# Model reduction of infinite dimensional systems: An application to TDS and linear PDE cases

I. Pontes Duff, P. Vuillemin, C. Poussot-Vassal, C. Seren & C. Briat

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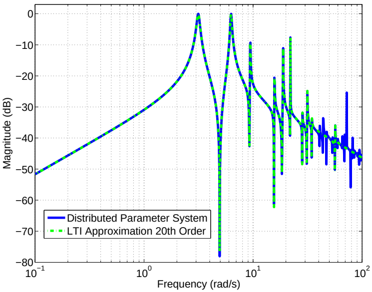

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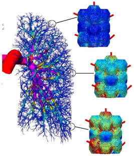
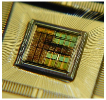
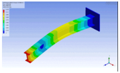
# Large-scale dynamical models

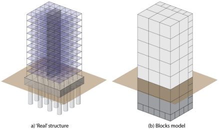
**... some motivating examples in the simulation & control domains**

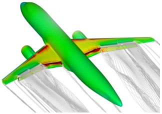
Large-scale systems are present in many engineering fields: aerospace, computational biology, building structure, VLI circuits, automotive, weather forecasting, fluid flow. . .

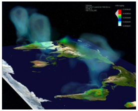










- difficulties with simulation & memory management (e.g. ODE solvers)
- difficulties with analysis (e.g. frequency response, $\mu_{ssv}$ and $\mathcal{H}_\infty$ computation . . . )
- difficulties with controller design (e.g. robust, optimal, predictive, . . . )

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# INTRODUCTION

## Context

Topics addressed in this presentation about model approximation:

- Some projection based methods in the finite dimensional case where a realization is available : IRKA/ITIA[1], IETIA[2] ...
- Interpolation method using Loewner framework [3] [4]
- Approximation of stability regions for large-scale time-delay systems [5] [6]

[1] S. Gugercin and A C. Antoulas and C. Beattie, *"$\mathcal{H}_2$ Model Reduction for Large Scale Linear Dynamical Systems"*, SIAM Journal on Matrix Analysis and Applications, vol. 30(2), 2008, pp. 609-638.

[2] C. Poussot-Vassal and P. Vuillemin, *"An Iterative Eigenvector Tangential Interpolation Algorithm for Large-Scale LTI and a Class of LPV Model Approximation"*, European Control Conference, 2013, pp. 4490-4495.

[3] A.J. Mayo and A.C. Antoulas, *"A framework for the solution of the generalized realization problem"*, Linear Algebra and its Applications 425(2-3), 2007, pp. 634-662.

[4] C. Beattie and S. Gugercin, *"Realization-independent $\mathcal{H}_2$-approximation"*, IEEE Conference on Decision and Control, 2012, pp. 4953-4958.

[5] I. Pontes Duff, P. Vuillemin, C. Poussot-Vassal, C. Seren and C. Briat, *"Approximation of stability regions for large-scale time-delay systems using model reduction techniques"*, submitted to ECC 2015.

[6] I. Pontes Duff, P. Vuillemin, C. Poussot-Vassal, C. Seren and C. Briat, *"Stability and Performance Analysis of a Large-Scale Aircraft Anti-Vibration Control Subject to Delays Using Model Reduction Techniques"*, submitted to EuroGNC 2015.

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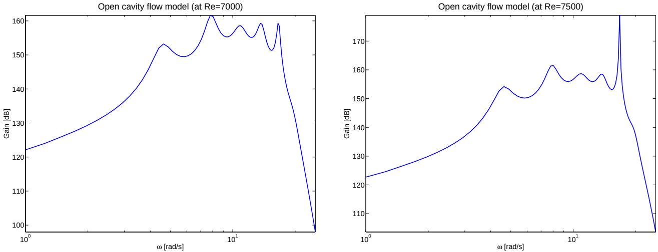
## INTRODUCTION

### Considered benchmarks

**Benchmark NSS :** Navier-Stokes equation in a open cavity flow : discretization and linearisation for different Reynolds Numbers[7]

$$\begin{aligned} E\dot{x}(t) &= A(Re)x(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \quad (1)$$




- Two Reynolds cases ($Re = 7000$ and $Re = 7500$)
- SISO DAE, 8 unstable modes, order $\approx$ 700,000 states

7 A. Barbagallo, D. Sipp and P. Schmid, *"Closed-loop control of an open cavity flow using reduced order models"*, Journal of Fluid Mechanics, 641, 2009, pp. 1-50.

## Considered benchmarks

**Benchmark TDS-#1: Feedback delay and controller gain**[8]
Let us consider

$$\dot{x}(t) = Ax(t) + Bu(t); \; y(t) = Cx(t), \tag{2}$$

where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & 10 & 0 & 0 \\ 5 & -15 & 0 & -0.25 \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \; C = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T. \tag{3}$$

We add to this model the delayed static output feedback $u(t) = -ky(t) + ky(t - \tau)$
The resulted model $\mathbf{H_{TDS1}}$ is governed by

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau) \tag{4}$$

where $A - BCk$ and $A_1 = BCk$.

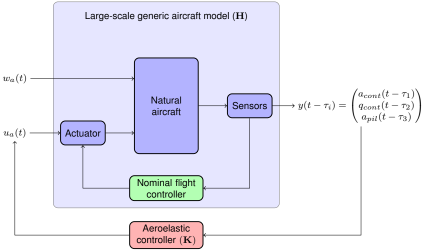
**Question:** Given $(k, \tau)$, what is the stability of (4)?

---

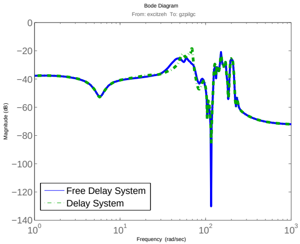
8 A. Seuret and F. Gouaisbaut, *"Hierarchy of LMI conditions for the stability analysis of time delay systems"*, Repport LAAS 14429.

## Considered benchmarks

### Benchmark TDS-#2: Multiple delays (in feedback) large-scale system [9]




- Vibration control of aircraft model.
- order$(\mathbf{H}) \approx 600$ states.
- Controller $\mathbf{H}$ designed without taking into account time-delays.
- Three output delays $\{\tau_1, \tau_2, \tau_3\}$.




**Question**: Stability function of $\{\tau_1, \tau_2, \tau_3\}$?
How to measure loss of performance ?

9 C. Poussot-Vassal and T. Loquen and P. Vuillemin and O. Cantinaud and J-P. Lacoste, *"Business Jet Large-Scale Model Approximation and Vibration Control"*, IFAC ALCOSP, 2013, pp. 199-204.

