Necessary and sufficient condition for stabilizability of discrete-time linear switched systems: A set-theory approach

Mirko Fiacchini[1] and Marc Jungers[2]

[1]GIPSA-lab, Grenoble. mirko.fiacchini@gipsa-lab.fr
[2]CRAN, Nancy. marc.jungers@univ-lorraine.fr

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Outline

1. Why set-theory for control?
2. Stabilizability of DT linear switched systems

1. Why set-theory for control?
2. Stabilizability of DT linear switched systems

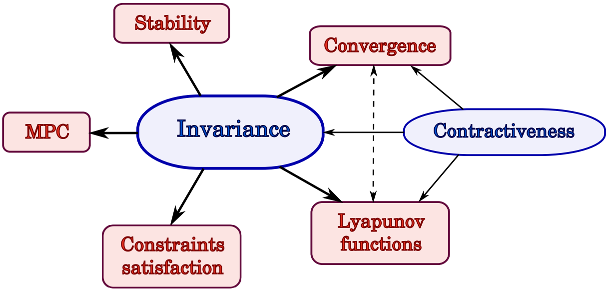
Set-theory and invariance for complex systems

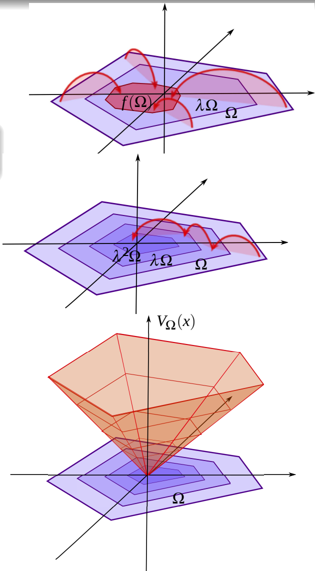
Set-theory: techniques concerning properties shared by all the elements of sets of the state space.

Invariance

Set $\Omega \subseteq \mathbb{R}^n$ is invariant if every trajectory with $x_0 \in \Omega$ stays in Ω.

Geometric condition: $\Omega \subseteq \mathbb{R}^n$ invariant iff $f(\Omega) \subseteq \Omega$.

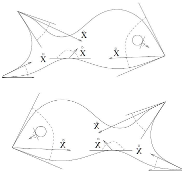




Set-theory and invariance for complex systems

For linear systems:

- well established theoretical and computational results,
- iterative procedures (mainly for discrete-time systems),
- boundary-type condition for invariance, also for discrete-time systems,
- set-induced Lyapunov functions,
- computationally suitable methods: convex analysis, optimization, LMI.



Problem

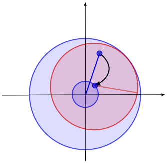
When moving from linear systems, useful properties related to linearity are lost $\Rightarrow$ adaptation of tools for linear systems to more complex systems is not trivial.

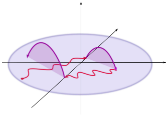
Objective

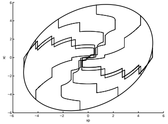
Extend and apply set-theory and invariance to complex (nonlinear, hybrid, interconnected, saturated, etc) systems.

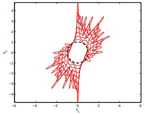
Why set-theory and invariance?

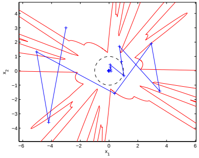
- Nice.
- Computationally-oriented $\Rightarrow$ useful.
- Intuitive.
- Different point of view on the problems $\Rightarrow$ original.

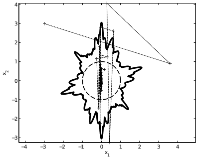


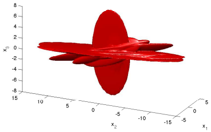












Stabilizability of DT linear switched systems

Joint work with Marc Jungers (CNRS researcher at CRAN, Nancy).

Discrete-time autonomous switched system

$$x_{k+1} = A_{\sigma(k)} x_k,$$

where $\sigma : \mathbb{N} \to \mathbb{N}_q$ selects the transition matrix $\{A_i\}_{i \in \mathbb{N}_q}$, and can be considered as:

- a perturbation: necessary and sufficient condition for asymptotic stability; existence of a polyhedral Lyapunov function *(Molchanov & Pyatnitskiy, SCL89; Blanchini, AUT95)*,
- or as a control input: sufficient condition for stabilizability, Lyapunov-Metzler inequality *(Geromel & Colanieri, IJC06)*.

