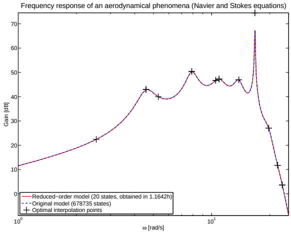
Fluid flow dynamical model approximation and control

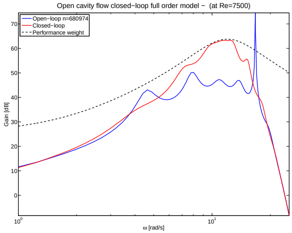
... a case-study on an open cavity flow

C. Poussot-Vassal & D. Sipp

THE FRENCH AEROSPACE LAB

Journée conjointe GT Contrôle de Décollement & GT MOSAR





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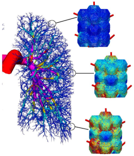
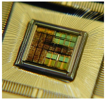
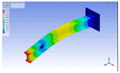
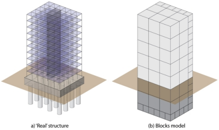
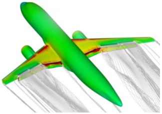
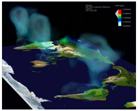
Large-scale dynamical models

... some motivating examples in the simulation & control domains

Large-scale systems are present in many engineering fields: aerospace, computational biology, building structure, VLI circuits, automotive, weather forecasting, fluid flow. . .



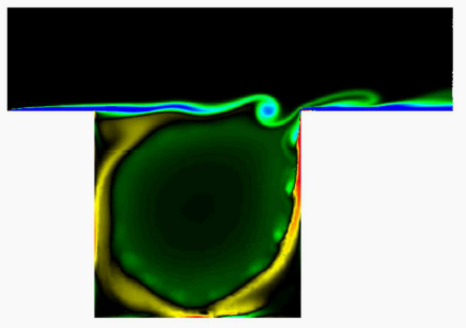
- difficulties with simulation & memory management (e.g. ODE solvers)
- difficulties with analysis (e.g. frequency response, μ_{ssv} and $\mathcal{H}_\infty$ computation . . .)
- difficulties with controller design (e.g. robust, optimal, predictive, . . .)

Large-scale dynamical models

... in fluid flow dynamical problems

Fluid flow dynamical models

- Complex phenomena describing the motion of fluid flows,
- described by Navier and Stokes equations,
- arising when modeling the weather, ocean currents, water flow in a pipe and air flow around a wing. . .



Some challenges arising

- Modeling and simulation
- Control turbulences

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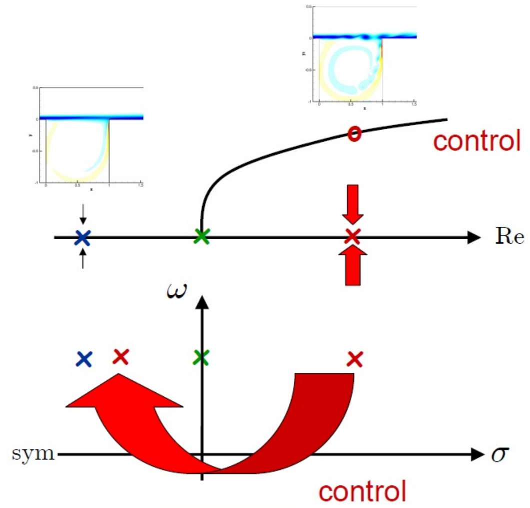
Open-cavity flow and Hopf bifurcation



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Outlines

Physical model and dynamical modeling
- Navier and Stokes equations and assumptions
- Linearisation and simplifications
- Reduce and control approach

Large-scale dynamical model approximation

Active closed-loop control design

Conclusions

Navier and Stokes equations

$$\partial_t u + u \cdot \nabla u = -\nabla p + \frac{1}{Re}\Delta u \quad (1)$$

$$\nabla \cdot u = 0 \quad (2)$$

or in a condensed way

$$\dot{x}(t) = f\big(x(t), Re\big) \quad (3)$$

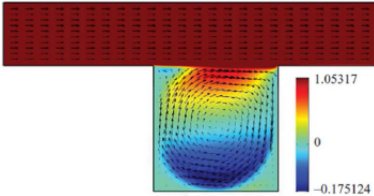
Existence of equilibrium points for a range of Reynolds numbers

Family of base-flows parametrized by the **Reynolds** number: $f\big(x_0, Re\big) = 0$

Physical model and dynamical modeling

Navier and Stokes equations and assumptions

Navier and Stokes equations



Existence of equilibrium points for a range of Reynolds numbers

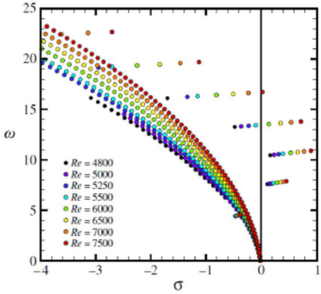
Family of base-flows parametrized by the **Reynolds** number: $f(x_0, Re) = 0$

Linearisation for different Reynolds Numbers

$$x(t) = x_0^{(Re)}(t) + \epsilon x_1^{(Re)}(t) \tag{1}$$

where ϵ is small

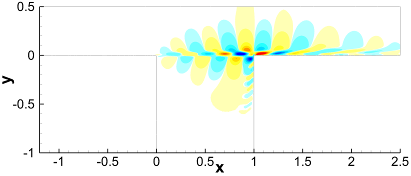
$$\dot{x}_1^{(Re)}(t) = \left.\frac{\partial f}{\partial x}\right|_{x_0^{(Re)}} x_1(t) = A(Re)x_1(t) \tag{2}$$

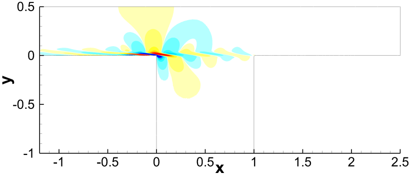


Physical model and dynamical modeling

Linearisation and simplifications - Eigenvectors

Right and left eigenvectors





Localization of sensor and actuator

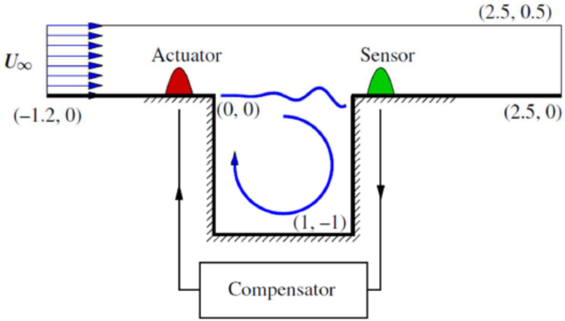
Physical model and dynamical modeling

Linearisation and simplifications - Dynamical model and control setting

Actuator/sensor

$$\begin{aligned} E\dot{x}(t) &= A(Re)x(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \quad (3)$$

