Approximate Dynamic Programming meets Statistical Learning Theory

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Journées "Méthodologies pour le contrôle de systèmes complexes"

INRIA Lille – Team SequeL

November 17, 2014

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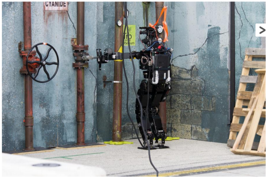

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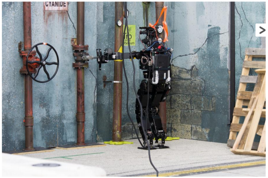

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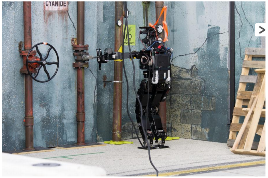


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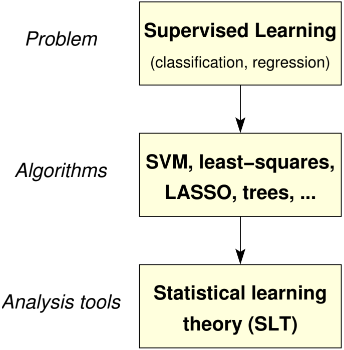

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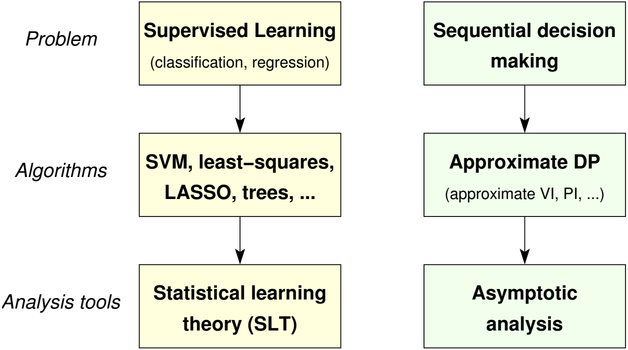

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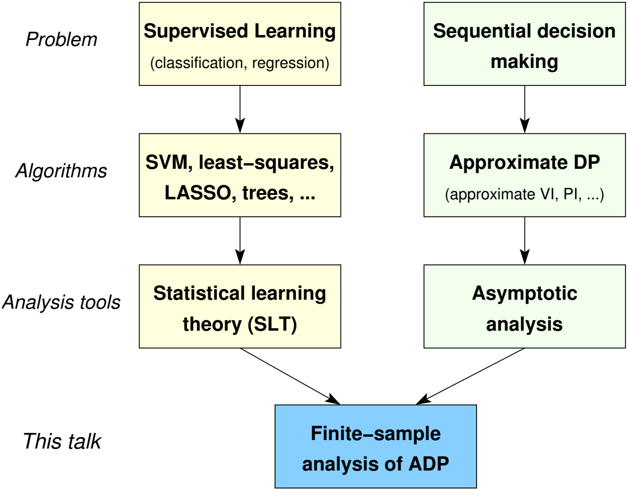

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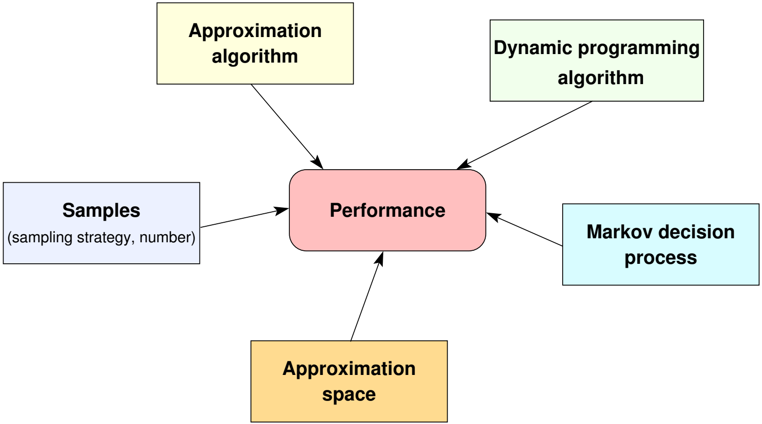