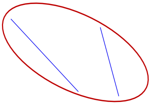
Open problem: necessary and sufficient condition for the stabilizability of switched linear systems, *(Lin & Antsaklis, TAC08)*.

Objectives and contributions *(F. & Jungers, IFAC13, AUT13)*:

- provide necessary and sufficient condition for stabilizability,
- set-theory and invariance based results,
- computational espects: algorithmic test,
- nonconvex control Lyapunov functions,
- highlight the duality with the perturbation case,
- characterize the class of stabilizing controls.

Preliminiaries

A C-set is a compact, convex set containing the origin in its interior.

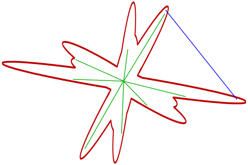


Definition

A set $\Omega \subseteq \mathbb{R}^n$ is a C*-set if it is compact, star-convex with respect to the origin and $0 \in \text{int}(\Omega)$.

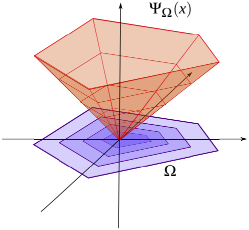
Notice a set is

- convex if $\forall x_0 \in \Omega$ and $\forall x \in \Omega$, then $\alpha x_0 + (1-\alpha)x \in \Omega,\ \forall \alpha \in [0,1]$.
- star-convex if $\exists x_0 \in \Omega$, such that $\forall x \in \Omega$, then $\alpha x_0 + (1-\alpha)x \in \Omega, \forall \alpha \in [0,1]$.



Minkowski function of a C*-set Ω: $\Psi_\Omega(x) = \min_\alpha \{\alpha \in \mathbb{R} : x \in \alpha\Omega\}$.

- Any C-set is a C*-set.
- Given a C*-set Ω, we have that $\alpha\Omega$ is a C*-set and $\alpha\Omega \subseteq \Omega$ for all $\alpha \in [0,1]$.
- $\Psi_\Omega(\cdot)$ is: defined on $\mathbb{R}^n$; homogenous of degree one; positive definite and radially unbounded. But nonconvex in general!



Theorem (*Blanchini, AUT95*)

There exists a Lyapunov function for the perturbed system if and only if there exists a C-set $\hat{\Omega}$ and a scalar $\lambda \in [0,1)$ such that $A_i\hat{\Omega} \subseteq \lambda\hat{\Omega}$, for all $i \in \mathbb{N}_q$.

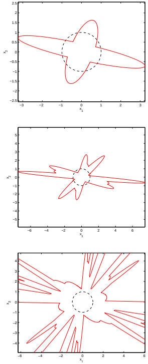
Idea: look for a C*-set whose Minkowski function is a control Lyapunov function.

Necessary and sufficient condition for stabilizability

Algorithm 1

Control λ-contractive C*-set for the switched system.

- **Initialization**: given the C*-set $\Omega \subseteq \mathbb{R}^n$, define $\Omega_0 = \Omega$ and $k = 0$;
- **Iteration** for $k \geq 0$:
$$\Omega^i_{k+1} = A_i^{-1}\Omega_k, \quad \forall i \in \mathbb{N}_q,$$
$$\Omega_{k+1} = \bigcup_{i \in \mathbb{N}_q} \Omega^i_{k+1};$$
- **Stop** if $\Omega \subseteq \text{int}\Big(\bigcup_{j \in \mathbb{N}_{k+1}} \Omega_j\Big)$; denote $\check{N} = k+1$ and $\check{\Omega} = \bigcup_{j \in \mathbb{N}_{\check{N}}} \Omega_j$.



Geometrical interpretation:

- the set Ω^i_k is the set of x that can be stirred in Ω in k steps by a switching sequence beginning with $i \in \mathbb{N}_q$;
- then Ω_k is the set of points that can be driven in Ω in k steps;
- and hence $\check{\Omega}$ the set of those which can reach Ω in $\check{N}$ or less steps, by an adequate switching law.

Necessary and sufficient condition for stabilizability.

Theorem

There exists a control Lyapunov function for the switched system if and only if the Algorithm 1 ends with finite $\check{N}$.