Actuator (volumic forcing in momentum equations) Sensor (shear stress)



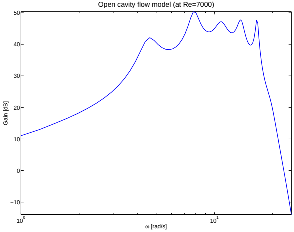
Physical model and dynamical modeling

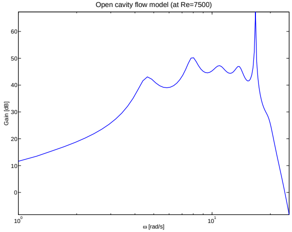
Linearisation and simplifications - Dynamical model and control setting

Actuator/sensor

$$\begin{array}{rcl} E\dot{x}(t) & = & A(Re)x(t) + Bu(t) \\ y(t) & = & Cx(t) \end{array} \quad (4)$$

- Two Reynolds cases ($Re = 7000$ and $Re = 7500$)
- Single Input Single Output Differential Algebraic Equations (SISO DAE)
- 8 unstable modes, order ≈ 650,000 states





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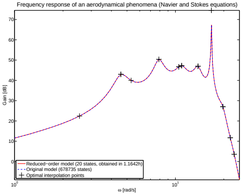
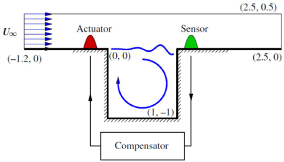
Physical model and dynamical modeling

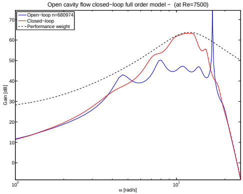
Reduce and control approach

Proposed procedure

- Approximate the large-scale dynamical model
- Design a stabilizing active closed loop control strategy







Challenge of simulating and controlling such high complexity system...

PHYSICAL MODEL AND DYNAMICAL MODELING

Reduce and control approach

Proposed procedure

- Approximate the large-scale dynamical model
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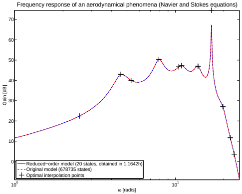
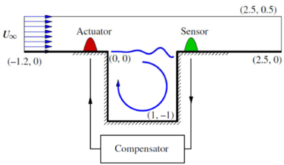
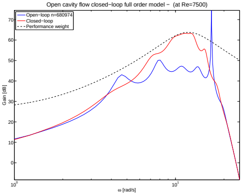






Challenge of simulating and controlling such high complexity system...

Outlines

Physical model and dynamical modeling

Large-scale dynamical model approximation
- Projection-based approximation framework
- Approximation in the $\mathcal{H}_2$, $\mathcal{H}_{2,\Omega}$ and $\mathcal{L}_2$-norm
- MIMO IRKA (or ITIA)
- IETIA
- Balanced Truncation POD
- Fluid flow dynamical model approximation

Active closed-loop control design

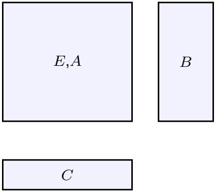
Conclusions

Large-scale dynamical model approximation

Projection-based approximation framework

Let $H:\mathbb{C}\rightarrow\mathbb{C}^{n_y\times n_u}$ be a n_u inputs n_y outputs, full order $\mathcal{H}_2^{n_y\times n_u}$ (or $\mathcal{L}_2^{n_y\times n_u}$) complex-valued function describing a **LTI** dynamical system as a DAE of order n, with realization $\mathbf{H}$:

$$\mathbf{H}:\left\{\begin{array}{rcl} E\dot{x}(t) & = & Ax(t)+Bu(t) \\ y(t) & = & Cx(t) \end{array}\right. \quad (5)$$



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LARGE-SCALE DYNAMICAL MODEL APPROXIMATION

Projection-based approximation framework

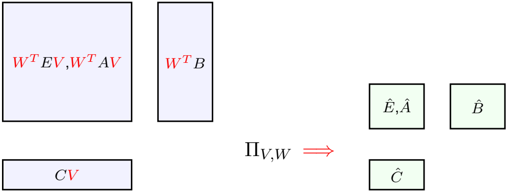
Let $H : \mathbb{C} \rightarrow \mathbb{C}^{n_y \times n_u}$ be a n_u inputs n_y outputs, full order $\mathcal{H}_2^{n_y \times n_u}$ (or $\mathcal{L}_2^{n_y \times n_u}$) complex-valued function describing a **LTI** dynamical system as a DAE of order n, with realization $\mathbf{H}$:

$$\mathbf{H} : \begin{cases} E\dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) \end{cases} \quad (5)$$

the approximation problem consists in finding $V, W \in \mathbb{R}^{n \times r}$ (with $r \ll n$) **spanning $\mathcal{V}$ and $\mathcal{W}$ subspaces** and forming a projector $\Pi_{V,W} = VW^T$, such that

$$\hat{\mathbf{H}} : \begin{cases} W^T EV\dot{\hat{x}}(t) & = & W^T AV\hat{x}(t) + W^T Bu(t) \\ \hat{y}(t) & = & CV\hat{x}(t) \end{cases} \quad (6)$$

well approximates $\mathbf{H}$.



LARGE-SCALE DYNAMICAL MODEL APPROXIMATION

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$$\hat{\mathbf{H}} : \begin{cases} W^T E V \dot{\hat{x}}(t) & = & W^T A V \hat{x}(t) + W^T B u(t) \\ \hat{y}(t) & = & CV\hat{x}(t) \end{cases} \tag{6}$$

well approximates $\mathbf{H}$.

- **Small approximation error and/or global error bound**
- **Stability / passivity preservation**
- **Numerically stable & efficient procedure**

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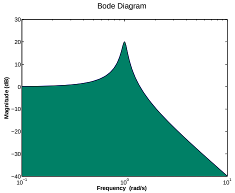
LARGE-SCALE DYNAMICAL MODEL APPROXIMATION

Approximation in the $\mathcal{H}_2$, $\mathcal{H}_{2,\Omega}$ and $\mathcal{L}_2$-norm [1, 2]

$\mathcal{H}_2$ model approximation

$$\hat{H} := \arg \min_{\substack{G \in \mathcal{H}_2^{n_y \times n_u} \\ \textbf{rank}(G) = r \ll n}} ||H - G||_{\mathcal{H}_2} \tag{7}$$



1 S. Gugercin and A C. Antoulas and C A. Beattie, *"$\mathcal{H}_2$ Model Reduction for Large Scale Linear Dynamical Systems"*, SIAM Journal on Matrix Analysis and Applications, vol. 30(2), June 2008, pp. 609-638.

