Randomized robust static anti-windup

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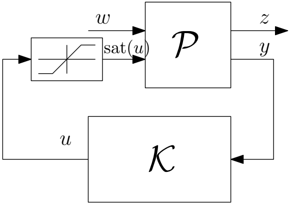
Performance with saturation depends on size of disturbance



- Saturation: an abrupt nonlinearity:
  - Small signals: $\mathrm{sat}(u) = u \Rightarrow$ no effect
  - Large signals: $\mathrm{sat}(u)$ bounded $\Rightarrow$ severe effect
- Signal size ($\mathcal{L}_2$ norm): $\|z\|_2 := \left(\int_0^\infty |z(t)|^2 dt \right)^{\frac{1}{2}}$
  - $z \in \mathcal{L}_2$ (square integrable) if $\|z\|_2 < \infty$
- Closed-loop performance measures:
  - Finite $\mathcal{L}_2$ gain (linear $\mathcal{H}_\infty$ norm): $\overline{\gamma}_{wz} \in \mathbb{R}_{\geq 0}$:

$$\|z\|_2 \leq \overline{\gamma}_{wz} \|w\|_2 \quad \text{for all } w \in \mathcal{L}_2$$

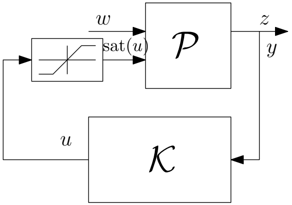
  - **Nonlinear $\mathcal{L}_2$ gain**: a function $s \mapsto \gamma_{wz}(s)$: Megretski [1996]

$$\|z\|_2 \leq \gamma_{wz}(s) \|w\|_2 \text{ for all } w \text{ satisfying } \|w\|_2 \leq s$$

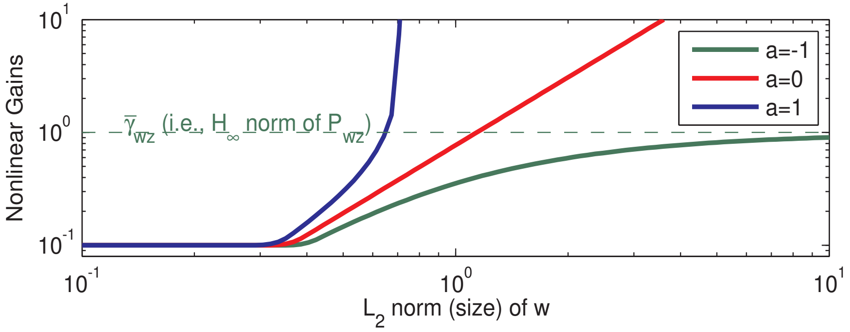
Example demonstrates relevance of nonlinear gains

Controller $\mathcal{K}$ **cancels the plant dynamics** and **stabilizes** *(before saturation)*

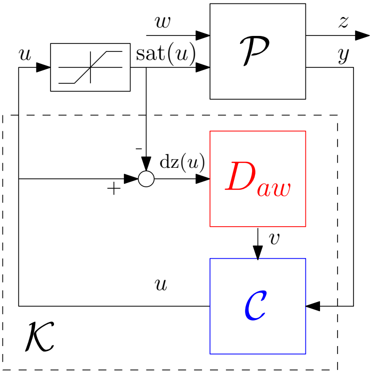
$$\mathcal{P}: \ \dot{z} = az + \text{sat}(u) + w$$
$$\mathcal{K}: \ u = -az - 10z$$



Three representative cases Sontag [1984], Lasserre [1992]



Optimal **nominal** static linear anti-windup design (LMI)