Stabilizing switching control law

Proposition

If Algorithm 1 ends with finite $\check{N}$ then $\Psi_{\check{\Omega}}(x)$ is a global control Lyapunov function and given the set-valued map

$$\check{\Sigma}(x) = \arg\min_{(i,k)}\{\Psi_{\Omega_k^i}(x) : i \in \mathbb{N}_q, k \in \mathbb{N}_{\check{N}}\} \subseteq \mathbb{N}_q \times \mathbb{N}_{\check{N}},$$

with $\check{\lambda} = \check{\lambda}(\Omega) = \min_{\lambda}\{\lambda \geq 0 : \Omega \subseteq \lambda\check{\Omega}\}$, then any switching law defined as $(\check{\sigma}(x), \check{k}(x)) \in \check{\Sigma}(x)$ is stabilizing and

$$\begin{cases} \Psi_{\check{\Omega}}(x_j^{\check{\sigma}}(x)) \leq \Psi_{\check{\Omega}}(x), & \forall j \in \mathbb{N}_{\check{k}(x)}, \\ \Psi_{\check{\Omega}}(x_{\check{k}(x)}^{\check{\sigma}}(x)) \leq \check{\lambda}\Psi_{\check{\Omega}}(x). \end{cases}$$

Corollary

If the Algorithm 1 ends with finite $\check{N}$ then the switching law is such that $\Psi_{\check{\Omega}}(x_{p\check{N}}^{\check{\sigma}}(x)) \leq \check{\lambda}^p \Psi_{\check{\Omega}}(x)$, for all $p \in \mathbb{N}$ and $x \in \mathbb{R}^n$.

- If the system is asymptotically stabilizable, then the algorithm ends with finite $\check{N}$ for all initial C*-set Ω.
- The value of $\check{N}$ and the complexity of the set $\check{\Omega}$ depends on the choice of Ω. But...
- If Ω is a (union of) ellipsoid $\Rightarrow$ also Ω_k^i, Ω_k and $\check{\Omega}$ are union of ellipsoids $\Rightarrow$ the switching law consists in finding the minimal $x^T P_j x$ with $j \in \check{M} = (q^{\check{N}+1} - q)/(q-1)$.
- If Ω is a (union of) polytope $\Rightarrow$ also Ω_k^i, Ω_k and $\check{\Omega}$ are union of polytopes $\Rightarrow$ the switching law consists in checking linear equalities.

Robustness-control duality

Uncertain linear systems

Robust λ-contractive C-set for an uncertain system.

- **Initialization**: given the C-set $\Gamma \subseteq \mathbb{R}^n$ and $\lambda \in [0,1]$, define $\Gamma_0 = \Gamma$ and $k = 0$;
- **Iteration** for $k \geq 0$:

$$\begin{aligned} \Gamma_{k+1}^i &= \lambda A_i^{-1} \Gamma_k, \quad \forall i \in \mathbb{N}_q, \\ \Gamma_{k+1} &= \Gamma \cap \bigcap_{i \in \mathbb{N}_q} \Gamma_{k+1}^i; \end{aligned}$$

- **Stop** if $\Gamma_k \subseteq \Gamma_{k+1}$; denote $\hat{N} = k$ and $\hat{\Gamma} = \Gamma_k$.

Theorem (*Blanchini, AUT95*)

There is a Lyapunov function for the parametric uncertain linear system if and only if there exists a polyhedral Lyapunov function for the system.

Then, the family of convex, homogeneous functions induced by a C-set are a class of universal Lyapunov functions for parametric uncertain linear systems.

Switched linear systems

Control λ-contractive C*-set for the switched system.

- **Initialization**: given the C*-set $\Omega \subseteq \mathbb{R}^n$, define $\Omega_0 = \Omega$ and $k = 0$;
- **Iteration** for $k \geq 0$:

$$\begin{aligned} \Omega_{k+1}^i &= A_i^{-1} \Omega_k, \quad \forall i \in \mathbb{N}_q, \\ \Omega_{k+1} &= \bigcup_{i \in \mathbb{N}_q} \Omega_{k+1}^i; \end{aligned}$$

- **Stop** if $\Omega \subseteq \operatorname{int}\Big(\bigcup_{j \in \mathbb{N}_{k+1}} \Omega_j \Big)$; denote $\check{N} = k+1$ and

$$\check{\Omega} = \bigcup_{j \in \mathbb{N}_{\check{N}}} \Omega_j .$$

Theorem (*F. & Jungers, AUT13*)

There exists a control Lyapunov function for the switched linear system if and only if the Algorithm ends with finite $\check{N}$.