2 K. A. Gallivan, A. Vanderope, and P. Van-Dooren, *"Model reduction of MIMO systems via tangential interpolation"*, SIAM Journal of Matrix Analysis and Application, vol. 26(2), February 2004, pp. 328-349.

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Large-scale dynamical model approximation

Approximation in the $\mathcal{H}_2$, $\mathcal{H}_{2,\Omega}$ and $\mathcal{L}_2$-norm[1 2]

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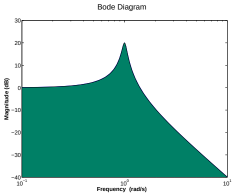

Energy to an impulse input

$$\begin{aligned}
||H||^2_{\mathcal{H}_2} &:= \textbf{trace}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\overline{H(i\nu)}H(i\nu)\right)d\nu\right) \\
&:= \textbf{trace}\left(C\mathcal{P}C^T\right) = \textbf{trace}\left(B^T\mathcal{Q}B\right) \\
&:= \sum_{i=1}^{n} \textbf{trace}\left(\phi_i H(-\lambda_i)^T\right)
\end{aligned}$$

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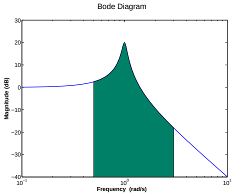
LARGE-SCALE DYNAMICAL MODEL APPROXIMATION

Approximation in the $\mathcal{H}_2$, $\mathcal{H}_{2,\Omega}$ and $\mathcal{L}_2$-norm[3] [4]

$\mathcal{H}_{2,\Omega}$ model approximation

$$\hat{H} := \arg \min_{\substack{G \in \mathcal{H}_\infty^{n_y \times n_u} \\ \textbf{rank}(G) = r \ll n}} ||H - G||_{\mathcal{H}_{2,\Omega}} \quad (8)$$



3 P. Vuillemin, C. Poussot-Vassal and D. Alazard, *"A Spectral Expression for the Frequency-Limited $\mathcal{H}_2$-norm"*, Available as http://arxiv.org/abs/1211.1858, 2012.

4 P. Vuillemin, C. Poussot-Vassal and D. Alazard, *"Spectral expression for the Frequency-Limited $\mathcal{H}_2$-norm of LTI Dynamical Systems with High Order Poles"*, European Control Conference, 2014, pp. 55-60.

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Large-scale dynamical model approximation

Approximation in the $\mathcal{H}_2$, $\mathcal{H}_{2,\Omega}$ and $\mathcal{L}_2$-norm[3] [4]

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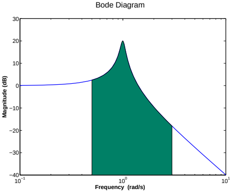

Energy (in a finite frequency) to an impulse input

$$\begin{aligned}
||H||^2_{\mathcal{H}_{2,\Omega}} &:= \textbf{trace}\left(\frac{1}{\pi}\int_{\Omega}\left(\overline{H(i\nu)}H(i\nu)\right)d\nu\right) \\
&:= \textbf{trace}\left(C\mathcal{P}_{\Omega}C^T\right) = \textbf{trace}\left(B^T\mathcal{Q}_{\Omega}B\right) \\
&:= \sum_{i=1}^{n}\textbf{trace}\left(\phi_i H(-\lambda_i)^T\right)\left[-\frac{2}{\pi}\textbf{atan}\left(\frac{\omega}{\lambda_i}\right)\right]
\end{aligned}$$

[3] P. Vuillemin, C. Poussot-Vassal and D. Alazard, *"A Spectral Expression for the Frequency-Limited $\mathcal{H}_2$-norm"*, Available as http://arxiv.org/abs/1211.1858, 2012.

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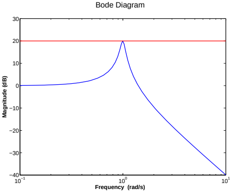
Large-scale dynamical model approximation

Approximation in the $\mathcal{H}_2$, $\mathcal{H}_{2,\Omega}$ and $\mathcal{L}_2$-norm[5]

$\mathcal{H}_\infty$ model approximation

$$\hat{H} := \arg \min_{\substack{G \in \mathcal{H}_\infty^{n_y \times n_u} \\ \textbf{rank}(G) = r \ll n}} ||H - G||_{\mathcal{H}_\infty} \tag{9}$$



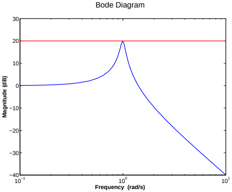
5 P. Vuillemin, C. Poussot-Vassal, D. Alazard, *"Two upper bounds on the $\mathcal{H}_\infty$-norm of LTI dynamical systems"*, 19th IFAC World Congress, pp. 5562-5567, 2014.

Large-scale dynamical model approximation

Approximation in the $\mathcal{H}_2$, $\mathcal{H}_{2,\Omega}$ and $\mathcal{L}_2$-norm[5]

$\mathcal{H}_\infty$ model approximation

$$\hat{H} := \arg \min_{\substack{G \in \mathcal{H}_\infty^{n_y \times n_u} \\ \textbf{rank}(G) = r \ll n}} ||H - G||_{\mathcal{H}_\infty} \tag{9}$$



Worst case to an impulse input (numerically complex to compute)

$$\begin{aligned} \|H\|_{\mathcal{H}_\infty} &:= \sup_{\omega \in \mathbb{R}} \overline{\sigma}\left(H(j\omega)\right) \\ &:= \max_{w \in L_2} \frac{||y||_2}{||u||_2} \end{aligned}$$

5 P. Vuillemin, C. Poussot-Vassal, D. Alazard, *"Two upper bounds on the $\mathcal{H}_\infty$-norm of LTI dynamical systems"*, 19th IFAC World Congress, pp. 5562-5567, 2014.