- Given $\mathcal{P}$ linear, $\mathcal{C}$ linear, **design** only
  - linear anti-windup gain $D_{aw} = \begin{bmatrix} D_{aw,1} \\ D_{aw,2} \end{bmatrix}$
- **Performance objective**:
  given s^*, minimize $\gamma_{dz}(s^*)$
- Linear **controller** $\mathcal{K}$ **equations**

$$\dot{x}_c = Ax_c + By + D_{aw,1}(u - \mathrm{sat}(u))$$
$$y_c = Cx_c + Dy + D_{aw,2}(u - \mathrm{sat}(u))$$

- LMI-based design Mulder et al. [2001], Gomes da Silva Jr and Tarbouriech [2005], Hu et al. [2008]

- **Preserve** of *small signal* response (D_{aw} multiplies $\mathrm{dz}(u) = u - \mathrm{sat}(u)$)
  Asymptotically recover *large signal* response (global not always possible)
- Robust designs follow a deterministic worst case paradigm, **imposing strong convexity** conditions Turner et al. [2007], Grimm et al. [2004]
- **This talk**: randomized analysis and synthesis of robust static anti-windup

Nonlinear $\mathcal{L}_2$ gains are estimated using Lyapunov functions

Hu et al. [2006], Dai et al. [2009], Garulli et al. [2013]

- **Quadratic** functions (LMIs Boyd et al. [1994])

$$V_1(x) = x^T P x$$

- **Max** of quadratics (BMIs)

$$V_2(x) = \max_{j \in \{1,\dots,J\}} x^T P_j x$$

- **Convex Hull** of quadratics (BMIs)

$$V_3(x) = \min_{\gamma_j \geq 0 : \sum_j \gamma_j = 1} x^T \left(\sum_j \gamma_j Q_j\right)^{-1} x$$

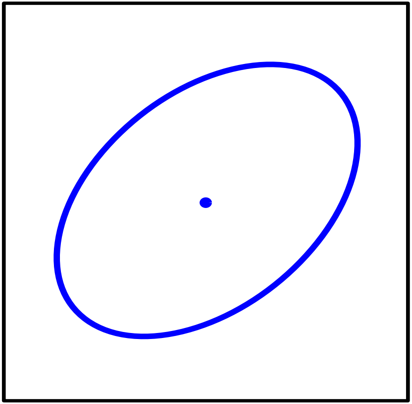
- **Piecewise** quadratic (LMI-BMI)

$$V_4(x) = \begin{bmatrix} x \\ \mathrm{dz}(u(x)) \end{bmatrix}^T \bar{P} \begin{bmatrix} x \\ \mathrm{dz}(u(x)) \end{bmatrix}$$

- Piecewise **Polynomial** (LMI-BMI)

$$V_5(x) = \begin{bmatrix} x \\ \mathrm{dz}(u(x)) \end{bmatrix}^{\{m\}T} \hat{P} \begin{bmatrix} x \\ \mathrm{dz}(u(x)) \end{bmatrix}^{\{m\}}$$

$$\dot{V} + \frac{1}{\gamma_{dz}(s)}|z|^2 - \gamma_{dz}(s)|w|^2 < 0$$



A possible level set

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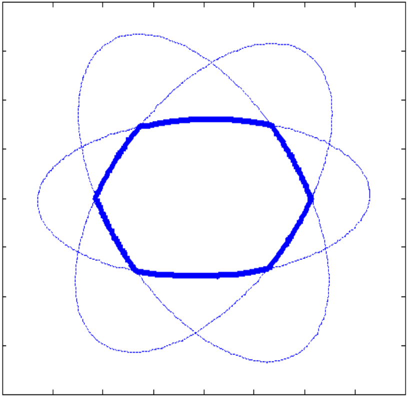

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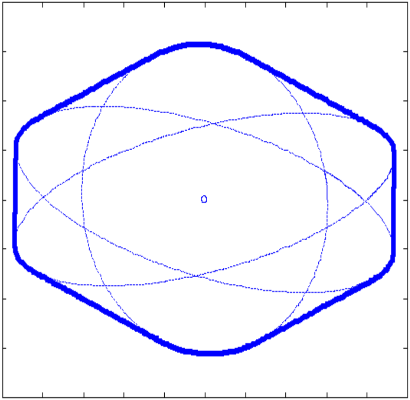