Outline

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Preliminaries

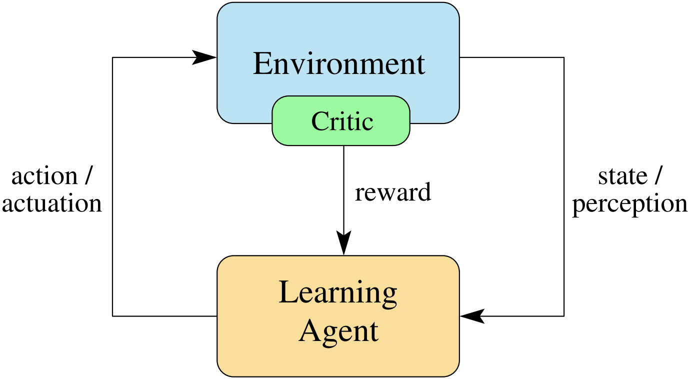
Approximate Dynamic Programming

Linear Fitted Q-Iteration

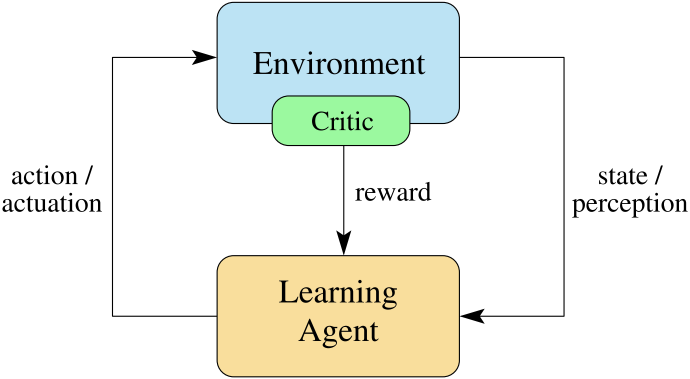
Least-Squares Policy Iteration (LSPI)

Discussion

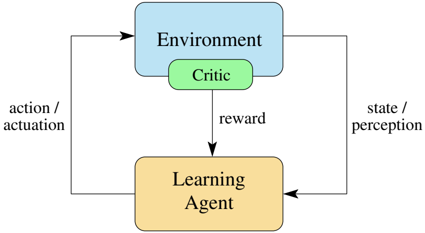
Markov Decision Process



The Reinforcement Learning Model



The Reinforcement Learning Model (Glossary)



- *Environment* = system to control
- *Agent* = controller
- *State* = fully observable state
- *Action* = control
- *Reward* = cost

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- *$p(y|x, a)$ is the (stationary and Markov) transition probability*

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Definition (Policy)

*A **policy** is a (stationary and deterministic) mapping*

$$\pi : X \to A$$

Infinite Time Horizon with Discount

Definition (Value functions)

For any policy π, the (action-) state value function $V^\pi : X \mapsto \mathbb{R}$ ($Q^\pi : X \times A \mapsto \mathbb{R}$) is

$$V^\pi(x) = \mathbb{E}\Big[\sum_{t=0}^{\infty} \gamma^t r(x_t, \pi(x_t)) \,|\, x_0 = x; \pi\Big]$$

$$Q^\pi(x, a) = \mathbb{E}\Big[\sum_{t=0}^{\infty} \gamma^t r(x_t, a_t) | x_0 = x, a_0 = a, a_t = \pi(x_t), \forall t \geq 1\Big]$$

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Definition (Optimal policy and optimal value function)

The solution to an MDP is an optimal policy π^* satisfying

$$\pi^* \in \arg\max_{\pi \in \Pi} V^\pi$$

and its value function is the optimal value function $V^* = V^{\pi^*}$.

The Bellman Operators

Notation. w.l.o.g. a discrete state space $|X| = N$ and $V^\pi \in \mathbb{R}^N$.

Definition

For any $W \in \mathbb{R}^N$, the Bellman operator $\mathcal{T}^\pi : \mathbb{R}^N \to \mathbb{R}^N$ is

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With abuse of notation $\mathcal{T}^\pi$ and $\mathcal{T}$ will be used for Q-functions as well.

The Bellman Operators

The Bellman operators are *γ-Contraction in L_∞-norm*: for any $W_1, W_2 \in \mathbb{R}^N$

$$\begin{aligned} \|\mathcal{T}^\pi W_1 - \mathcal{T}^\pi W_2\|_\infty &\leq \gamma \|W_1 - W_2\|_\infty, \\ \|\mathcal{T} W_1 - \mathcal{T} W_2\|_\infty &\leq \gamma \|W_1 - W_2\|_\infty. \end{aligned}$$

Thus

V^π is the *unique fixed point* of $\mathcal{T}^\pi$,

V^* is the *unique fixed point* of $\mathcal{T}$.