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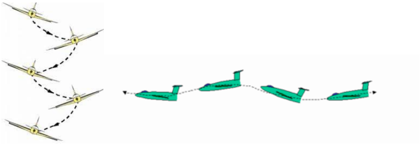
Large-scale dynamical model approximation

Approximation in the $\mathcal{H}_2$, $\mathcal{H}_{2,\Omega}$ and $\mathcal{L}_2$-norm[6]

Mismatch objective and eigenvector preservation

$$\hat{H} := \arg \min_{\substack{G \in \mathcal{L}_2^{n_y \times n_u} \\ \textbf{rank}(G) = r \ll n \\ \lambda_k(G) \subseteq \lambda(H) \;\; k=1,\dots,q_1 < r}} ||H - G||_{\mathcal{H}_2} \quad (10)$$

- More than a $\mathcal{H}_2$ (sub-optimal) criteria
- Keep some user defined eigenvalues... *e.g.* the unstable/well known ones



6 C. Poussot-Vassal and P. Vuillemin, *"An Iterative Eigenvector Tangential Interpolation Algorithm for Large-Scale LTI and a Class of LPV Model Approximation"*, European Control Conference, 2013, pp. 4490-4495.

Large-scale dynamical model approximation

Approximation in the $\mathcal{H}_2$, $\mathcal{H}_{2,\Omega}$ and $\mathcal{L}_2$-norm

MIMO Iterative Krylov Interpolation Algorithm (or ITIA)

- $\mathcal{H}_2$-optimal, but still do not theoretically preserves stability
- Numerically very efficient (*e.g.* with sparse methods, $Ax = b$)

Iterative Eigenvector Tangential Interpolation Algorithm (IETIA)

- $\mathcal{H}_2$ sub-optimal, but still do not theoretically preserves stability
- Numerically very efficient (*e.g.* with sparse methods, $Ax = b$ and $AV = EV\lambda$)
- Applicable to $\mathcal{L}_2$ dynamical systems

Balanced Truncation Proper Orthogonal Decomposition (BT POD)

- Provides a $\mathcal{H}_\infty$-norm mismatch error (not tight), preserves stability
- Costly to compute, but a Matrix free version alleviate this problem by replacing by simulation (direct and adjoint)

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Large-scale dynamical model approximation

MIMO IRKA (or ITIA) - $\mathcal{H}_2$ optimality conditions (Tangential subspace approach) [7] [8]

Given $H(s)$, let $V \in \mathbb{C}^{n \times r}$ and $W \in \mathbb{C}^{n \times r}$ be matrices of full column rank r such that $W^* V = I_r$. If, for $j = 1, \dots, r$,

$$\left[(\sigma_j E - A)^{-1} B \hat{b}_j\right] \in \textbf{span}(V) \text{ and } \left[(\sigma_j E - A^T)^{-1} C^T \hat{c}_j^*\right] \in \textbf{span}(W) \quad (11)$$

where $\sigma_j \in \mathbb{C}$, $\hat{b}_j \in \mathbb{C}^{n_u}$ and $\hat{c}_j \in \mathbb{C}^{n_y}$, be given sets of interpolation points and left and right tangential directions, respectively.

[7] P. Van-Dooren, K. A. Gallivan, and P. A. Absil, *"$\mathcal{H}_2$-optimal model reduction of MIMO systems"*, Applied Mathematics Letters, vol. 21(12), December 2008, pp. 53-62.

[8] S. Gugercin and A C. Antoulas and C A. Beattie, *"$\mathcal{H}_2$ Model Reduction for Large Scale Linear Dynamical Systems"*, SIAM Journal on Matrix Analysis and Applications, vol. 30(2), June 2008, pp. 609-638.

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LARGE-SCALE DYNAMICAL MODEL APPROXIMATION

MIMO IRKA (or ITIA) - $\mathcal{H}_2$ optimality conditions (Tangential subspace approach) [7] [8]

Given $H(s)$, let $V \in \mathbb{C}^{n\times r}$ and $W \in \mathbb{C}^{n\times r}$ be matrices of full column rank r such that $W^*V = I_r$. If, for $j = 1, \ldots, r$,

$$\left[(\sigma_j E - A)^{-1} B\hat{b}_j\right] \in \textbf{span}(V) \text{ and } \left[(\sigma_j E - A^T)^{-1} C^T \hat{c}_j^*\right] \in \textbf{span}(W) \quad (11)$$

where $\sigma_j \in \mathbb{C}$, $\hat{b}_j \in \mathbb{C}^{n_u}$ and $\hat{c}_j \in \mathbb{C}^{n_y}$, be given sets of interpolation points and left and right tangential directions, respectively. Then, the reduced order system $\hat{H}(s)$ satisfies the tangential interpolation conditions

$$\begin{array}{rcl} H(-\hat{\sigma}_j)\hat{b}_j & = & \hat{H}(-\hat{\sigma}_j)\hat{b}_j \\ \hat{c}_j^* H(-\hat{\sigma}_j) & = & \hat{c}_j^* \hat{H}(-\hat{\sigma}_j) \\ \hat{c}_j^* \left.\frac{d}{ds}H(s)\right|_{s=-\hat{\sigma}_j} \hat{b}_j & = & \hat{c}_j^* \left.\frac{d}{ds}\hat{H}(s)\right|_{s=-\hat{\sigma}_j} \hat{b}_j \end{array} \quad (12)$$

7 P. Van-Dooren, K. A. Gallivan, and P. A. Absil, *"$\mathcal{H}_2$-optimal model reduction of MIMO systems"*, Applied Mathematics Letters, vol. 21(12), December 2008, pp. 53-62.

8 S. Gugercin and A C. Antoulas and C A. Beattie, *"$\mathcal{H}_2$ Model Reduction for Large Scale Linear Dynamical Systems"*, SIAM Journal on Matrix Analysis and Applications, vol. 30(2), June 2008, pp. 609-638.