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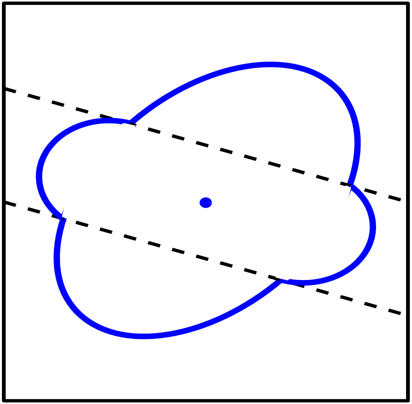

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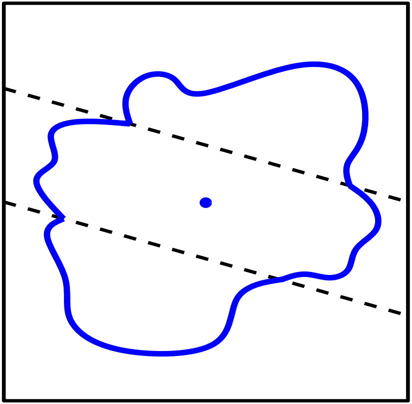
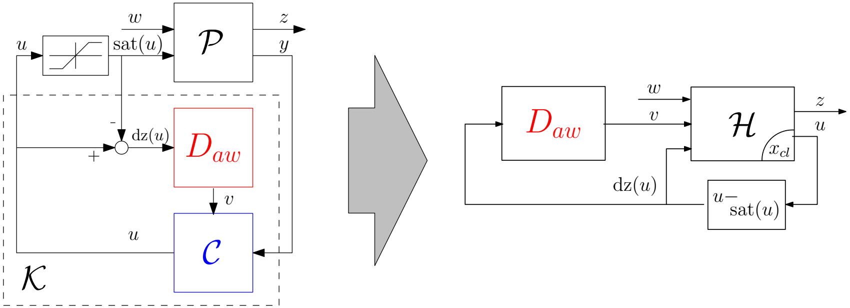


A possible level set

Compact representation of the closed-loop system



$$\mathcal{H}: \begin{cases} \dot{x}_{cl} = A_{cl}x_{cl} + B_{cl,d}(u - \mathrm{sat}(u)) + B_{cl,v}v + B_{cl,w}w \\ u = C_{cl,u}x_p + D_{cl,ud}(u - \mathrm{sat}(u)) + D_{cl,uv}v + D_{cl,uw}w \\ z = C_{cl,z}x_p + D_{cl,zd}\underbrace{(u - \mathrm{sat}(u))}_{\mathrm{dz}(u)} + D_{cl,zv}v + D_{cl,zw}w, \end{cases}$$

Quadratic analysis conditions are convex

Mulder et al. [2001], Gomes da Silva Jr and Tarbouriech [2005], Hu et al. [2008]

Proposition: Given the NOMINAL system and $s > 0$, if the LMI problem

$$\hat{\gamma}^2(s) = \min_{\{\gamma^2, Q, Y, U\}} \gamma^2 \text{ subject to } Q = Q^T > 0, \; U > 0 \text{ diagonal},$$

$$\mathrm{He}\begin{bmatrix} A_{cl}Q & B_{cl,d}U + B_{cl,v}D_{aw}U + Y^T & B_{cl,w} & 0 \\ C_{cl,u}Q & D_{cl,ud}U + D_{cl,uv}D_{aw}U - U & D_{cl,uw} & 0 \\ 0 & 0 & -I/2 & 0 \\ C_{cl,z}Q & D_{cl,zd}U + D_{cl,zv}D_{aw}U & D_{cl,zw} & -\frac{\gamma^2}{2}I \end{bmatrix} \prec 0, \quad \begin{bmatrix} Q & Y_{[k]}{}^T \\ Y_{[k]} & \bar{u}_k^2/s^2 \end{bmatrix} \succeq 0, \quad k = 1, \ldots, n_u$$

is feasible, then the following holds for the saturated closed-loop:

1. **[Stab]** the origin is **locally exponentially stable** with region of attraction containing the set $\mathcal{E}(Q, s) := \{x : \; x^T Q^{-1} x \le s^2\}$;
2. **[Reach]** the **reachable set** from $x(0) = 0$ with $\|w\|_2 \le s$ is contained in $\mathcal{E}(Q, s)$;
3. **[$\mathcal{L}_2$Perf]** for each w such that $\|w\|_2 \le s$, the zero state solution satisfies the $\mathcal{L}_2$ gain bound:

$$\|z\|_2 \le \hat{\gamma}(s)\|w\|_2$$

Quadratic analysis conditions easily lead to synthesis

Mulder et al. [2001], Gomes da Silva Jr and Tarbouriech [2005], Hu et al. [2008]

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is feasible, then, selecting the static AW gain as

$$D_{aw} = XU^{-1}$$

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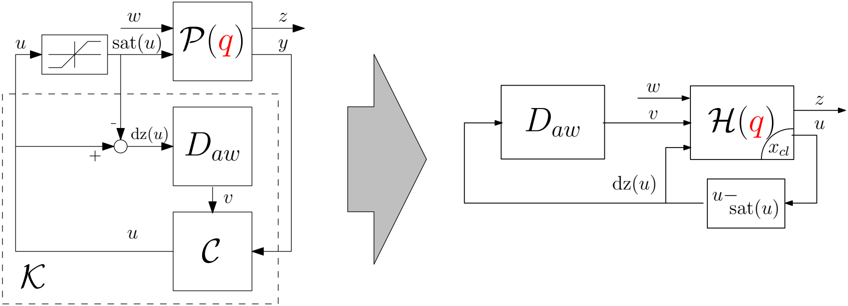
Closed loop now depends on uncertain parameter $q \in \mathbb{Q}$



$$\mathcal{H}(q): \begin{cases} \dot{x}_{cl} &= A_{cl}(q)x_{cl} + B_{cl,d}(q)(u - \mathrm{sat}(u)) + B_{cl,v}(q)v + B_{cl,w}(q)w \\ u &= C_{cl,u}(q)x_p + D_{cl,ud}(q)(u - \mathrm{sat}(u)) + D_{cl,uv}(q)v + D_{cl,uw}(q)w \\ z &= C_{cl,z}(q)x_p + D_{cl,zd}(q)\underbrace{(u - \mathrm{sat}(u))}_{\mathrm{dz}(u)} + D_{cl,zv}(q)v + D_{cl,zw}(q)w, \end{cases}$$

Robust static anti-windup synthesis: unviable formulation

- To solve the robust synthesis problem, **may** look for $\theta = \{\gamma^2, Q, Y, U, X\}$ s.t. the LMI holds for all **for all** $q \in \mathbb{Q}$

  Given a scalar $s > 0$, if the **nonconvex** optimization problem is feasible

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  then, selecting the static AW gain as

$$D_{aw} = XU^{-1}$$

  [WP], **[LocStab]**, and **[$\mathcal{L}_2$Perf]** are robustly guaranteed

- This construction is hard due to general dependence on q.
  $\Rightarrow$ Can use scenario (or sequential) randomized approach

Robust static anti-windup synthesis is a classical problem

- To solve the robust synthesis problem, **may** look for $\theta = \{\gamma^2, Q, Y, U, X\}$ s.t. the LMI holds for all **for all** $q \in \mathbb{Q}$

