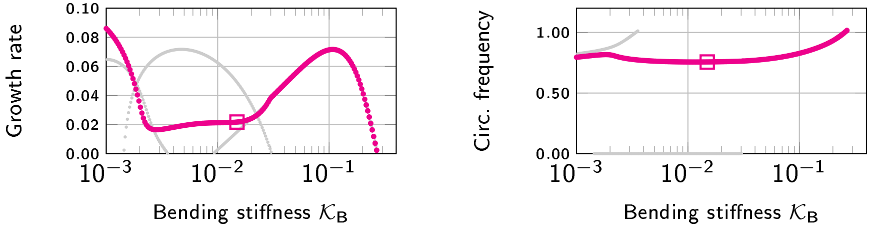
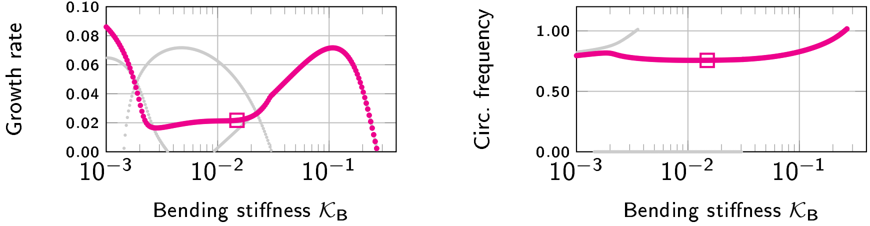


Stability on the deviated state

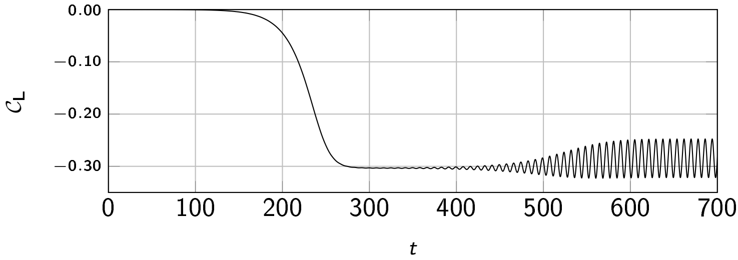
Now we compute the linear global modes of the **deviated** state $\mathbf{q}_{\mathrm{dev}}(\mathbf{x})$:



Only one unstable, unsteady branch now.

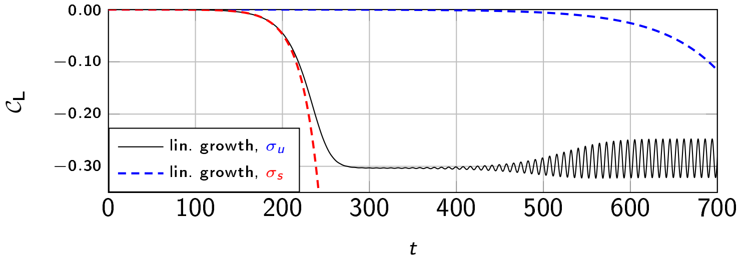
Modes interaction

Evolution of the lift coefficient in the non-linear solution, $\mathcal{K}_\mathrm{B} = 0.015$



Modes interaction

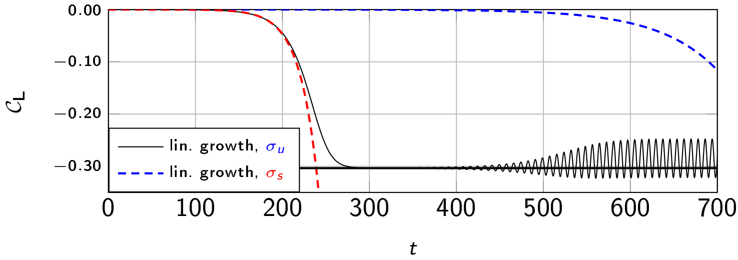
Evolution of the lift coefficient in the non-linear solution, $\mathcal{K}_{\mathrm{B}} = 0.015$