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Large-scale dynamical model approximation

Require: $\mathbf{H} = (E, A, B, C)$, $\{\sigma_1^{(0)}, \dots, \sigma_{q_2}^{(0)}\} \in \mathbb{C}^{q_2}$, $\{\hat{b}_1, \dots, \hat{b}_{q_2}\} \in \mathbb{C}^{n_u \times q_2}$, $\{\hat{c}_1, \dots, \hat{c}_{q_2}\} \in \mathbb{C}^{n_y \times q_2}$ and $r \in \mathbb{N}$

Construct,

$$\textbf{span}\left(V(\sigma_j^{(0)}, \hat{b}_j)\right) \text{ and } \textbf{span}\left(W(\sigma_j^{(0)}, \hat{c}_j^*)\right) \tag{13}$$

Compute $W \leftarrow W(V^T W)^{-1}$

while Stopping criteria **do**

$k \leftarrow k + 1$

$\hat{E} = W^T E V$, $\hat{A} = W^T A V$, $\hat{B} = W^T B$, $\hat{C} = CV$

Compute $\hat{A}R = \Lambda(\hat{A}, \hat{E})R$ and $L\hat{A} = \Lambda(\hat{A}, \hat{E})L$

Compute $\{\hat{b}_1, \dots, \hat{b}_r\} = \hat{B}^T L$ and $\{\hat{c}_1^*, \dots, \hat{c}_r^*\} = \hat{C}R$

Set $\sigma^{(i)} = -\Lambda(\hat{A}, \hat{E})$

Construct,

$$\textbf{span}\left(V(\sigma_j^{(k)}, \hat{b}_j)\right) \text{ and } \textbf{span}\left(W(\sigma_j^{(k)}, \hat{c}_j^*)\right) \tag{14}$$

Compute $W \leftarrow W(V^T W)^{-1}$

end while

Construct $\hat{\mathbf{H}} := (W^T E V, W^T A V, W^T B, CV)$

Ensure: $V, W \in \mathbb{R}^{n \times r}$, $W^T V = I_r$

Large-scale dynamical model approximation

Require: $\mathbf{H} = (E, A, B, C)$, $\{\sigma_1^{(0)}, \dots, \sigma_{q_2}^{(0)}\} \in \mathbb{C}^{q_2}$, $\{\hat{b}_1, \dots, \hat{b}_{q_2}\} \in \mathbb{C}^{n_u \times q_2}$, $\{\hat{c}_1, \dots, \hat{c}_{q_2}\} \in \mathbb{C}^{n_y \times q_2}$ and $r \in \mathbb{N}$

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   Compute $\{\hat{b}_1, \dots, \hat{b}_r\} = \hat{B}^T L$ and $\{\hat{c}_1^*, \dots, \hat{c}_r^*\} = \hat{C}R$

   Set $\sigma^{(i)} = -\Lambda(\hat{A}, \hat{E})$

   Construct,

$$\textbf{span}\left(V(\sigma_j^{(k)}, \hat{b}_j)\right) \text{ and } \textbf{span}\left(W(\sigma_j^{(k)}, \hat{c}_j^*)\right) \tag{14}$$

   Compute $W \leftarrow W(V^T W)^{-1}$

end while

Construct $\hat{\mathbf{H}} := (W^T E V, W^T A V, W^T B, CV)$

Ensure: $V, W \in \mathbb{R}^{n \times r}$, $W^T V = I_r$

Large-scale dynamical model approximation

IETIA - $\mathcal{H}_2$ & spectral optimality conditions (Tangential subspace approach) [9]

Given $H(s)$, let $V \in \mathbb{C}^{n\times r}$ and $W \in \mathbb{C}^{n\times r}$ be matrices of full column rank $r = q_1 + q_2$ such that $W^*V = I_r$. If, for $i = 1, \dots, q_1$ and $j = 1, \dots, q_2$,

$$\left[r_i^\star \;\; (\sigma_j E - A)^{-1} B \hat{b}_j \right] \in \textbf{span}(V) \text{ and } \left[l_i^\star \;\; (\sigma_j E - A^T)^{-1} C^T \hat{c}_j^* \right] \in \textbf{span}(W) \quad (15)$$

$l_i^\star \in \mathbb{C}^n$ and $r_i^\star \in \mathbb{C}^n$ are left and right eigenvectors associated to $\lambda_i^\star \in \mathbb{C}$ eigenvalues associated to A, E and $\sigma_j \in \mathbb{C}$, $\hat{b}_j \in \mathbb{C}^{n_u}$ and $\hat{c}_j \in \mathbb{C}^{n_y}$, be given sets of interpolation points and left and right tangential directions, respectively.

9 C. Poussot-Vassal and P. Vuillemin, *"An Iterative Eigenvector Tangential Interpolation Algorithm for Large-Scale LTI and a Class of LPV Model Approximation"*, European Control Conference, 2013, pp. 4490-4495.

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LARGE-SCALE DYNAMICAL MODEL APPROXIMATION

IETIA - $\mathcal{H}_2$ & spectral optimality conditions (Tangential subspace approach) [9]

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$l_i^\star \in \mathbb{C}^n$ and $r_i^\star \in \mathbb{C}^n$ are left and right eigenvectors associated to $\lambda_i^\star \in \mathbb{C}$ eigenvalues associated to A, E and $\sigma_j \in \mathbb{C}$, $\hat{b}_j \in \mathbb{C}^{n_u}$ and $\hat{c}_j \in \mathbb{C}^{n_y}$, be given sets of interpolation points and left and right tangential directions, respectively. Then, the reduced order system $\hat{H}(s)$ satisfies the eigenvalue conditions,

$$\{\lambda_1^\star,\dots,\lambda_{q_1}^\star\} \subset \Lambda(\hat{A},\hat{E}) \quad (16)$$

and the tangential interpolation conditions

$$\begin{aligned} H(-\hat{\sigma}_j)\hat{b}_j &= \hat{H}(-\hat{\sigma}_j)\hat{b}_j \\ \hat{c}_j^* H(-\hat{\sigma}_j) &= \hat{c}_j^*\hat{H}(-\hat{\sigma}_j) \\ \hat{c}_j^* \frac{d}{ds}H(s)\Big|_{s=-\hat{\sigma}_j}\hat{b}_j &= \hat{c}_j^*\frac{d}{ds}\hat{H}(s)\Big|_{s=-\hat{\sigma}_j}\hat{b}_j \end{aligned} \quad (17)$$

[9] C. Poussot-Vassal and P. Vuillemin, *"An Iterative Eigenvector Tangential Interpolation Algorithm for Large-Scale LTI and a Class of LPV Model Approximation"*, European Control Conference, 2013, pp. 4490-4495.