## Considered benchmarks

**Benchmark PDE: Example string vibration with dissipation**[10]
Vibrating string of length $L = 1$ whose ends are fixed with control and observation are both distributed along the string.

$$\frac{\partial^2 z(x,t)}{\partial t^2} + \varepsilon \langle \frac{\partial z(x,t)}{\partial t}, \mathbf{1}_{[0,\frac{1}{2}]} \rangle \mathbf{1}_{[0,\frac{1}{2}]}(x) = \frac{\partial^2 z(x,t)}{\partial x^2} + \mathbf{1}_{[0,\frac{1}{2}]}(x) u(t), \quad 0 < x < 1, \ t \geq 0 \tag{5}$$

where, $\mathbf{1}_{[0,\frac{1}{2}]}(x) = \begin{cases} 1 = & 0 \leq x \leq 1/2 \\ 0 = & 1/2 < x \leq 1 \end{cases}$, with

$$z(0,t) = 0, z(1,t) = 0$$

, and

$$y(t) = \int_0^1 \frac{\partial z(x,t)}{\partial t} \mathbf{1}_{[0,\frac{1}{2}]}(x) dx.$$

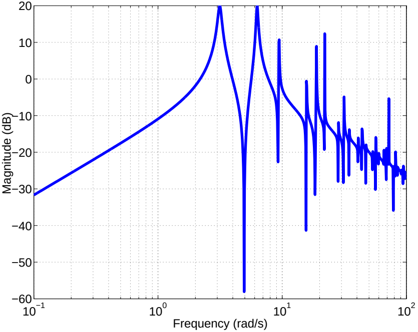
10 R. Curtain and K. Morris, *"Transfer functions of distributed parameter systems: A tutorial"*, Automatica, 45(5), 2009, pp. 1101-1116.

# INTRODUCTION

## Considered benchmarks

**Benchmark PDE: Example string vibration with dissipation**
The transfer function of this model is given by

$$H(s) = \frac{\frac{s}{2}\sinh(s) + 2\cosh(\frac{s}{2}) - 3\cosh^2(\frac{s}{2}) + 1}{s(s+\frac{1}{2})\sinh(s) + 2\cosh(\frac{s}{2}) - 3\cosh^2(\frac{s}{2}) + 1} \quad (6)$$



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# Optimal model approximation

## Approximation in the $\mathcal{H}_2$, $\mathcal{H}_{2,\Omega}$ and $\mathcal{L}_2$-norm[11]

### $\mathcal{L}_2$ model approximation

$$\hat{H} := \arg \min_{\substack{G \in \mathcal{L}_2^{n_y \times n_u} \\ \dim(G) = r}} ||H - G||_{\mathcal{L}_2} \tag{7}$$

$$||H||^2_{\mathcal{L}_2} := \textbf{trace}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\overline{H(i\nu)}H(i\nu)\right) d\nu\right) \tag{8}$$

- $\mathcal{L}_2(i\mathbb{R})$ the Hilbert space of matrix-valued functions $\mathbb{C} \to \mathbb{C}^{n_y \times n_u}$ satisfying $\int_{\mathbb{R}} \textbf{trace}[\overline{F(i\omega)}F(i\omega)^T] d\omega < \infty$.
- $\mathcal{L}_2(i\mathbb{R}) = \mathcal{H}_2(\mathbb{C}^-) \bigoplus \mathcal{H}_2(\mathbb{C}^+)$

[11] C. Magruder and C A. Beattie and S. Gugercin, *"Rational Krylov methods for optimal $\mathcal{L}_2$ model reduction"*, IEEE Conference on Decision and Control, 2010.

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# Optimal model approximation

### Approximation in the $\mathcal{H}_2$, $\mathcal{H}_{2,\Omega}$ and $\mathcal{L}_2$-norm[12 13]

## $\mathcal{H}_2$ model approximation

$$\hat{H} := \arg \min_{\substack{G \in \mathcal{H}_2^{n_y \times n_u} \\ \dim(G) = r}} ||H - G||_{\mathcal{H}_2} \quad (9)$$

$$\begin{aligned} ||H||_{\mathcal{H}_2}^2 &:= \textbf{trace}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\overline{H(i\nu)}H(i\nu)\right)d\nu\right) \\ &:= \textbf{trace}\left(C\mathcal{P}C^T\right) = \textbf{trace}\left(B^T\mathcal{Q}B\right) \\ &:= \sum_{i=1}^{n}\textbf{trace}\left(\phi_i H(-\lambda_i)^T\right) \end{aligned} \quad (10)$$

12 S. Gugercin and A C. Antoulas and C A. Beattie, *"$\mathcal{H}_2$ Model Reduction for Large Scale Linear Dynamical Systems"*, SIAM Journal on Matrix Analysis and Applications, vol. 30(2), June 2008, pp. 609-638.

13 K. A. Gallivan, A. Vanderope, and P. Van-Dooren, *"Model reduction of MIMO systems via tangential interpolation"*, SIAM Journal of Matrix Analysis and Application, vol. 26(2), February 2004, pp. 328-349.

# $\mathcal{H}_2$ OPTIMALITY CONDITIONS [14]

Assume that $\mathbf{H}$ and $\hat{\mathbf{H}}$ have semi-simple poles and suppose that $\hat{\mathbf{H}}$ is a $r^{th}$-order finite-dimensional model with transfer function

$$\hat{H}(s) = \sum_{k=1}^{r} \frac{\hat{c}_k \hat{b}_k^T}{s - \hat{\lambda}_k}. \quad (11)$$

## $\mathcal{H}_2$-optimality conditions

If $\mathbf{H}, \hat{\mathbf{H}} \in \mathcal{H}_2$ and $\hat{\mathbf{H}}$ is a local minimum of the $\mathcal{H}_2$ approximation problem, then the following interpolations equations hold

$$H(-\hat{\lambda}_k)\hat{b}_k = \hat{H}(-\hat{\lambda}_k)\hat{b}_k, \;\; \hat{c}_k^T H(-\hat{\lambda}_k) = \hat{c}_k^T \hat{H}(-\hat{\lambda}_k) \quad (12)$$

$$\hat{c}_k^T \left.\frac{dH}{ds}\right|_{s=-\hat{\lambda}_k} \hat{b}_k = \hat{c}_k^T \left.\frac{d\hat{H}}{ds}\right|_{s=-\hat{\lambda}_k} \hat{b}_k, \quad (13)$$

for all $k = 1, \ldots, r$ where $\hat{\lambda}_k$ are the poles of $\hat{\mathbf{H}}$ and $\hat{b}_k$ and $\hat{c}_k$ are its tangential directions, respectively.

[14] S. Gugercin and A C. Antoulas and C. Beattie, *"$\mathcal{H}_2$ Model Reduction for Large Scale Linear Dynamical Systems"*, SIAM Journal on Matrix Analysis and Applications, vol. 30(2), 2008, pp. 609-638.