1. Modes σ_s and σ_u grow on the symmetric steady solution,

Modes interaction

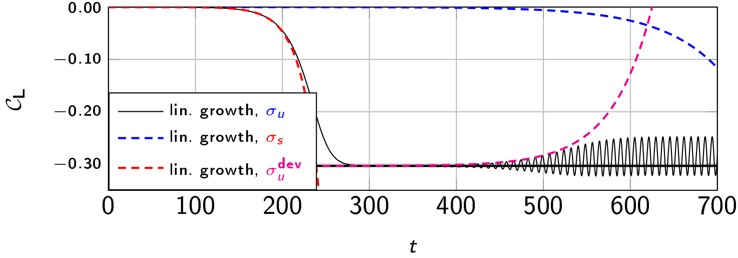
Evolution of the lift coefficient in the non-linear solution, $\mathcal{K}_\mathrm{B} = 0.015$



1. Modes σ_s and σ_u grow on the symmetric steady solution,
2. Non-linear, steady deviated solution,

Modes interaction

Evolution of the lift coefficient in the non-linear solution, $\mathcal{K}_\mathrm{B} = 0.015$



1. Modes σ_s and σ_u grow on the symmetric steady solution,
2. Non-linear, steady deviated solution,
3. The mode σ_u^{dev} develops on the deviated solution.

Conclusion

- Linearized fluid/solid stability analysis can predict well the instability thresholds,
- The static fluid/solid bifurcation mechanism is a solid stiffness cancellation due to negative fluid added stiffness.

Thank you – Questions

Appendix – Parameters

Non-dimensional parameters :

$$\mathcal{R}\mathrm{e} = \frac{DU_\infty}{\nu}, \quad \mathcal{K}_\mathrm{B} = \frac{EH^3}{12\rho_f U_\infty^2 L^3}, \quad \mathcal{M}_\rho = \rho_s/\rho_f$$

In this study :

- $H/L \sim 1\%$
- $\mathcal{R}\mathrm{e} \sim 100$
- $\mathcal{K}_\mathrm{B} \sim 10^{-3} - 1$
- $\mathcal{M}_\rho = 1$

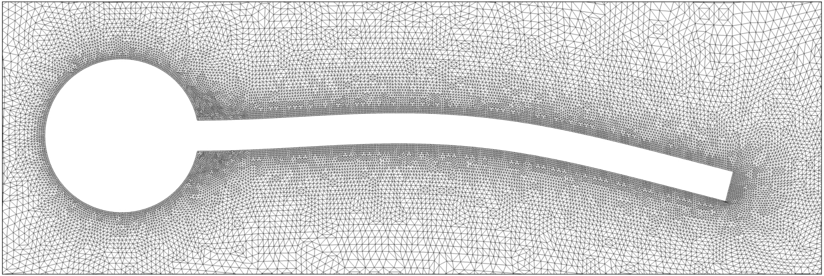
Corresponding values in the real world (e.g. in a water chanel) :

- $D \sim 1$ mm, $U_\infty \sim 0.1$ m/s, $E \sim 1 - 100$ MPa (rubber)

$\rightarrow$ Haemodynamics, small water-swimmers,...

Appendix – Spatial discretization

- Domain $x/D \in [-20, 50]$, $y/D \in [-20, 20]$
- Finite elements P2 for velocity/displacement, P1 for the pressure
- Conformal mesh with $\sim$ 30k elements $\rightarrow$ 200k d.o.f. in the stability problem



Appendix – Non-linear ALE system

Local equations[10]

$$\mathrm{J}\frac{\partial \mathbf{u}}{\partial t} + \nabla \mathbf{u} \mathbf{\Phi} \left(\mathbf{u} - \frac{\partial \boldsymbol{\xi}}{\partial t} \right) - \nabla \cdot \mathbf{\Sigma}^f = 0, \qquad \text{fluid momentum eq.}$$

$$\nabla \mathbf{u} : \mathbf{\Phi}^\mathsf{T} = 0, \qquad \text{fluid continuity eq.}$$

$$\nabla^2 \boldsymbol{\xi} = 0, \qquad \text{extension eq.}$$

$$\mathcal{M}_\rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} - \nabla \cdot \mathbf{\Sigma}^s = 0 \qquad \text{solid momentum eq.}$$