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Large-scale dynamical model approximation

Require: $\mathbf{H} = (E, A, B, C)$, $\{\lambda_1^\star, \dots, \lambda_{q_1}^\star\} \in \mathbb{C}^{q_1}$, $\{\sigma_1^{(0)}, \dots, \sigma_{q_2}^{(0)}\} \in \mathbb{C}^{q_2}$, $\{\hat{b}_1, \dots, \hat{b}_{q_2}\} \in \mathbb{C}^{n_u \times q_2}$, $\{\hat{c}_1, \dots, \hat{c}_{q_2}\} \in \mathbb{C}^{n_y \times q_2}$ and $r = q_1 + q_2 \in \mathbb{N}$

Compute $\{l_1^\star, \dots, l_{q_1}^\star\}$ and $\{r_1^\star, \dots, r_{q_1}^\star\}$, eigenvectors of $\{\lambda_1^\star, \dots, \lambda_{q_1}^*\}$

Construct,

$$\textbf{span}\left(V(l_i^\star, \sigma_j^{(0)}, \hat{b}_j)\right) \text{ and } \textbf{span}\left(W(r_i^\star, \sigma_j^{(0)}, \hat{c}_j^*)\right) \tag{18}$$

Compute $W \leftarrow W(V^T W)^{-1}$

while Stopping criteria **do**

$k \leftarrow k + 1$

$\hat{E} = W^T E V$, $\hat{A} = W^T A V$, $\hat{B} = W^T B$, $\hat{C} = CV$

Compute $\hat{A}R = \hat{E}\Lambda(\hat{A}, \hat{E})R$ and $L\hat{A} = \Lambda(\hat{A})L$

Compute $\{\hat{b}_1, \dots, \hat{b}_{q_2}\} = \hat{B}^T L$ and $\{\hat{c}_1^*, \dots, \hat{c}_{q_2}^*\} = \hat{C}R$

Set $\sigma^{(i)} = -\Lambda(\hat{A}, \hat{E})$

Construct,

$$\textbf{span}\left(V(l_i^\star, \sigma_j^{(k)}, \hat{b}_j)\right) \text{ and } \textbf{span}\left(W(r_i^\star, \sigma_j^{(k)}, \hat{c}_j^*)\right) \tag{19}$$

Compute $W \leftarrow W(V^T W)^{-1}$

end while

Construct $\hat{\mathbf{H}} := (W^T E V, W^T A V, W^T B, CV)$

Ensure: $V, W \in \mathbb{R}^{n \times r}$, $W^T V = I_r$, $\{\lambda_1^\star, \dots, \lambda_{q_1}^\star\} \subset \Lambda(\hat{A}, \hat{E})$

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LARGE-SCALE DYNAMICAL MODEL APPROXIMATION

Require: $\mathbf{H} = (E, A, B, C)$, $\{\lambda_1^\star, \dots, \lambda_{q_1}^\star\} \in \mathbb{C}^{q_1}$, $\{\sigma_1^{(0)}, \dots, \sigma_{q_2}^{(0)}\} \in \mathbb{C}^{q_2}$, $\{\hat{b}_1, \dots, \hat{b}_{q_2}\} \in \mathbb{C}^{n_u \times q_2}$, $\{\hat{c}_1, \dots, \hat{c}_{q_2}\} \in \mathbb{C}^{n_y \times q_2}$ and $r = q_1 + q_2 \in \mathbb{N}$

Compute $\{l_1^\star, \dots, l_{q_1}^\star\}$ and $\{r_1^\star, \dots, r_{q_1}^\star\}$, eigenvectors of $\{\lambda_1^\star, \dots, \lambda_{q_1}^*\}$

Construct,

$$\textbf{span}\left(V(l_i^\star, \sigma_j^{(0)}, \hat{b}_j)\right) \text{ and } \textbf{span}\left(W(r_i^\star, \sigma_j^{(0)}, \hat{c}_j^*)\right) \tag{18}$$

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$\hat{E} = W^T E V$, $\hat{A} = W^T A V$, $\hat{B} = W^T B$, $\hat{C} = CV$

Compute $\hat{A}R = \hat{E}\Lambda(\hat{A}, \hat{E})R$ and $L\hat{A} = \Lambda(\hat{A})L$

Compute $\{\hat{b}_1, \dots, \hat{b}_{q_2}\} = \hat{B}^T L$ and $\{\hat{c}_1^*, \dots, \hat{c}_{q_2}^*\} = \hat{C}R$

Set $\sigma^{(i)} = -\Lambda(\hat{A}, \hat{E})$

Construct,

$$\textbf{span}\left(V(l_i^\star, \sigma_j^{(k)}, \hat{b}_j)\right) \text{ and } \textbf{span}\left(W(r_i^\star, \sigma_j^{(k)}, \hat{c}_j^*)\right) \tag{19}$$

Compute $W \leftarrow W(V^T W)^{-1}$

end while

Construct $\hat{\mathbf{H}} := (W^T E V, W^T A V, W^T B, C V)$

Ensure: $V, W \in \mathbb{R}^{n \times r}$, $W^T V = I_r$, $\{\lambda_1^\star, \dots, \lambda_{q_1}^\star\} \subset \Lambda(\hat{A}, \hat{E})$

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Large-scale dynamical model approximation

Balanced Truncation POD - Idea

Assume a stable system, the impulse response, $t \geq 0$ such that $h(t) = Ce^{At}B$

- Input-to-state map $x_c(t) = e^{At}B$
- State-to-output map $x_o(t) = Ce^{At} = (e^{A^*t}C^*)^*$

Corresponding to Gramian:

$$
\begin{array}{rclcl}
\mathcal{P} & = & \sum_t x_c(t)x_c^*(t) & = & \int_0^\infty e^{At}BB^*e^{A^*t}dt \\
\mathcal{Q} & = & \sum_t x_o^*(t)x_o(t) & = & \int_0^\infty e^{A^*t}C^*Ce^{At}dt
\end{array}
\quad (20)
$$

solution of the Lyapunov equations,

$$
\left\{
\begin{array}{rcl}
A\mathcal{P} + \mathcal{P}A^* + BB^* & = & 0 \\
A^*\mathcal{Q} + \mathcal{Q}A + C^*C & = & 0
\end{array}
\right.
\quad (21)
$$

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Large-scale dynamical model approximation

Balanced Truncation POD - in its Balanced basis $T = [T_1, \dots, T_n]$

Meaning of Gramians:

- $x_f^* \mathcal{P}^{-1} x_f$, is the minimal energy required to steer the state from 0 to x_f as $t \rightarrow \infty$.
- $x_0^* \mathcal{Q} x_0$ is the maximal energy produced by observing the output of the system corresponding to an initial state x_0 when no input is applied.

Balanced basis $T = [T_1, \dots, T_n]$:

$$\mathcal{P} = \mathcal{Q} = S = \textbf{diag}(\sigma_1, \dots, \sigma_n) \text{ with } \sigma_1 > \sigma_2 > \dots > \sigma_n \quad (22)$$

- $T_1^* \mathcal{P}^{-1} T_1 = \frac{1}{\sigma_1}$ (easily controllable) and $T_1^* \mathcal{Q} T_1 = \sigma_1$ (easily observable)
- $T_n^* \mathcal{P}^{-1} T_n = \frac{1}{\sigma_n}$ (weakly controllable) and $T_n^* \mathcal{Q} T_n = \sigma_n$ (weakly observable)

moreover,

- stability is preserved
- error between original and reduced systems is upper-bounded by,

$$\sigma_r \leq ||H - \hat{H}||_{\mathcal{H}_\infty} \leq 2(\sigma_{r+1} + \dots + \sigma_n) \quad (23)$$

where σ_i, $i = 1, \dots n$ are the Hankel singular values.