## $\mathcal{L}_2$ OPTIMALITY CONDITIONS [15]

In the case where $\mathbf{H} \in \mathcal{L}_2(i\mathbb{R})$ is a SISO LTI system and $\mathbf{H} = \mathbf{H}^+ + \mathbf{H}^-$ where $\mathbf{H}^+ \in \mathcal{H}(\mathbb{C}^+)$ and $\mathbf{H}^- \in \mathcal{H}(\mathbb{C}^-)$, it is possible to state the following result:

### $\mathcal{L}_2$ optimality conditions

Given $\mathbf{H} \in \mathcal{L}_2(i\mathbb{R})$ and its decomposition $\mathbf{H} = \mathbf{H}^+ + \mathbf{H}^-$ where $\mathbf{H}^+ \in \mathcal{H}(\mathbb{C}^+)$ and $\mathbf{H}^- \in \mathcal{H}(\mathbb{C}^-)$. Let $\hat{\mathbf{H}}$ be the local minimizer of order $r$ whose poles are all simple $\{\hat{\lambda}_1, \ldots, \hat{\lambda}_k\} \in \mathbb{C}^-$ and $\{\hat{\lambda}_{k+1}, \ldots, \hat{\lambda}_r\} \in \mathbb{C}^+$. If $\hat{H}(s)$ is given as (11) and if it is a local minimal of the $\mathcal{L}_2$ approximation problem, then following hold for $i = 1, \ldots, k$

$$H^+(-\hat{\lambda}_i) = \hat{H}^+(-\hat{\lambda}_i), \ \frac{dH^+}{ds}\bigg|_{s=-\hat{\lambda}_i} = \frac{d\hat{H}^+}{ds}\bigg|_{s=-\hat{\lambda}_i} \tag{14}$$

and for $i = k+1, \ldots, r$,

$$H^-(-\hat{\lambda}_i) = \hat{H}^-(\hat{\lambda}_i), \ \frac{dH^-}{ds}\bigg|_{s=-\hat{\lambda}_i} = \frac{d\hat{H}^-}{ds}\bigg|_{s=-\hat{\lambda}_i}. \tag{15}$$

[15] C. Magruder and C A. Beattie and S. Gugercin, *"Rational Krylov methods for optimal $\mathcal{L}_2$ model reduction"*, IEEE Conference on Decision and Control, 2010.

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# Optimal model approximation

## Approximation in the $\mathcal{H}_{2,\Omega}$-norm [16] [17]

### $\mathcal{H}_{2,\Omega}$ model approximation

$$\hat{H} := \arg \min_{\substack{G \in \mathcal{H}_\infty^{n_y \times n_u} \\ \dim(G) = r}} ||H - G||_{\mathcal{H}_{2,\Omega}} \quad (16)$$

$$\begin{aligned} ||H||^2_{\mathcal{H}_{2,\Omega}} &:= \textbf{trace}\left(\frac{1}{\pi}\int_\Omega \left(\overline{H(i\nu)}H(i\nu)\right)d\nu\right) \\ &:= \textbf{trace}\left(C\mathcal{P}_\Omega C^T\right) = \textbf{trace}\left(B^T\mathcal{Q}_\Omega B\right) \\ &:= \sum_{i=1}^{n} \textbf{trace}\left(\phi_i H(-\lambda_i)^T\right)\left[-\frac{2}{\pi}\textbf{atan}\left(\frac{\omega}{\lambda_i}\right)\right] \end{aligned} \quad (17)$$

16 P. Vuillemin, C. Poussot-Vassal and D. Alazard, *"A Spectral Expression for the Frequency-Limited $\mathcal{H}_2$-norm"*, Available as http://arxiv.org/abs/1211.1858, 2012.

17 P. Vuillemin, C. Poussot-Vassal and D. Alazard, *"Spectral expression for the Frequency-Limited $\mathcal{H}_2$-norm of LTI Dynamical Systems with High Order Poles"*, European Control Conference, 2014, pp. 55-60.

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[16] P. Vuillemin, C. Poussot-Vassal and D. Alazard, *"A Spectral Expression for the Frequency-Limited $\mathcal{H}_2$-norm"*, Available as http://arxiv.org/abs/1211.1858, 2012.

[17] P. Vuillemin, C. Poussot-Vassal and D. Alazard, *"Spectral expression for the Frequency-Limited $\mathcal{H}_2$-norm of LTI Dynamical Systems with High Order Poles"*, European Control Conference, 2014, pp. 55-60.

# Optimal model approximation

## $\mathcal{H}_2$ and $\mathcal{L}_2$ optimality conditions[18]

### Mismatch objective and eigenvector preservation

$$\hat{H} := \arg \min_{\substack{G \in \mathcal{L}_2^{n_y \times n_u} \\ \dim(G) = r \ll n \\ \lambda_k(G) \subseteq \lambda(H) \;\; k=1,\dots,q_1 < r}} ||H - G||_{\mathcal{H}_2} \tag{18}$$

- More than a $\mathcal{H}_2$ (sub-optimal) criteria
- Keep some user defined eigenvalues... *e.g.* the unstable ones

18 C. Poussot-Vassal and P. Vuillemin, *"An Iterative Eigenvector Tangential Interpolation Algorithm for Large-Scale LTI and a Class of LPV Model Approximation"*, European Control Conference, 2013, pp. 4490-4495.

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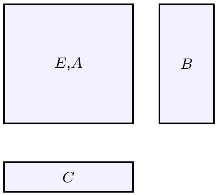
# Projection-based approximation framework

## Projectors

Let $H : \mathbb{C} \to \mathbb{C}^{n_y \times n_u}$ be a $n_u$ inputs $n_y$ outputs, full order $\mathcal{H}_2^{n_y \times n_u}$ (or $\mathcal{L}_2^{n_y \times n_u}$) complex-valued function describing a **LTI** dynamical system as a DAE of order $n$, with realization $\mathbf{H}$:

$$\mathbf{H} : \left\{ \begin{array}{rcl} E\dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) \end{array} \right. \quad (19)$$



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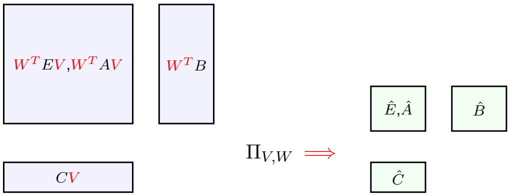
$$\mathbf{H} : \begin{cases} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{cases} \quad (19)$$

the approximation problem consists in finding $V, W \in \mathbb{R}^{n \times r}$ (with $r \ll n$) **spanning $\mathcal{V}$ and $\mathcal{W}$ subspaces** and forming a projector $\Pi_{V,W} = VW^T$, such that

$$\hat{\mathbf{H}} : \begin{cases} W^T EV\dot{\hat{x}}(t) &= W^T AV\hat{x}(t) + W^T Bu(t) \\ \hat{y}(t) &= CV\hat{x}(t) \end{cases} \quad (20)$$

well approximates $\mathbf{H}$.



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well approximates $\mathbf{H}$.