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From the *contraction* property of $\mathcal{T}$

$$\begin{aligned} ||Q_{k+1} - Q^*||_\infty = ||\mathcal{T} Q_k - \mathcal{T} Q^*||_\infty &\leq \gamma ||Q_k - Q^*||_\infty \\ &\leq \gamma^{k+1} ||Q_0 - Q^*||_\infty \rightarrow 0 \end{aligned}$$

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From the Bellman operators

$$V^{\pi_{k+1}} \geq V^{\pi_k}$$

Limitation of Exact Dynamic Programming

Dynamic programming algorithms require

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Approximate DP relaxes these requirements by using

- ***Samples*** $\{x_i, a_i, x'_i, r_i\}_i$ obtained from a generative model (e.g., a simulator) of the MDP,
- An ***approximation space*** $\mathcal{F} = \{f : X \times A \to \mathbb{R}\}$ to approximate Q-functions.

Approximate Value Iteration

Input: *samples* $\mathcal{D} = \{x_i, a_i, x'_i, r_i\}_{i=1}^n$, *approximation space* $\mathcal{F}$

1. Let Q_0 be any Q-function
2. At each iteration $k = 1, 2, \ldots, K$
   - Compute $\widehat{Q}_{k+1} \approx \mathcal{T} Q_k$ (using $\mathcal{D}$ and $\mathcal{F}$)
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Problem: $||Q^* - \widehat{Q}_{k+1}|| \overset{?}{\leq} \gamma ||Q^* - Q_k||$

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Problem: $V^{\pi_k} \overset{?}{\geq} V^{\pi_{k-1}}$

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Question: what is the performance of the policy π_K returned by an ADP algorithm?

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Statistical Learning Theory in ADP

Solution:

- supervised learning methods (regression, classification) appear in the inner-loop of ADP algorithms
- SLT tools used to analyze supervised learning methods can be used in ADP!

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The specific nature of ADP makes things not so straightforward...

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Linear Fitted Q-iteration

Linear space (used to approximate action–value functions)

$$\mathcal{F} = \Big\{ f(x,a) = \sum_{j=1}^{d} \alpha_j \varphi_j(x,a), \ \ \alpha \in \mathbb{R}^d \Big\}$$

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with features

$$\varphi_j : \mathcal{X} \times \mathcal{A} \to [0, L] \qquad \phi(x,a) = [\varphi_1(x,a) \ldots \varphi_d(x,a)]^\top$$

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$$f_{\widehat{\alpha}_k} = \arg\min_{f_\alpha \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^n \big(f_\alpha(x_i, a_i) - y_i\big)^2$$

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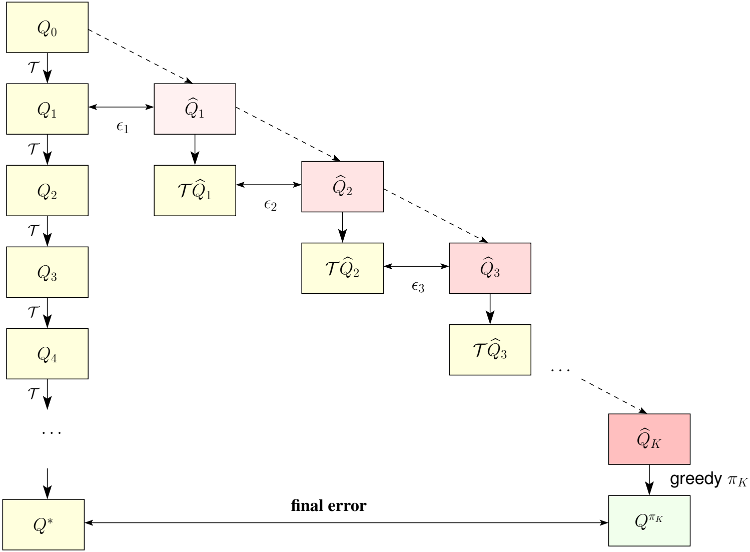
Return $\pi_K(\cdot) = \arg\max_a \widehat{Q}_K(\cdot, a)$ (*greedy policy*)

Sketch of the Analysis



Theoretical Objectives

Objective: derive a bound on the performance (*quadratic*) loss w.r.t. a *testing* distribution μ

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Sub-Objective 2: analyze how the *error at each iteration* is *propagated* through iterations

$$||Q^* - Q^{\pi_K}||_\mu \leq \textit{propagation}(||\mathcal{T}\widehat{Q}_{k-1} - \widehat{Q}_k||_\rho)$$

The Sources of Error

- *Desired* solution

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$$f_{\alpha_k^*} = \arg \inf_{f_\alpha \in \mathcal{F}} ||f_\alpha - Q_k||_\rho$$