Then, the family of nonconvex, homogeneous functions induced by a C*–set are a class of universal Lyapunov functions for switched systems.

Sufficient condition for non-stabilizability

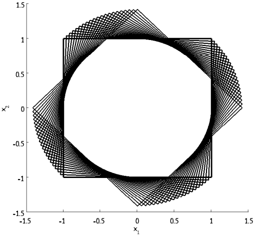
Algorithm 2

Non-stabilizability test for the switched system.

- **Initialization**: given the C*-set $\Omega \subseteq \mathbb{R}^n$, define $\Omega_0 = \Omega$ and $k = 0$;
- **Iteration** for $k \geq 0$ get Ω^i_{k+1} and Ω_{k+1} as above and define:

$$\hat{\Omega}_{k+1} = \Big(\bigcup_{j \in \mathbb{N}_{k+1}} \Omega_j \Big) \cup \Omega.$$

- **Stop** if $\Omega_{k+1} \subseteq \hat{\Omega}_k$; denote $\hat{N} = k$ and $\hat{\Omega} = \hat{\Omega}_{\hat{N}}$.



Geometrical interpretation:

- if the new set Ω_{k+1} is contained in the union of the former ones and the initial set Ω, then the following sets will not increase $\Rightarrow$ non-stabilizable.

Sufficient condition for non-stabilizability.

Theorem

If the Algorithm 2 ends with finite $\hat{N}$ then there is no switching law stabilizing the switched system.

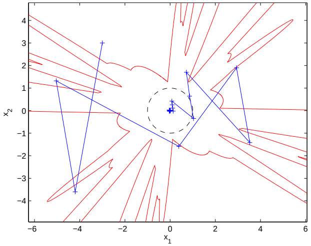
If the system is not stabilizable the algorithm can terminate of not.

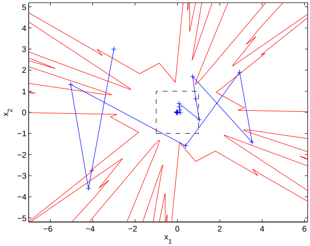
Example Single mode linear system with $A_1 = R(\beta\pi)$ with $R(\beta\pi)$ rotation matrix, $\beta \in \mathbb{R} \setminus \mathbb{Q}$ and $\beta \in (0,\ 0.5)$.

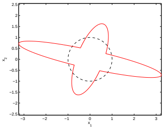
Example 1

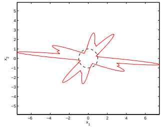
Non-Schur switched system with $q = n = 2$.

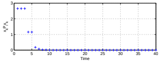
$$A_1 = \begin{bmatrix} 1.2 & 0 \\ -1 & 0.8 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.6 & -2 \\ 0 & -1.2 \end{bmatrix},$$

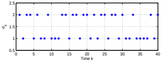










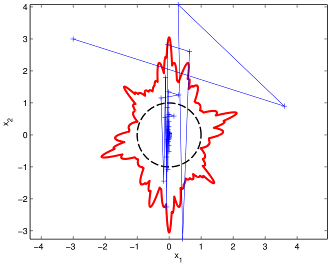


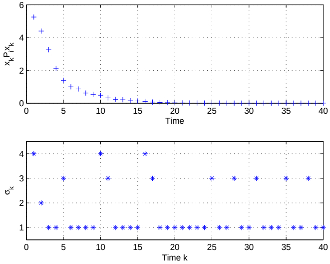
Example 2

System with $q = 4$, $n = 2$ and

$$A_1 = \begin{bmatrix} 1.5 & 0 \\ 0 & -0.8 \end{bmatrix}, \qquad A_2 = 1.1 R(\tfrac{2\pi}{5})$$
$$A_3 = 1.05 R(\tfrac{2\pi}{5} - 1), \qquad A_4 = \begin{bmatrix} -1.2 & 0 \\ 1 & 1.3 \end{bmatrix}.$$

The matrices A_i, with $i \in \mathbb{N}_4$, are not Schur. Notice: only one stable eigenvalue!