- **Small approximation error and/or global error bound**
- **Stability / passivity preservation**
- **Numerically stable & efficient procedure**

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# PROJECTION-BASED APPROXIMATION FRAMEWORK

### MIMO IRKA (or ITIA) - $\mathcal{H}_2$ optimality conditions (Tangential subspace approach) [19] [20]

Given $H(s)$, let $V \in \mathbb{C}^{n\times r}$ and $W \in \mathbb{C}^{n\times r}$ be matrices of full column rank $r = q_2$ such that $W^*V = I_r$. If, for $j = 1, \ldots, q_2$,

$$\left[(\sigma_j E - A)^{-1} B\hat{b}_j\right] \in \textbf{span}(V) \text{ and } \left[(\sigma_j E - A^T)^{-1} C^T \hat{c}_j^*\right] \in \textbf{span}(W) \tag{21}$$

where $\sigma_j \in \mathbb{C}$, $\hat{b}_j \in \mathbb{C}^{n_u}$ and $\hat{c}_j \in \mathbb{C}^{n_y}$, be given sets of interpolation points and left and right tangential directions, respectively.

[19] P. Van-Dooren, K. A. Gallivan, and P. A. Absil, *"$\mathcal{H}_2$-optimal model reduction of MIMO systems"*, Applied Mathematics Letters, vol. 21(12), December 2008, pp. 53-62.

[20] S. Gugercin and A C. Antoulas and C A. Beattie, *"$\mathcal{H}_2$ Model Reduction for Large Scale Linear Dynamical Systems"*, SIAM Journal on Matrix Analysis and Applications, vol. 30(2), June 2008, pp. 609-638.

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where $\sigma_j \in \mathbb{C}$, $\hat{b}_j \in \mathbb{C}^{n_u}$ and $\hat{c}_j \in \mathbb{C}^{n_y}$, be given sets of interpolation points and left and right tangential directions, respectively. Then, the reduced order system $\hat{H}(s)$ satisfies the tangential interpolation conditions

$$\begin{array}{rcl} H(-\hat{\sigma}_j)\hat{b}_j & = & \hat{H}(-\hat{\sigma}_j)\hat{b}_j \\ \hat{c}_j^* H(-\hat{\sigma}_j) & = & \hat{c}_j^* \hat{H}(-\hat{\sigma}_j) \\ \hat{c}_j^* \dfrac{d}{ds} H(s)\Big|_{s=-\hat{\sigma}_j} \hat{b}_j & = & \hat{c}_j^* \dfrac{d}{ds}\hat{H}(s)\Big|_{s=-\hat{\sigma}_j} \hat{b}_j \end{array} \quad (22)$$

19 P. Van-Dooren, K. A. Gallivan, and P. A. Absil, *"$\mathcal{H}_2$-optimal model reduction of MIMO systems"*, Applied Mathematics Letters, vol. 21(12), December 2008, pp. 53-62.

20 S. Gugercin and A C. Antoulas and C A. Beattie, *"$\mathcal{H}_2$ Model Reduction for Large Scale Linear Dynamical Systems"*, SIAM Journal on Matrix Analysis and Applications, vol. 30(2), June 2008, pp. 609-638.

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# Projection-based approximation framework

**Require:** $\mathbf{H} = (E, A, B, C)$, $\{\sigma_1^{(0)}, \ldots, \sigma_{q_2}^{(0)}\} \in \mathbb{C}^{q_2}$, $\{\hat{b}_1, \ldots, \hat{b}_{q_2}\} \in \mathbb{C}^{n_u \times q_2}$, $\{\hat{c}_1, \ldots, \hat{c}_{q_2}\} \in \mathbb{C}^{n_y \times q_2}$ and $r = q_2 \in \mathbb{N}$

1: Construct,

$$\textbf{span}\left(V(\sigma_j^{(0)}, \hat{b}_j)\right) \text{ and } \textbf{span}\left(W(\sigma_j^{(0)}, \hat{c}_j^*)\right) \tag{23}$$

2: Compute $W \leftarrow W(V^T W)^{-1}$

3: **while** Stopping criteria **do**

4: $k \leftarrow k + 1$

5: $\hat{E} = W^T E V$, $\hat{A} = W^T A V$, $\hat{B} = W^T B$, $\hat{C} = CV$

6: Compute $\hat{A} R = \Lambda(\hat{A}, \hat{E}) R$ and $L \hat{A} = \Lambda(\hat{A}, \hat{E}) L$

7: Compute $\{\hat{b}_1, \ldots, \hat{b}_{q_2}\} = \hat{B}^T L$ and $\{\hat{c}_1^*, \ldots, \hat{c}_{q_2}^*\} = \hat{C} R$

8: Set $\sigma^{(i)} = -\Lambda(\hat{A}, \hat{E})$

9: Construct,

$$\textbf{span}\left(V(\sigma_j^{(k)}, \hat{b}_j)\right) \text{ and } \textbf{span}\left(W(\sigma_j^{(k)}, \hat{c}_j^*)\right) \tag{24}$$

10: Compute $W \leftarrow W(V^T W)^{-1}$

11: **end while**

12: Construct $\hat{\mathbf{H}} := (W^T E V, W^T A V, W^T B, C V)$

**Ensure:** $V, W \in \mathbb{R}^{n \times r}$, $W^T V = I_r$

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# Projection-based approximation framework

**Require:** $\mathbf{H} = (E, A, B, C)$, $\{\sigma_1^{(0)}, \ldots, \sigma_{q2}^{(0)}\} \in \mathbb{C}^{q2}$, $\{\hat{b}_1, \ldots, \hat{b}_{q2}\} \in \mathbb{C}^{n_u \times q2}$, $\{\hat{c}_1, \ldots, \hat{c}_{q2}\} \in \mathbb{C}^{n_y \times q2}$ and $r = q_2 \in \mathbb{N}$

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  Compute $\hat{A}R = \Lambda(\hat{A}, \hat{E})R$ and $L\hat{A} = \Lambda(\hat{A}, \hat{E})L$

  Compute $\{\hat{b}_1, \ldots, \hat{b}_{q2}\} = \hat{B}^TL$ and $\{\hat{c}_1^*, \ldots, \hat{c}_{q2}^*\} = \hat{C}R$

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  Construct,

$$\textbf{span}\big(V(\sigma_j^{(k)}, \hat{b}_j)\big) \text{ and } \textbf{span}\big(W(\sigma_j^{(k)}, \hat{c}_j^*)\big) \tag{24}$$

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**end while**

Construct $\hat{\mathbf{H}} := (W^TEV, W^TAV, W^TB, CV)$

**Ensure:** $V, W \in \mathbb{R}^{n \times r}$, $W^TV = I_r$

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# PROJECTION-BASED APPROXIMATION FRAMEWORK

## IETIA - $\mathcal{H}_2$ & spectral optimality conditions (Tangential subspace approach) [21]

Given $H(s)$, let $V \in \mathbb{C}^{n\times r}$ and $W \in \mathbb{C}^{n\times r}$ be matrices of full column rank $r = q_1 + q_2$ such that $W^*V = I_r$. If, for $i = 1,\dots,q_1$ and $j = 1,\dots,q_2$,

$$\left[r_i^\star \quad (\sigma_j E - A)^{-1} B\hat{b}_j\right] \in \textbf{span}(V) \text{ and } \left[l_i^\star \quad (\sigma_j E - A^T)^{-1} C^T \hat{c}_j^*\right] \in \textbf{span}(W) \quad (25)$$

$l_i^\star \in \mathbb{C}^n$ and $r_i^\star \in \mathbb{C}^n$ are left and right eigenvectors associated to $\lambda_i^\star \in \mathbb{C}$ eigenvalues associated to $A, E$ and $\sigma_j \in \mathbb{C}$, $\hat{b}_j \in \mathbb{C}^{n_u}$ and $\hat{c}_j \in \mathbb{C}^{n_y}$, be given sets of interpolation points and left and right tangential directions, respectively.