$\Rightarrow$ ***Error*** from the approximation space $\mathcal{F}$

The Sources of Error

- *Desired* solution

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$\Rightarrow$ ***Error*** from the (random) samples

Per-Iteration Error

Theorem

*At each iteration k, Linear-FQI returns an approximation $\widehat{Q}_k$ such that (**Sub-Objective 1**)*

$$
\begin{aligned}
||Q_k - \widehat{Q}_k||_\rho \leq 4||Q_k - f_{\alpha_k^*}||_\rho & \\
& + O\left(\left(V_{\max} + L||\alpha_k^*||\right)\sqrt{\frac{\log 1/\delta}{n}}\right) \\
& + O\left(V_{\max}\sqrt{\frac{d \log n/\delta}{n}}\right),
\end{aligned}
$$

with probability $1-\delta$.

Tools: concentration of measure inequalities, covering space, linear algebra, union bounds, special tricks for linear spaces, ...

Per-Iteration Error

$$
\begin{aligned}
||Q_k - \widehat{Q}_k||_\rho \leq 4||Q_k - f_{\alpha_k^*}||_\rho \\
&+ O\left(\left(V_{\max} + L||\alpha_k^*||\right)\sqrt{\frac{\log 1/\delta}{n}}\right) \\
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Per-Iteration Error

$$\begin{aligned}||Q_k - \widehat{Q}_k||_\rho \leq\ & 4||Q_k - f_{\alpha_k^*}||_\rho \\ & + O\Big(\big(V_{\max} + L||\alpha_k^*||\big)\sqrt{\frac{\log 1/\delta}{n}}\Big) \\ & + O\Big(V_{\max}\sqrt{\frac{d\log n/\delta}{n}}\Big)\end{aligned}$$

Remarks

- No algorithm can do better
- Constant 4
- Depends on the space $\mathcal{F}$
- Changes with the iteration k

Per-Iteration Error

$$||Q_k - \widehat{Q}_k||_\rho \leq 4||Q_k - f_{\alpha_k^*}||_\rho + O\bigg(\big(V_{\max} + L||\alpha_k^*||\big)\sqrt{\frac{\log 1/\delta}{n}}\bigg) + O\bigg(V_{\max}\sqrt{\frac{d \log n/\delta}{n}}\bigg)$$

Remarks

- Vanishing to zero as $O(n^{-1/2})$
- Depends on the features (L) and on the best solution ($||\alpha_k^*||$)

Per-Iteration Error

$$
\begin{aligned}
||Q_k - \widehat{Q}_k||_\rho \leq 4||Q_k - f_{\alpha_k^*}||_\rho \\
&+ O\bigg((V_{\max} + L||\alpha_k^*||) \sqrt{\frac{\log 1/\delta}{n}} \bigg) \\
&+ O\bigg(V_{\max} \sqrt{\frac{d \log n/\delta}{n}} \bigg)
\end{aligned}
$$

Remarks

- Vanishing to zero as $O(n^{-1/2})$
- Depends on the dimensionality of the space (d) and the number of samples (n)

Error Propagation

Objective

$$||Q^* - Q^{\pi_K}||_\mu$$

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Objective

$$||Q^* - Q^{\pi_K}||_\mu$$

- **Problem 1**: the test norm μ is different from the sampling norm ρ
- **Problem 2**: we have bounds for $\widehat{Q}_k$ not for the performance of the corresponding π_k
- **Problem 3**: we have bounds for one single iteration

Error Propagation

Transition kernel for a fixed policy P^π.

- m-step (worst-case) concentration of future state distribution

$$c(m) = \sup_{\pi_1 \ldots \pi_m} \left\| \frac{d(\mu P^{\pi_1} \ldots P^{\pi_m})}{d\rho} \right\|_\infty < \infty$$

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- Average (discounted) concentration

$$C_{\mu,\rho} = (1-\gamma)^2 \sum_{m \geq 1} m \gamma^{m-1} c(m) < +\infty$$

Error Propagation

Remark: relationship to top-Lyapunov exponent

$$L^+ = \sup_\pi \lim \sup_{m \to \infty} \frac{1}{m} \log^+ \left(||\rho P^{\pi_1} P^{\pi_2} \cdots P^{\pi_m}|| \right)$$

If $L^+ \leq 0$ (*stable system*), then $c(m)$ has a growth rate which is polynomial and $C_{\mu,\rho} < \infty$ is ***finite***