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Large-scale dynamical model approximation

Balanced Truncation POD

Require: $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_u}$, $C \in \mathbb{R}^{n_y\times n}$, $r \in \mathbb{N}^*$

1: Solve $A\mathcal{P} + \mathcal{P}A^T + BB^T = 0$

2: Solve $A^T\mathcal{Q} + \mathcal{Q}A + C^TC = 0$

3: $\mathcal{P} = UU^T$ and $\mathcal{Q} = LL^T$

4: SVD decomposition: $[Z, S, Y] = \textbf{SVD}(U^TL)$

5: Set $V = UZS^{-1/2}$

6: Set $W = LYS^{-1/2}$

7: Apply projectors V and W and obtain

$$\mathbf{H}_\perp = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] \tag{24}$$

8: Approximation is obtained by $\hat{\mathbf{H}} = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$

Ensure: Small approximation error

Balanced Truncation POD - Factorization of Gramians with snapshot method

Numerical approximation

Solving Lyapunov equations is memory/time consuming.

$$\mathcal{P} = UU^* \tag{25}$$

usually done with Cholesky Factorization, but noticing that (i finite)

$$\begin{aligned} \mathcal{P} &= \int_0^\infty e^{At} BB^* e^{A^*t} \\ &\approx \sum_i x_c(t_i) x_c^*(t_i) \Delta t \\ &\approx \begin{bmatrix} x_c(t_1)\sqrt{\Delta t} & x_c(t_2)\sqrt{\Delta t} & \dots \end{bmatrix} \begin{bmatrix} x_c^*(t_1)\sqrt{\Delta t} \\ x_c^*(t_2)\sqrt{\Delta t} \\ \vdots \end{bmatrix} \end{aligned} \tag{26}$$

where $x_c(t) = e^{At}B$ (here use of the linearized model, $\dot{x}(t) = Ax(t)$, $x(0) = B$).
Same process for $\mathcal{Q}$ with adjoint simulation $\dot{x}(t) = A^* x(t)$, $x(0) = C^*$.

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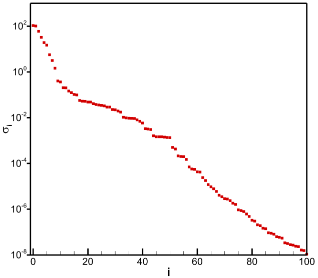
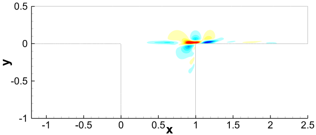
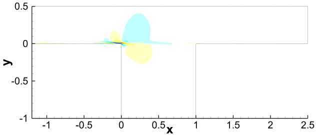
Large-scale dynamical model approximation

Balanced Truncation POD - Hankel singular values and bpod structures)

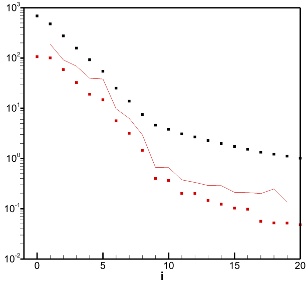


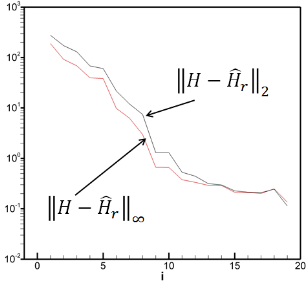




Large-scale dynamical model approximation

Balanced Truncation POD - Properties (lower/upper bounds & mismatch errors)





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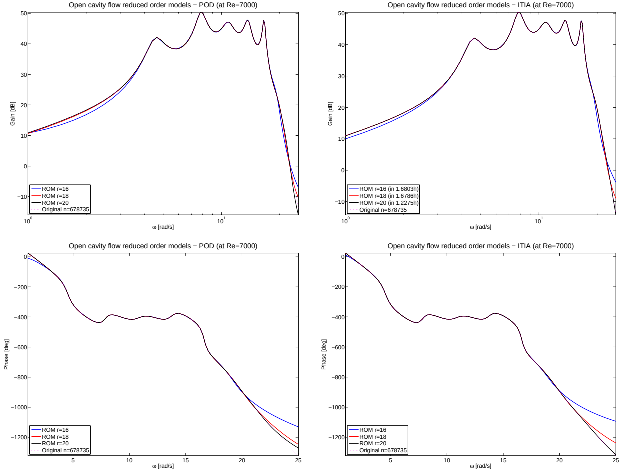
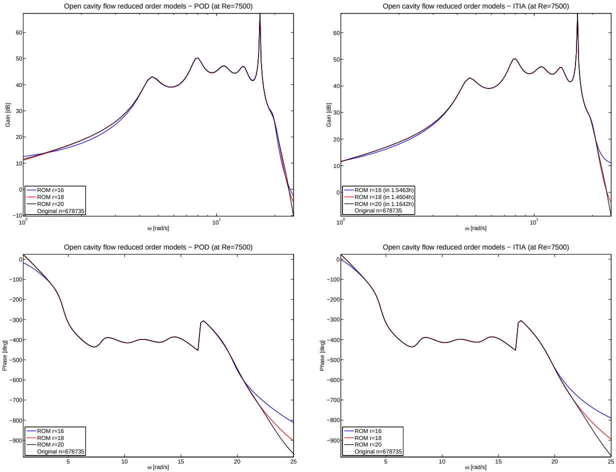
Large-scale dynamical model approximation

Fluid flow dynamical model approximation - Re=7000



LARGE-SCALE DYNAMICAL MODEL APPROXIMATION

Fluid flow dynamical model approximation - Re=7500



OUTLINES

Physical model and dynamical modeling

Large-scale dynamical model approximation

Active closed-loop control design
- Objectives
- Results

Conclusions

Active closed-loop control design

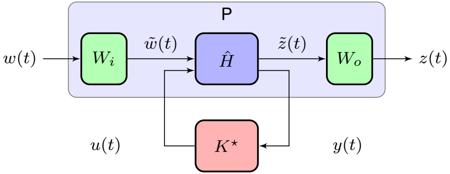
Objectives

Objectives and $\mathcal{H}_\infty$ control approach

- Stabilize the system
- Damp modes
- Potentially attenuate the $\mathcal{H}_\infty$-norm
- Engineering appealing / structured in view of RT implementation