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21 C. Poussot-Vassal and P. Vuillemin, *"An Iterative Eigenvector Tangential Interpolation Algorithm for Large-Scale LTI and a Class of LPV Model Approximation"*, European Control Conference, 2013, pp. 4490-4495.

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# Projection-based approximation framework

## IETIA - $\mathcal{H}_2$ & spectral optimality conditions (Tangential subspace approach) [21]

Given $H(s)$, let $V \in \mathbb{C}^{n\times r}$ and $W \in \mathbb{C}^{n\times r}$ be matrices of full column rank $r = q_1 + q_2$ such that $W^*V = I_r$. If, for $i = 1, \dots, q_1$ and $j = 1, \dots, q_2$,

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$$\{\lambda_1^\star, \dots, \lambda_{q_1}^\star\} \subset \Lambda(\hat{A}, \hat{E}) \quad (26)$$

and the tangential interpolation conditions

$$\begin{aligned} H(-\hat{\sigma}_j)\hat{b}_j &= \hat{H}(-\hat{\sigma}_j)\hat{b}_j \\ \hat{c}_j^* H(-\hat{\sigma}_j) &= \hat{c}_j^* \hat{H}(-\hat{\sigma}_j) \\ \hat{c}_j^* \frac{d}{ds} H(s)\Big|_{s=-\hat{\sigma}_j} \hat{b}_j &= \hat{c}_j^* \frac{d}{ds}\hat{H}(s)\Big|_{s=-\hat{\sigma}_j} \hat{b}_j \end{aligned} \quad (27)$$

[21] C. Poussot-Vassal and P. Vuillemin, *"An Iterative Eigenvector Tangential Interpolation Algorithm for Large-Scale LTI and a Class of LPV Model Approximation"*, European Control Conference, 2013, pp. 4490-4495.

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# Projection-based approximation framework

**Require:** $\mathbf{H} = (E, A, B, C)$, $\{\lambda_1^\star, \ldots, \lambda_{q_1}^\star\} \in \mathbb{C}^{q_1}$, $\{\sigma_1^{(0)}, \ldots, \sigma_{q_2}^{(0)}\} \in \mathbb{C}^{q_2}$, $\{\hat{b}_1, \ldots, \hat{b}_{q_2}\} \in \mathbb{C}^{n_u \times q_2}$, $\{\hat{c}_1, \ldots, \hat{c}_{q_2}\} \in \mathbb{C}^{n_y \times q_2}$ and $r = q_1 + q_2 \in \mathbb{N}$

1: Compute $\{l_1^\star, \ldots, l_{q_1}^\star\}$ and $\{r_1^\star, \ldots, r_{q_1}^\star\}$, eigenvectors of $\{\lambda_1^\star, \ldots, \lambda_{q_1}^*\}$

2: Construct,

$$\textbf{span}\left(V(l_i^\star, \sigma_j^{(0)}, \hat{b}_j)\right) \text{ and } \textbf{span}\left(W(r_i^\star, \sigma_j^{(0)}, \hat{c}_j^*)\right) \tag{28}$$

3: Compute $W \leftarrow W(V^T W)^{-1}$

4: **while** Stopping criteria **do**

5: $k \leftarrow k + 1$

6: $\hat{E} = W^T E V$, $\hat{A} = W^T A V$, $\hat{B} = W^T B$, $\hat{C} = CV$

7: Compute $\hat{A}R = \hat{E}\Lambda(\hat{A}, \hat{E})R$ and $L\hat{A} = \Lambda(\hat{A})L$

8: Compute $\{\hat{b}_1, \ldots, \hat{b}_{q_2}\} = \hat{B}^T L$ and $\{\hat{c}_1^*, \ldots, \hat{c}_{q_2}^*\} = \hat{C}R$

9: Set $\sigma^{(i)} = -\Lambda(\hat{A}, \hat{E})$

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$$\textbf{span}\left(V(l_i^\star, \sigma_j^{(k)}, \hat{b}_j)\right) \text{ and } \textbf{span}\left(W(r_i^\star, \sigma_j^{(k)}, \hat{c}_j^*)\right) \tag{29}$$

11: Compute $W \leftarrow W(V^T W)^{-1}$

12: **end while**

13: Construct $\hat{\mathbf{H}} := (W^T E V, W^T A V, W^T B, C V)$

**Ensure:** $V, W \in \mathbb{R}^{n \times r}$, $W^T V = I_r$, $\{\lambda_1^\star, \ldots, \lambda_{q_1}^\star\} \subset \Lambda(\hat{A}, \hat{E})$

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# Projection-based approximation framework

**Require:** $\mathbf{H} = (E, A, B, C)$, $\{\lambda_1^\star, \dots, \lambda_{q1}^\star\} \in \mathbb{C}^{q_1}$, $\{\sigma_1^{(0)}, \dots, \sigma_{q2}^{(0)}\} \in \mathbb{C}^{q_2}$, $\{\hat{b}_1, \dots, \hat{b}_{q2}\} \in \mathbb{C}^{n_u \times q_2}$, $\{\hat{c}_1, \dots, \hat{c}_{q2}\} \in \mathbb{C}^{n_y \times q_2}$ and $r = q_1 + q_2 \in \mathbb{N}$

Compute $\{l_1^\star, \dots, l_{q1}^\star\}$ and $\{r_1^\star, \dots, r_{q1}^\star\}$, eigenvectors of $\{\lambda_1^\star, \dots, \lambda_{q1}^\star\}$

Construct,

$$\textbf{span}\left(V(l_i^\star, \sigma_j^{(0)}, \hat{b}_j)\right) \text{ and } \textbf{span}\left(W(r_i^\star, \sigma_j^{(0)}, \hat{c}_j^*)\right) \tag{28}$$

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  Set $\sigma^{(i)} = -\Lambda(\hat{A},\hat{E})$

  Construct,

$$\textbf{span}\left(V(l_i^\star, \sigma_j^{(k)}, \hat{b}_j)\right) \text{ and } \textbf{span}\left(W(r_i^\star, \sigma_j^{(k)}, \hat{c}_j^*)\right) \tag{29}$$

  Compute $W \leftarrow W(V^TW)^{-1}$

**end while**

Construct $\hat{\mathbf{H}} := (W^TEV, W^TAV, W^TB, CV)$

**Ensure:** $V, W \in \mathbb{R}^{n\times r}$, $W^TV = I_r$, $\{\lambda_1^\star, \dots, \lambda_{q1}^\star\} \subset \Lambda(\hat{A},\hat{E})$