Error Propagation

Proposition

Let $\varepsilon_k = Q_k - \widehat{Q}_k$ *be the propagation error at each iteration, then after* K *iteration the* performance loss *of the greedy policy* π_K *is*

$$||Q^* - Q^{\pi_K}||_\mu^2 \leq \left[\frac{2\gamma}{(1-\gamma)^2}\right]^2 C_{\mu,\rho} \max_k ||\varepsilon_k||_\rho^2 + O\left(\frac{\gamma^K}{(1-\gamma)^3} V_{\max}^2\right)$$

The Final Bound

Bringing everything together...

$$||Q^* - Q^{\pi_K}||_\mu^2 \leq \left[\frac{2\gamma}{(1-\gamma)^2}\right]^2 C_{\mu,\rho} \max_k ||\varepsilon_k||_\rho^2 + O\left(\frac{\gamma^K}{(1-\gamma)^3} V_{\max}^2\right)$$

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$$\begin{aligned} ||\varepsilon_k||_\rho = ||Q_k - \widehat{Q}_k||_\rho \leq 4||Q_k - f_{\alpha_k^*}||_\rho & \\ + O\left(\left(V_{\max} + L||\alpha_k^*||\right)\sqrt{\frac{\log 1/\delta}{n}}\right) & \\ + O\left(V_{\max}\sqrt{\frac{d \log n/\delta}{n}}\right) & \end{aligned}$$

The Final Bound

Theorem (see e.g., Munos,'03)

LinearFQI with a space $\mathcal{F}$ of d features, with n samples at each iteration returns a policy π_K after K iterations such that

$$||Q^* - Q^{\pi_K}||_\mu \leq \frac{2\gamma}{(1-\gamma)^2}\sqrt{C_{\mu,\rho}}\left(4d(\mathcal{F},\mathcal{T}\mathcal{F}) + O\left(V_{\max}\big(1+\frac{L}{\sqrt{\omega}}\big)\sqrt{\frac{d\log n/\delta}{n}}\right)\right) + O\left(\frac{\gamma^K}{(1-\gamma)^3}V_{\max}^2\right)$$

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The *propagation* (and different norms) makes the problem *more complex*
$\Rightarrow$ how do we choose the *sampling distribution*?

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The *approximation* error is *worse* than in regression

The Final Bound

The inherent Bellman error

$$
\begin{aligned}
||Q_k - f_{\alpha_k^*}||_\rho &= \inf_{f \in \mathcal{F}} ||Q_k - f||_\rho \\
&= \inf_{f \in \mathcal{F}} ||\mathcal{T}\widehat{Q}_{k-1} - f||_\rho \\
&\leq \inf_{f \in \mathcal{F}} ||\mathcal{T} f_{\alpha_{k-1}} - f||_\rho \\
&\leq \sup_{g \in \mathcal{F}} \inf_{f \in \mathcal{F}} ||\mathcal{T} g - f||_\rho = d(\mathcal{F}, \mathcal{T}\mathcal{F})
\end{aligned}
$$

Question: how to design $\mathcal{F}$ to make it "compatible" with the Bellman operator?

The Final Bound

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LinearFQI with a space $\mathcal{F}$ of d features, with n samples at each iteration returns a policy π_K after K iterations such that

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The dependency on γ is worse than at each iteration

$\Rightarrow$ is it possible to *avoid* it?

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The error decreases exponentially in K

$\Rightarrow K \approx \varepsilon/(1-\gamma)$

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Theorem

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The smallest eigenvalue of the Gram matrix

$\Rightarrow$ design the features so as to be *orthogonal* w.r.t. ρ

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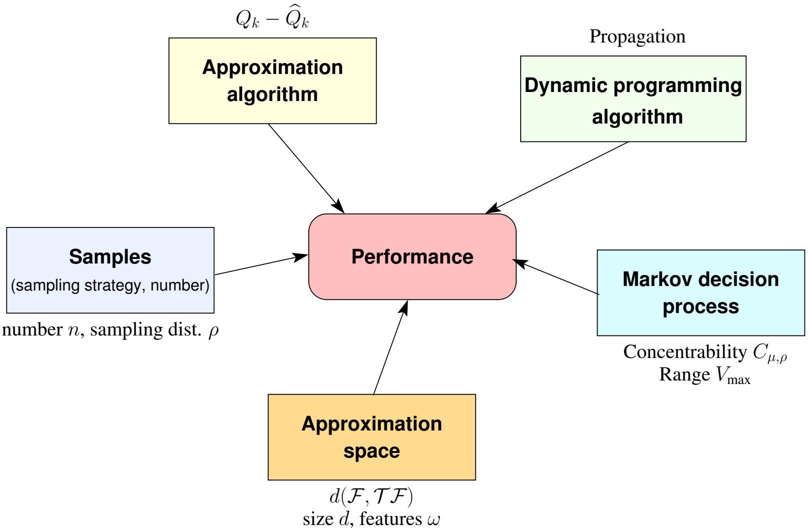
The asymptotic rate $O(d/n)$ is the same as for regression