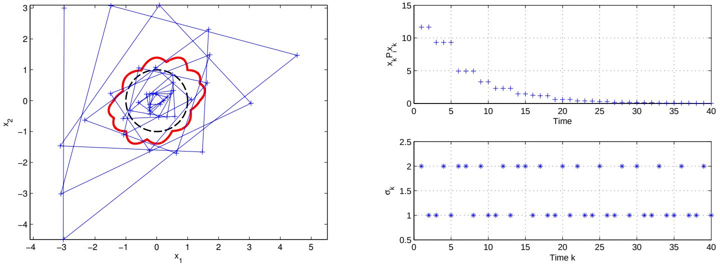
Example 3

Switched system with

$$A_1 = \begin{bmatrix} 0 & -1.01 \\ 1 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & -1.01 \\ 1 & -0.5 \end{bmatrix}.$$

The product of the eigenvalues of every convex combination of the matrices is always 1.01 and the technique based on Lyapunov-Metzler inequalities *(Geromel & Colanieri, IJC06)* is NOT applicable.

Nevertheless...

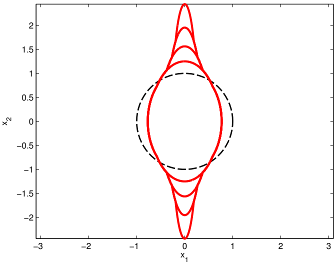


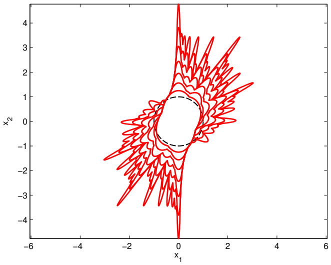
Example 4

Switched system with

$$A_1 = \begin{bmatrix} 1.3 & 0 \\ 0 & 0.9 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1.4 & 0 \\ 0 & 0.8 \end{bmatrix},$$

for $\theta = 0$ (left) and $\theta = \frac{\pi}{5}$ (right).





Example 5

Sufficient condition for non-stabilizability.

Theorem (*F. & Jungers, AUT13*)

If

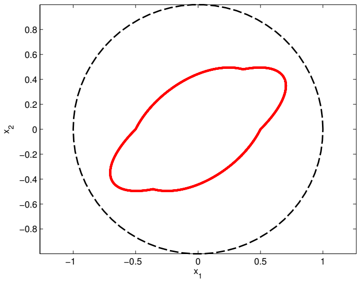
$$\Omega_{k+1} = ri \bigcup_{i \in \mathbb{N}_q} A_i^{-1} \Omega_k \subseteq \bigcup_{j \in \mathbb{N}_k} \Omega_j \cup \Omega,$$

then there is no switching law stabilizing the switched system.

Consider

$$A_1 = 2 \begin{bmatrix} 0 & -1.01 \\ 1 & -1 \end{bmatrix}, \quad A_2 = 2 \begin{bmatrix} 0 & -1.01 \\ 1 & -0.5 \end{bmatrix}.$$

The criterion is attained in only one step, then the system is not stabilizable.

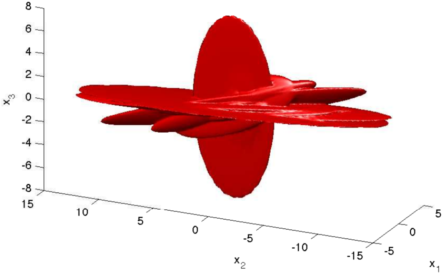


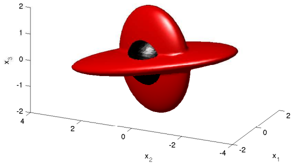
Example 6

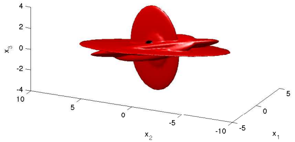
Switched system with $q = 2$, $n = 3$ and

$$A_1 = \begin{bmatrix} 1.2 & 0 & 0 \\ -1 & 0.8 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.7 & 0 & 0 \\ 0 & -0.6 & -2 \\ 0 & 0 & -1.2 \end{bmatrix}.$$

A_1 and A_2 are not Schur. The ball $\mathbb{B}^3$ is chosen as initial set.







Conclusions

Results:

- necessary and sufficient condition for the stabilizability of discrete-time linear switched systems;
- constructive method based on set-theory: nonconvex control Lyapunov functions;
- computational approach: iterative algorithm;
- evident duality: robustness-control, for all-existence, intersection-union, C-set-C*-set...
- characterize "non-stabilizability".

Open problems and future works:

- complexity analisys and computational issues;
- more genaral cases: nonautonomous, nonlinear switched systems,...

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