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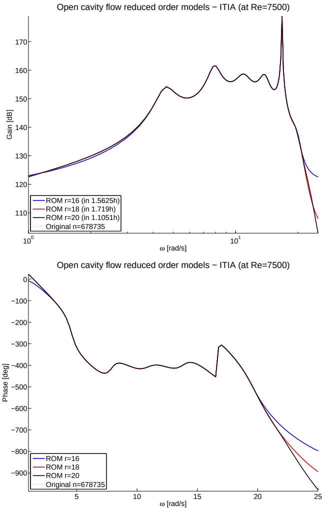
# Projection-based approximation framework

## Fluid flow dynamical model approximation - Re=7000 and Re=7500



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# Rational interpolation Loewner framework

## Rational interpolation

Given $H(s)$, complex points $\sigma_1, \ldots, \sigma_n$, and tangential directions $\hat{b}_1, \ldots, \hat{b}_n$, $\hat{c}_1, \ldots, \hat{c}_n$, one constructs $(\hat{E}, \hat{A}, \hat{B}, \hat{C})$ such that the transfer function $\hat{H}(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B}$ satisfies the tangential interpolation conditions :

$$\begin{aligned} H(\sigma_j)\hat{b}_j &= \hat{H}(\sigma_j)\hat{b}_j \\ \hat{c}_j^T H(\sigma_j) &= \hat{c}_j^T \hat{H}(\sigma_j) \\ \hat{c}_j^T \left.\frac{d}{ds} H(s)\right|_{s=\sigma_j} \hat{b}_j &= \hat{c}_j^T \left.\frac{d}{ds} \hat{H}(s)\right|_{s=\sigma_j} \hat{b}_j \end{aligned} \quad (30)$$

This is possible thanks to Loewner matrices.

# Rational interpolation Loewner framework

## Loewner approach[22]

The rational function $\hat{C}(s\hat{E}-\hat{A})^{-1}\hat{B}$ interpolates $H(s)$ at points $\sigma_i$ and directions $\hat{b}_i$ and $\hat{c}_i$ iff.

$$(\hat{E})_{ij} = \begin{cases} -\dfrac{\hat{c}_i^T(H(\sigma_i)-H(\sigma_j))\hat{b}_j}{\sigma_i-\sigma_j} & i \neq j \\ -\hat{c}_i^T H'(\sigma_i)\hat{b}_i & i = j \end{cases}$$

$$(\hat{A})_{ij} = \begin{cases} -\dfrac{\hat{c}_i^T(\sigma_i H(\sigma_i)-\sigma_j H(\sigma_j))\hat{b}_j}{\sigma_i-\sigma_j} & i \neq j \\ -\hat{c}_i^T (sH(s))'|_{s=\sigma_i}\hat{b}_i & i = j \end{cases}$$

$$\hat{C} = [H(\sigma_1)\hat{b}_1, \ldots, H(\sigma_r)\hat{b}_r] \text{ and } \hat{B} = \begin{bmatrix} \hat{c}_1^T H(\sigma_1) \\ \vdots \\ \hat{c}_r^T H(\sigma_r) \end{bmatrix}.$$

- An analogous to IRKA iterative method was proposed.

22 A.J. Mayo and A.C. Antoulas, *"A framework for the solution of the generalized realization problem"*, Linear Algebra and its Applications 425(2-3) 634 - 662, 2007.

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# Rational interpolation Loewner framework

## TF-IRKA algorithm[23]

1: **Initialization:** transfer function $H(s)$, dimension $r$, $\sigma^0 = \{\sigma_1^0, \ldots, \sigma_r^0\} \in \mathbb{C}$ initial interpolation points and tangential directions $\{b_1, \ldots, b_r\} \in \mathbb{C}^{n_u \times 1}$ and $\{c_1, \ldots, c_r\} \in \mathbb{C}^{n_y \times 1}$.

2: **while** not convergence **do**

3: **Build** $\hat{E}$, $\hat{A}$, $\hat{B}$ and $\hat{C}$ using *Loewner Matrices*.

4: Solve the generalized eigenvalue problem $\hat{A}^{(k)} x_i^{(k)} = \lambda_i^{(k)} \hat{E}^{(k)} x_i^{(k)}$ and $y_i^{(k)}$ such that $y_i^{(k)*} \hat{E}^{(k)} x_j^{(k)} = \delta_{i,j}$.

5: *Set* $\sigma_i^{(k+1)} \leftarrow -\lambda_i^{(k)}$, $b_i^{(k+1)T} \leftarrow y_i^{(k)} \hat{B}^{(k)}$ and $c_i^{(k+1)} \leftarrow \hat{C}^{(k)} x_i^{(k)}$, for $i = 1, \ldots, r$.

6: **end while**

7: **Ensure** conditions (31) are satisfied.

8: **Build** $\hat{E}$, $\hat{A}$, $\hat{B}$ and $\hat{C}$.

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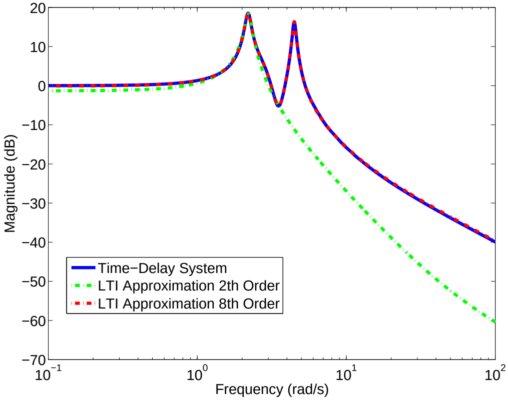
23 C. Beattie and S. Gugercin, *"Realization-independent $\mathcal{H}_2$-approximation"*, Proceedings of the 51st IEEE Conference on Decision and Control, 2012.

# Rational interpolation Loewner framework

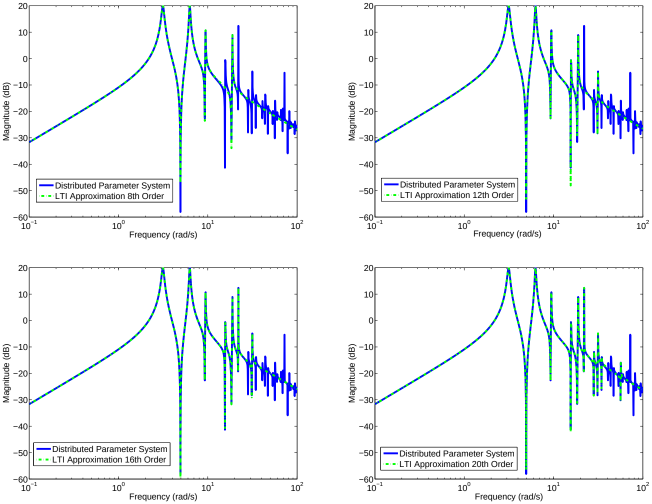
## Examples TDS and DPS

Example TDS-#1 for $k = 1$ and $\tau = 3$



# Rational interpolation Loewner framework

## Example DPS



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## Stability regions estimation