Summary


$Q_k - \widehat{Q}_k$
Approximation algorithm
Propagation
Dynamic programming algorithm
Samples
(sampling strategy, number)
number n, sampling dist. ρ
Performance
Markov decision process
Concentrability $C_{\mu,\rho}$
Range $V_{\max}$
Approximation space
$d(\mathcal{F}, \mathcal{T}\mathcal{F})$
size d, features ω

Summary

The STL recipe for ADP:

1. Take your favorite learning algorithm.

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Not always so easy...

Outline

Least-Squares Policy Iteration (LSPI)

LSPI uses

- Linear space to approximate value functions

$$\mathcal{F} = \Big\{ f(x) = \sum_{j=1}^{d} \alpha_j \varphi_j(x), \;\; \alpha \in \mathbb{R}^d \Big\}$$

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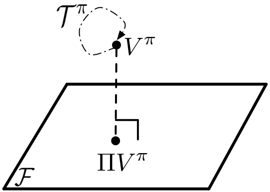
- Least-Squares Temporal Difference (LSTD) algorithm for *policy evaluation*.

Least-Squares Temporal-Difference Learning (LSTD)

- V^π is the fixed-point of $\mathcal{T}^\pi$ $\qquad V^\pi = \mathcal{T}^\pi V^\pi$
- V^π may not belong to $\mathcal{F}$ $\qquad V^\pi \notin \mathcal{F}$
- Best approximation of V^π in $\mathcal{F}$ is

$$\Pi V^\pi = \arg\min_{f \in \mathcal{F}} ||V^\pi - f||$$

(Π is the projection onto $\mathcal{F}$)



Least-Squares Temporal-Difference Learning (LSTD)

- LSTD searches for the fixed-point of $\Pi_? \mathcal{T}^\pi$ instead ($\Pi_?$ is a projection into $\mathcal{F}$ w.r.t. $L_?$-norm)

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  - L_∞-projection is numerically expensive when the number of states is large or infinite

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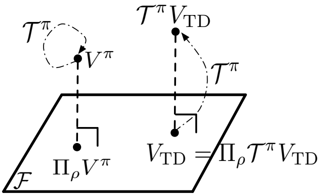
- $\Pi_\infty \mathcal{T}^\pi$ is a contraction in L_∞-norm
  - L_∞-projection is numerically expensive when the number of states is large or infinite
- LSTD searches for the fixed-point of $\Pi_{2,\rho} \mathcal{T}^\pi$

$$\Pi_{2,\rho}\, g = \arg\min_{f \in \mathcal{F}} ||g - f||_{2,\rho}$$

Least-Squares Temporal-Difference Learning (LSTD)

When the fixed-point of $\Pi_\rho \mathcal{T}^\pi$ exists, we call it the LSTD solution

$$V_{\mathsf{TD}} = \Pi_\rho \mathcal{T}^\pi V_{\mathsf{TD}}$$



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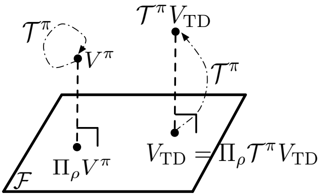

$$\langle \mathcal{T}^\pi V_{\mathsf{TD}} - V_{\mathsf{TD}}, \varphi_i \rangle_\rho = 0, \qquad i = 1, \ldots, d$$

$$\underbrace{\langle r^\pi, \varphi_i \rangle_\rho}_{b_i} - \sum_{i=1}^{d} \underbrace{\langle \varphi_j - \gamma P^\pi \varphi_j, \varphi_i \rangle_\rho}_{A_{ij}} \cdot \alpha_{\mathsf{TD}}^{(j)} = 0 \quad \Longrightarrow \quad A\, \alpha_{\mathsf{TD}} = b$$

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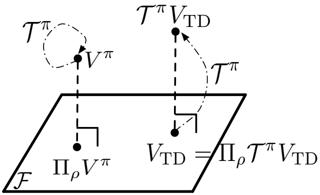

- **Problem:** In general, $\Pi_\rho \mathcal{T}^\pi$ is ***not a contraction*** and does not have a fixed-point.
- **Solution:** If $\rho = \rho^\pi$ (*stationary dist. of π*) then $\Pi_{\rho^\pi} \mathcal{T}^\pi$ has a unique fixed-point.