Constitutive relations

$$\mathbf{\Sigma}^f = \left(-p\mathbf{I} + \frac{1}{\mathcal{R}e} \frac{1}{\mathrm{J}} \left(\nabla \mathbf{u} \mathbf{\Phi} + \mathbf{\Phi}^\mathsf{T} \nabla \mathbf{u}^\mathsf{T} \right) \right) \mathbf{\Phi}^\mathsf{T}, \qquad \text{viscous fluid}$$

$$\mathbf{\Sigma}^s = \lambda \operatorname{tr} \left(\mathbf{F}^\mathsf{T}\mathbf{F} - \mathbf{I} \right) \mathbf{I} + 2\mu \left(\mathbf{F}^\mathsf{T}\mathbf{F} - \mathbf{I} \right) \qquad \text{Saint-Venant Kirchhoff solid}$$

Interface conditions : stress, displacement and velocity continuity.

10. $\mathbf{\Phi} = \mathrm{J}\,\mathbf{F}^{-1}$, $\mathbf{F} = \mathbf{I} + \nabla \boldsymbol{\xi}$, $\mathrm{J} = \det \mathbf{F}$ (geometric transformation operators)

Appendix – Stationary solver

Problem : Solve the stationary *non-linear* system :

$$\mathcal{R}(\mathbf{q}) = 0.$$

Approach : Newton method $\rightarrow$ solve successively *linear* systems :

$$\left.\frac{\partial \mathcal{R}}{\partial \mathbf{q}}\right|_{\mathbf{q}^k} \left(\mathbf{q}^{k+1} - \mathbf{q}^k\right) = -\mathcal{R}(\mathbf{q}^k).$$

- Monolithic approach : the fluid, solid and extension equation are solved simultaneously.
- *Exact Jacobian* implementation : an analytical derivation of $\partial\mathcal{R}/\partial\mathbf{q}$ is done rather than finite-differences $\rightarrow$ gain in convergence.
- Direct solver `MUMPS` for linear systems.

Appendix – Eigenvalue solver

Problem : Solve the eigenvalue problem :

$$\sigma \mathcal{B}\hat{\mathbf{q}} - \left.\frac{\partial \mathcal{R}}{\partial \mathbf{q}}\right|_{\mathbf{q}_b} \hat{\mathbf{q}} = 0$$

Unknowns (on the whole domain) in $\mathbf{q}$: 2×velocity + 4×displacement + 4×displacement velocity (fluid/solid, overlapping) + 1×pressure + interface Lagrange multipliers → symmetric $\mathcal{B}$ matrix.

Procedures :

- Exact Jacobian implementation,
- Shift-and-invert & ARNOLDI method,
- MUMPS solver for LU matrix decomposition,
- ARPACK interface in `FreeFem++`.

Appendix – DNS

Approach : ALE equations in reference conf. + monolithic weak form [11] :

- \+ No mesh-moving procedure (except for post-treatments)
- \+ No numerical added-mass effect
- – Strongly non-linear

Procedure : At each time-step n the fully implicit problem

$$\left.\frac{\partial \mathbf{q}}{\partial t}\right|^{n+1} + \mathcal{R}(\mathbf{q}^{n+1}, \mathbf{q}^{n}, \ldots) = 0$$

is solved using `FreeFem++`, with

- a (evtl. shifted) Crank-Nicholson time-discretization scheme (2[nd] order),
- a Newton method + exact Jacobian expressions,
- parallel matrix assembly & MUMPS direct solver.

Validated by comparison with popular FSI benchmarks (Turek & Hron)

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11. Richter, T. et Wick, T. *Finite elements for fluid-structure interaction in ALE and fully Eulerian coordinates* Comp. Meth. in Appl. Mech. Eng., 2010