### Proposed Strategy - Approximation & Eigenvalues (accompanied with proofs)

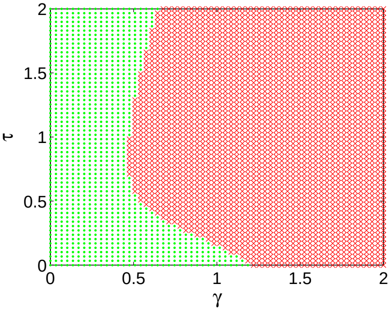
- **Procedure**: Estimating stability regions using model approximation & eigenvalues
- **Arguments for proof**: Provide some arguments why this procedure is valid

# STABILITY REGIONS ESTIMATION

## Proposed Strategy - Approximation & Eigenvalues (accompanied with proofs)

- **Procedure**: Estimating stability regions using model approximation & eigenvalues
- **Arguments for proof**: Provide some arguments why this procedure is valid

Example: Let us consider the model described by the transfer function $H(s) = \frac{1}{1+e^{\tau s}+2e^{\gamma s}}$, with $\tau, \gamma \in [0, 2]$. After discretizing $[0, 2]$ and finding LTI approximation via TF-IRKA, the stability of the reduced model is plotted



# Stability regions estimation

## Results about stability approximation in $\mathcal{L}_2$

- $\mathcal{L}_2(i\mathbb{R})$ the Hilbert space of matrix-valued functions $\mathbb{C} \to \mathbb{C}^{n_y \times n_u}$ satisfying $\int_{\mathbb{R}}$ **trace**$[\overline{F(i\omega)}F(i\omega)^T]d\omega < \infty$.
- $\langle \mathbf{H}, \mathbf{G} \rangle_{\mathcal{L}_2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}\left(\overline{H(i\omega)}G(i\omega)^T\right)d\omega.$
- $\mathcal{H}_2(\mathbb{C}^+)$ ($\mathcal{H}_2(\mathbb{C}^-)$) closed subspace of $\mathcal{L}_2(i\mathbb{R})$ containing the matrix functions $F(s)$ analytic in the open right-half plane (open left-half plane).
- $\mathcal{L}_2(i\mathbb{R}) = \mathcal{H}_2(\mathbb{C}^-) \bigoplus \mathcal{H}_2(\mathbb{C}^+)$
- $\mathcal{L}_2(i\mathbb{R}) \backslash \mathcal{H}_2(\mathbb{C}^+)$ set of unstable LTI systems
- **Remark: TF-IRKA** allows us to obtain a system of order $r$ which satisfies the interpolation conditions :

$$H(-\hat{\lambda}_k)\hat{b}_k = \hat{H}(-\hat{\lambda}_k)\hat{b}_k, \; \hat{c}_k^T H(-\hat{\lambda}_k) = \hat{c}_k^T \hat{H}(-\hat{\lambda}_k) \tag{31}$$

$$\hat{c}_k^T \left.\frac{dH}{ds}\right|_{s=-\hat{\lambda}_k} \hat{b}_k = \hat{c}_k^T \left.\frac{d\hat{H}}{ds}\right|_{s=-\hat{\lambda}_k} \hat{b}_k, \tag{32}$$

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# Stability regions estimation

## Results about stability approximation in $\mathcal{L}_2$

### Proposition 1

If $\mathbf{H} \in \mathcal{H}_2(\mathbb{C}^+)$ and there exists a global minimizer $\hat{\mathbf{H}} \in \mathcal{L}_2(i\mathbb{R})$ of the $\mathcal{L}_2$ approximation problem, then $\hat{\mathbf{H}} \in \mathcal{H}_2(\mathbb{C}^+)$. Similarly, if $\mathbf{H} \in \mathcal{H}_2(\mathbb{C}^-)$ and there exists a global minimizer $\hat{\mathbf{H}} \in \mathcal{L}_2(i\mathbb{R})$ of the $\mathcal{L}_2$ approximation problem, then $\hat{\mathbf{H}} \in \mathcal{H}_2(\mathbb{C}^-)$.

### Proof.

Let $\hat{\mathbf{H}} \in \mathcal{L}_2(i\mathbb{R})$ be the global minimizer of $\mathcal{L}_2$ approximation problem. Since $\mathbf{H} \in \mathcal{H}_2(\mathbb{C}^+)$, one has $\mathbf{H}^- = \mathbf{0}$. Seeing that $\mathcal{L}_2(i\mathbb{R}) = \mathcal{H}_2(\mathbb{C}^-) \bigoplus \mathcal{H}_2(\mathbb{C}^+)$ and this an orthogonal decomposition, thus

$$\|\mathbf{H} - \hat{\mathbf{H}}\|_{\mathcal{L}_2}^2 = \|\mathbf{H}^+ - \hat{\mathbf{H}}^+\|_{\mathcal{L}_2}^2 + \|\mathbf{0} - \hat{\mathbf{H}}^-\|_{\mathcal{L}_2}^2 \tag{33}$$

Thus, $\hat{\mathbf{H}}^- = \mathbf{0}$, otherwise $\hat{\mathbf{H}}$ is not a global minimizer. □

## Results about stability approximation in $\mathcal{L}_2$

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Let $\hat{\mathbf{H}} \in \mathcal{L}_2(i\mathbb{R})$ be the global minimizer of $\mathcal{L}_2$ approximation problem. Since $\mathbf{H} \in \mathcal{H}_2(\mathbb{C}^+)$, one has $\mathbf{H}^- = \mathbf{0}$. Seeing that $\mathcal{L}_2(i\mathbb{R}) = \mathcal{H}_2(\mathbb{C}^-) \bigoplus \mathcal{H}_2(\mathbb{C}^+)$ and this an orthogonal decomposition, thus

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Thus, $\hat{\mathbf{H}}^- = \mathbf{0}$, otherwise $\hat{\mathbf{H}}$ is not a global minimizer. □

# Stability regions estimation

## Results about stability approximation in $\mathcal{L}_2$

**Proposition 2**

For every unstable system $\mathbf{H}$, there exists a neighborhood $V$ of $\mathbf{H}$ such that if $\mathbf{G} \in V$, $\mathbf{G}$ is also unstable. In order words, the set of unstable systems $(\mathcal{L}_2(i\mathbb{R})\backslash\mathcal{H}_2(\mathbb{C}^+))$ is open for the $\mathcal{L}_2$ norm.

Proof.

Since $\mathcal{H}_2(\mathbb{C}^+)$ is a closed set, its complement $(\mathcal{H}_2(\mathbb{C}^+))^c = \mathcal{L}_2(i\mathbb{R})\backslash\mathcal{H}_2(\mathbb{C}^+)$ is open. □

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# Stability regions estimation

## Results about stability approximation in $\mathcal{L}_2$

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# Stability regions estimation

## Results about stability approximation in $\mathcal{L}_2$

**Theorem 1**

Given a unstable system $\mathbf{H} \in \mathcal{L}_2(i\mathbb{R})\backslash\mathcal{H}_2(\mathbb{C}^+)$, there exists $n \in \mathbb{N}^*$ for which the minimizer $G_k$ of order $k \in \mathbb{N}^*$, $k > n$, obtained from the $\mathcal{L}_2$-approximation problem is also unstable.