Least-Squares Temporal-Difference Learning (LSTD)

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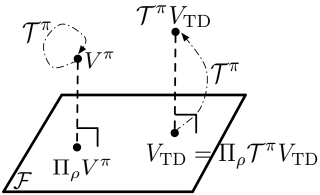

- **Problem:** In general, $\Pi_\rho \mathcal{T}^\pi$ cannot be computed (because ***unknown***)
- **Solution:** Use *samples* coming from a "trajectory" of π.

Least-Squares Policy Iteration (LSPI)

Input: space $\mathcal{F}$, iterations K, sampling distribution ρ, num of samples n

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For $k = 1, \ldots, K$

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For $k = 1, \ldots, K$

- Generate a trajectory of length n from the stationary dist. ρ^{π_k}

$$(x_1, \pi_k(x_1), r_1, x_2, \pi_k(x_2), r_2, \ldots, x_{n-1}, \pi_k(x_{n-1}), r_{n-1}, x_n)$$

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Return the last policy π_K

LSTD Algorithm

When $n \to \infty$ then $\widehat{A} \to A$ and $\widehat{b} \to b$, and thus,

$$\widehat{\alpha}_{\mathsf{TD}} \to \alpha_{\mathsf{TD}} \text{ and } \widehat{V}_{\mathsf{TD}} \to V_{\mathsf{TD}}$$

Proposition (LSTD Performance)

If LSTD is used to estimate the value of π with an ***infinite*** number of samples drawn from the stationary distribution ρ^π then

$$||V^\pi - V_{\mathsf{TD}}||_{\rho^\pi} \leq \frac{1}{\sqrt{1-\gamma^2}} \inf_{V \in \mathcal{F}} ||V^\pi - V||_{\rho^\pi}$$

LSTD Algorithm

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Problem: we don't have an infinite number of samples...

LSTD Algorithm

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Problem: we don't have an infinite number of samples...
Problem 2: V_{TD} is a fixed point solution and not a standard machine learning problem...

LSTD Error Bound

Assumption: The Markov chain induced by the policy π_k has a stationary distribution ρ^{π_k} and it is ergodic and β-mixing.

LSTD Error Bound

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Theorem (LSTD Error Bound)

At any iteration k, if LSTD uses n samples obtained from a single trajectory of π and a d-dimensional space, then with probability $1-\delta$

$$||V^{\pi_k} - \widehat{V}_k||_{\rho^{\pi_k}} \leq \frac{c}{\sqrt{1-\gamma^2}} \inf_{f\in\mathcal{F}} ||V^{\pi_k} - f||_{\rho^{\pi_k}} + O\left(\sqrt{\frac{d\log(d/\delta)}{n\,\nu}}\right)$$

LSTD Error Bound

$$||V^\pi - \widehat{V}||_{\rho^\pi} \leq \frac{c}{\sqrt{1-\gamma^2}} \underbrace{\inf_{f \in \mathcal{F}} ||V^\pi - f||_{\rho^\pi}}_{\text{approximation error}} + \underbrace{O\left(\sqrt{\frac{d \log(d/\delta)}{n\,\nu}}\right)}_{\text{estimation error}}$$

- **Approximation error:** it depends on how well the function space $\mathcal{F}$ can approximate the value function V^π
- **Estimation error:** it depends on the number of samples n, the dim of the function space d, the smallest eigenvalue of the Gram matrix ν, the mixing properties of the Markov chain (hidden in O)

LSTD Error Bound

$$||V^{\pi_k} - \widehat{V}_k||_{\rho^{\pi_k}} \leq \frac{c}{\sqrt{1-\gamma^2}} \underbrace{\inf_{f \in \mathcal{F}} ||V^{\pi_k} - f||_{\rho^{\pi_k}}}_{\text{approximation error}} + \underbrace{O\left(\sqrt{\frac{d \log(d/\delta)}{n\, \nu_k}}\right)}_{\text{estimation error}}$$

- n number of samples and d dimensionality

LSTD Error Bound

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- $\nu_k =$ the smallest eigenvalue of the Gram matrix $(\int \varphi_i \, \varphi_j \, d\rho^{\pi_k})_{i,j}$ (**Assumption:** *eigenvalues of the Gram matrix are strictly positive - existence of the model-based LSTD solution*)
- β-mixing coefficients are hidden in the $O(\cdot)$ notation

LSPI Error Bound

Theorem (LSPI Error Bound)

If LSPI is run over K iterations, then the performance loss policy π_K is

$$||V^* - V^{\pi_K}||_\mu \leq \frac{4\gamma}{(1-\gamma)^2} \left\{ \sqrt{CC_{\mu,\rho}} \left[E_0(\mathcal{F}) + O\left(\sqrt{\frac{d \log(dK/\delta)}{n\,\nu_\rho}} \right) \right] + \gamma^K R_{\max} \right\}$$

with probability $1 - \delta$.