A standard control approach:

$$K^\star = \arg \min_{K \subseteq \mathcal{K}} ||\mathcal{F}_l(\hat{H}, K)||_{\mathcal{H}_\infty} \tag{27}$$



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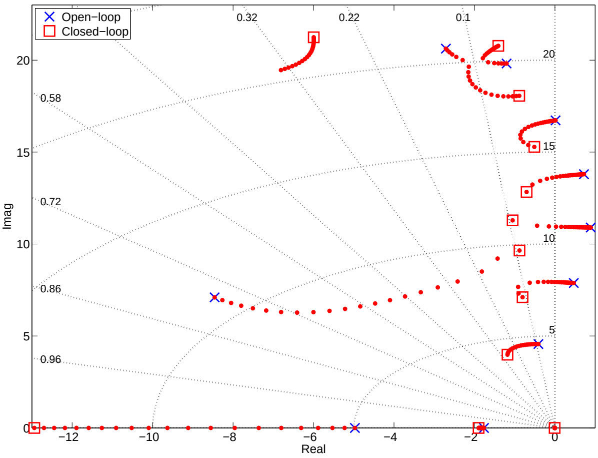
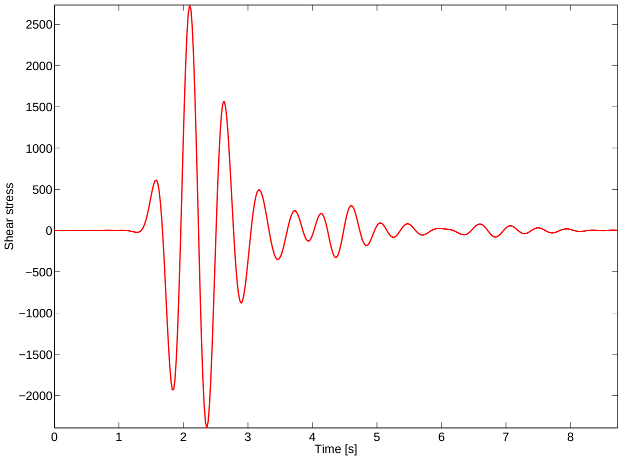
Active closed-loop control design

Results - Spectral (ROM)



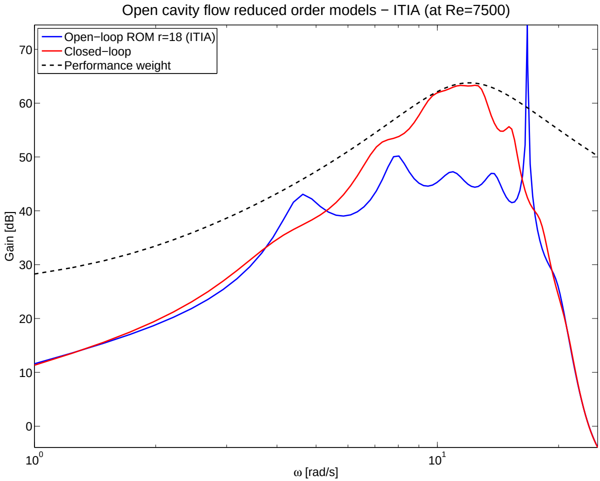
Active closed-loop control design

Results - Impulse (un-normalized ROM)



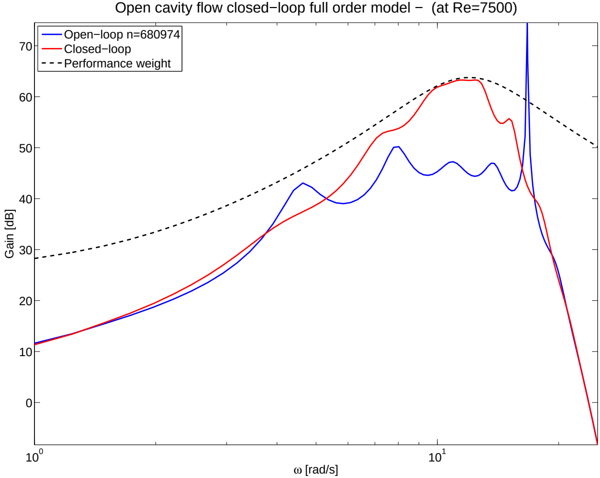
Active closed-loop control design

Results - Frequency-domain (un-normalized ROM)



Active closed-loop control design

Results - Frequency-domain (original LSS)



Outlines

Physical model and dynamical modeling

Large-scale dynamical model approximation

Active closed-loop control design

Conclusions
- About fluid flow control
- About model approximation

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Conclusions

About fluid flow control

About today's presentation

- Good performance of two model approximation techniques
- First attempt of $\mathcal{H}_\infty$ control synthesis (controller of order 6)
- Application on Navier and Stokes equations for an open cavity flow

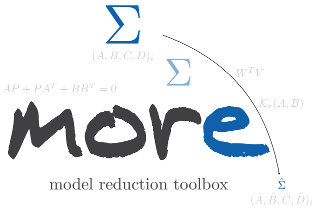
Some perspectives

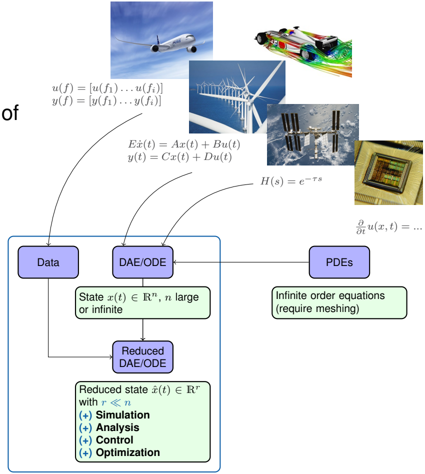
- Extension to robust analysis / parameter dependent control
- Apply the realization-less approaches (*e.g.* handle delays)
- Include learning policy for (on-line) model accuracy enhancement?

Conclusions

About model approximation - MORE toolbox [10]

- Successful application of advanced model approximation techniques
- both full and sparse
- on a complex unstable aerodynamical set of equations





10 C. Poussot-Vassal and P. Vuillemin, *"Introduction to MORE: a MOdel REduction Toolbox"*, IEEE Multi Systems Conference, pp. 776-781, 2012.

Fluid flow dynamical model approximation and control

... a case-study on an open cavity flow

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