*Proof.*

Proposition 2 states that if a system is sufficiently close to a unstable system in the $\mathcal{L}_2(i\mathbb{R})$ norm, it is also unstable. Furthermore, the subspace of rational functions which represents the finite LTI systems is dense in $\mathcal{L}_2(i\mathbb{R})$. Hence, for a given LTI unstable system $\mathbf{H} \in \mathcal{L}_2(i\mathbb{R})\backslash\mathcal{H}_2(\mathbb{C}^+)$, a sequence $\mathbf{G}_k$ of systems of order $k$ which satisfies the tangential interpolation conditions, will converge to $\mathbf{H}$. Thus, due to Proposition proposition 2, there exists an order $n \in \mathbb{N}^*$ such that if $k \geq n$, $G_k$ will be unstable as well. □

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# Stability regions estimation

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Given a unstable system $\mathbf{H} \in \mathcal{L}_2(i\mathbb{R})\backslash\mathcal{H}_2(\mathbb{C}^+)$, there exists $n \in \mathbb{N}^*$ for which the minimizer $G_k$ of order $k \in \mathbb{N}^*$, $k > n$, obtained from the $\mathcal{L}_2$-approximation problem is also unstable.

### Proof.

Proposition 2 states that if a system is sufficiently close to a unstable system in the $\mathcal{L}_2(i\mathbb{R})$ norm, it is also unstable. Furthermore, the subspace of rational functions which represents the finite LTI systems is dense in $\mathcal{L}_2(i\mathbb{R})$. Hence, for a given LTI unstable system $\mathbf{H} \in \mathcal{L}_2(i\mathbb{R})\backslash\mathcal{H}_2(\mathbb{C}^+)$, a sequence $\mathbf{G}_k$ of systems of order $k$ which satisfies the tangential interpolation conditions, will converge to $\mathbf{H}$. Thus, due to Proposition proposition 2, there exists an order $n \in \mathbb{N}^*$ such that if $k \geq n$, $G_k$ will be unstable as well. □

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# Stability regions estimation

## Results about stability approximation in $\mathcal{L}_2$

### Proposition 3

For every stable system $\mathbf{H} \in \mathcal{H}_2(\mathbb{C}^+)$, there exists a sequence of unstable systems $\mathbf{G_k} \in \mathcal{L}_2(i\mathbb{R}) \backslash \mathcal{H}_2(\mathbb{C}^+)$, $k \in \mathbb{N}^*$, such that

$$\|\mathbf{H} - \mathbf{G_k}\|_{\mathcal{L}_2(i\mathbb{R})} \to 0, \quad \text{when } k \to \infty \tag{34}$$

In other words, the set $\mathcal{H}_2(\mathbb{C}^+)$ is not an open set of $\mathcal{L}_2(i\mathbb{R})$.

### Proof.

Given $\mathbf{H} \in \mathcal{H}_2(\mathbb{C}^+)$, let $\mathbf{h} \in \mathcal{H}_2(\mathbb{C}^-)$ be an element such that $\|\mathbf{h}\|_{\mathcal{L}_2(i\mathbb{R})} = 1$. The system $\mathbf{G_k} = \mathbf{H} + \frac{1}{k}\mathbf{h} \in \mathcal{L}_2(i\mathbb{R}) \backslash \mathcal{H}_2(\mathbb{C}^+)$ and $\|\mathbf{H} - \mathbf{G_k}\|_{\mathcal{L}_2(i\mathbb{R})} = \frac{1}{k}\|\mathbf{h}\|_{\mathcal{L}_2(i\mathbb{R})} \to 0$ when $k \to \infty$. □

# Stability regions estimation

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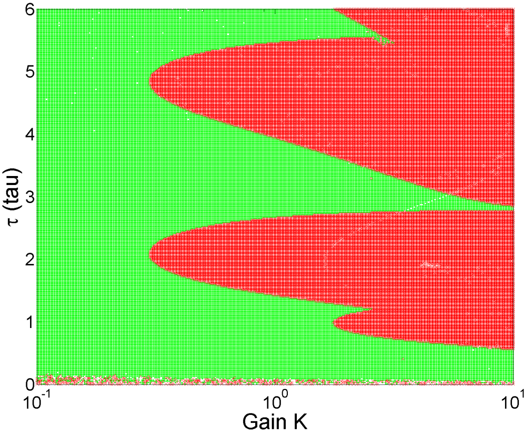
# Stability regions estimation

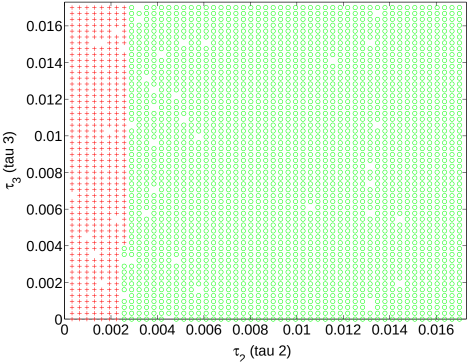
## Benchmark TDS-#1




- $\approx 0.13$s for each approximation
- Approximation of order $r = 6$.

# Stability regions estimation

## Benchmark TDS-#2




- $\tau_1$ fixed as $17$ms.
- $\approx 30$s for approximation
- Approximation of order $r = 12$.

# Table of Contents

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# Conclusions

## Conclusion

- Projection model approximation method using realization.
- Loewner interpolation method using transfer function.
- Method to estimate the stability of large-scale TDS and PDE is proposed.
- Some arguments are given to justify this method
- No borne of estimation error.

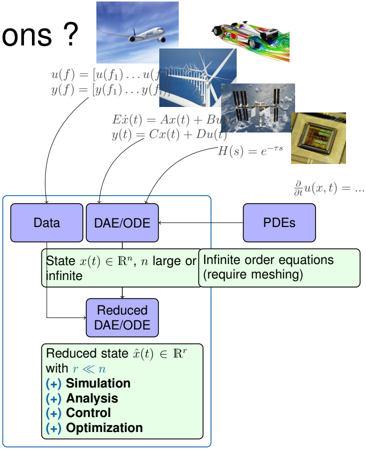
## Perspectives

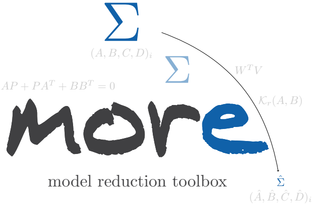
- Algorithm 'branch and bound' to find borders.
- $\mathcal{H}_2$-LPV model reduction.
- LSS reduction to TDS-system.

# CONCLUSIONS

## - MORE toolbox [24]

Thanks for your Attention. Questions ?




24 C. Poussot-Vassal and P. Vuillemin, *"Introduction to MORE: a MOdel REduction Toolbox"*, IEEE Multi Systems Conference, pp. 776-781.