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- **Estimation error:** depends on n, d, ν_ρ, K
- **Initialization error:** error due to the choice of the initial value function or initial policy $|V^* - V^{\pi_0}|$

LSPI Error Bound

LSPI Error Bound

$$||V^* - V^{\pi_K}||_\mu \leq \frac{4\gamma}{(1-\gamma)^2}\left\{\sqrt{CC_{\mu,\rho}}\left[cE_0(\mathcal{F}) + O\left(\sqrt{\frac{d\log(dK/\delta)}{n\,\nu_\rho}}\right)\right] + \gamma^K R_{\max}\right\}$$

Lower-Bounding Distribution

There exists a distribution ρ such that for any policy $\pi \in \mathcal{G}(\widetilde{\mathcal{F}})$, we have $\rho \leq C\rho^\pi$, where $C < \infty$ is a constant and ρ^π is the stationary distribution of π. Furthermore, we can define the concentrability coefficient $C_{\mu,\rho}$ as before.

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- $\nu_\rho =$ the smallest eigenvalue of the Gram matrix $(\int \varphi_i\, \varphi_j\, d\rho)_{i,j}$

Outline

Other Finite-Sample Analysis Results in ADP

Approximate Value Iteration

- Fitted value iteration ***(Munos & Szepesvari 2008)***
- L_2-Regularized Fitted Q-Iteration ***(Farahmand et al. 2009)***
- Transfer of samples in Fitted Q-Iteration ***(L, Restelli, 2010)***
- Multi-task Sparse Fitted Q-Iteration ***(Calandriello, L, Restelli, 2014)***

Other Finite-Sample Analysis Results in ADP

Approximate Policy Iteration

- LSTD and LSPI ***(L, Ghavamzadeh, Munos 2010, 2012)***
- Bellman Residual Minimization ***(Maillard, Munos, L, Ghavamzadeh 2010)***
- Modified Bellman Residual Minimization ***(Antos et al. 2008)***
- Classification-based Policy Iteration ***(Fern et al. 2006; L, Ghavamzadeh, Munos et al. 2010; Gabillon, L, Ghavamzadeh, Scherrer 2011)***
- Conservative Policy Iteration ***(Kakade & Langford 2002; Kakade 2003)***
- ℓ_1-regularize LSTD ***(Ghavamzadeh, L, Munos, 2011, Hoffman, L, Ghavamzadeh, Munos, 2012, Geist, Scherrer, L, Ghavamzadeh, 2012)***
- LSTD (LSPI) with Random Projections ***(Ghavamzadeh, L, Maillard, Munos, 2010)***

Comparison to Supervised Learning

Similarity: The convergence rate is the same (optimal) rate of statistical learning theory.

Difference

- dependency on $1/(1-\gamma)$ (sequential nature of the problem)
- the approximation error is more complex (iterative nature of the algorithms)
- the propagation of error (concentrability) (control problem)
- the sampling problem (how to choose ρ – exploration problem)

Practical Lessons

- Tuning the parameters (given a fixed accuracy ϵ)
  - number of samples (inverting the bound) $n \geq \widetilde{\Omega}(\frac{d}{\epsilon})$
  - number of iterations (inverting the bound) $K \approx \epsilon/(1-\gamma)$
- choice of function $\mathcal{F}$ and/or policy space Π
  - features $\{\varphi_i\}_{i=1}^d$ to be linearly independent given the sampling distribution ρ (on-policy – off-policy sampling)
- tradeoff between approximation and estimation errors

Open Problems

Control the propagation of error

- Improve the *sampling* distribution
- Refine the analysis of *concentrability* terms
- Off-policy learning
- No-regret algorithms
- Find “easier” MDPs

Control the approximation error

- Non-parametric approaches
- Smooth MDPs
- Automatic construction of basis functions
- Representation learning

Approximate Dynamic Programming meets Statistical Learning Theory

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