Given a scalar $s > 0$, if the **nonconvex** optimization problem is feasible

$$\hat{\gamma}^2(s) = \min_{\theta} c^\top \theta \text{ subject to } f_s(\theta, q) \leq 0, \ \forall q \in \mathbb{Q}$$

where

1. $\theta \in \mathbb{R}^{n_\theta}$ are the design variables;
2. $q \in \mathbb{Q}$ are the uncertain parameters;
3. $f_s(\theta, q) \leq 0$ are the problem constraints

then, selecting the static AW gain as

$$D_{aw} = XU^{-1}$$

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Robust and chance constrained optimization

Problem (Robust optimization - RO)

Given an objective vector $c \in \mathbb{R}^{n_\theta}$*, solve*

$$\min_{\theta} \; c^\top \theta \qquad \text{subject to}$$

$$f(\theta, q) \leq 0, \textbf{ for all } q \in \mathbb{Q}$$

Problem (Chance constrained optimization - CC)

Let a distribution over $\mathbb{Q}$ *be given, and let* $\varepsilon \in (0,1)$ *be a (small) probability level. Given an objective vector* $c \in \mathbb{R}^{n_\theta}$*, solve*

$$\min_{\theta} \; c^\top \theta \qquad \text{subject to}$$

$$\underbrace{\mathrm{Prob}\{q \in \mathbb{Q} : f(\theta, q) > 0\}}_{\mathsf{Viol}(\theta)} \leq \varepsilon$$

Randomized algorithms are a viable trade-off

Calafiore et al. [2011], Tempo et al. [2013], Petersen and Tempo [2014]

Both problems are very hard in general

- Robust optimization is hard whenever the uncertainty enters in a nonlinear way
- Chance-constrained optimization is a even more difficult non-convex problem (it involves hard integral evaluations)

Proposed solution approach: **Randomized algorithms**

- A Randomized Algorithm is an algorithm that makes random choices during its execution to produce a result

- Randomized algorithms entail a (pre determined) probability of failure

Scenario approach amounts to a simple extraction

Calafiore and Campi [2006]

- Scenario techniques provide a simple and theoretically sound way to approximately solve the two problems RO and CC
- The idea is to replace these hard optimization problems with the following sampled counterpart (**random convex program**)

Problem (Scenario optimization)

*Extract N i.i.d. samples (**scenarios**) $q^{(1)}, \dots, q^{(N)}$, and solve*

$$\min_{\theta} \; c^\top \theta \qquad \text{subject to}$$

$$f(\theta, q^{(i)}) \leq 0, \quad i = 1, \dots, N$$

- The scenario problem is a **standard convex optimization problem** with a finite number of constraints

Probability of violation is bounded by number of samples

Assumption (Basic assumptions)

$f(\theta, q)$ is continuous and convex in θ for any fixed $q \in \mathbb{Q}$. For any multisample extraction $\mathbf{q} = \{q^{(1)}, \ldots, q^{(M_k)}\} \in \mathbb{Q}$, the scenario problem is feasible and attains a unique optimal solution

Theorem (violation of scenario solutions Campi and Garatti [2008])

Let $\varepsilon \in (0, 1)$ be a given probability level and let $N \geq n_\theta$. Under **convexity**, **uniqueness** *and* **feasibility** *assumptions, the scenario solution θ_{sc} satisfies*

$$\Pr\left\{\mathrm{Viol}(\theta_{sc}) > \varepsilon\right\} \leq \mathrm{B}(N, \varepsilon, n_\theta)$$

where

$$\mathrm{B}(N, \varepsilon, n_\theta) = \sum_{k=0}^{n_\theta - 1} \binom{N}{k} \varepsilon^k (1-\varepsilon)^{N-k}.$$

Robust static anti-windup synthesis based on scenario

Theorem (Robust static AW synthesis using scenario Formentin et al. [2013])

Fix *a positive value* $s \geq \|w\|_2$, $\varepsilon \in (0,1)$, $\beta \in (0,1)$, *and select* N *satisfying*

$$\mathrm{B}(N, \varepsilon, n_\theta) \leq \beta,$$

with $n_\theta = 1 + n(n+1)/2 + nn_u + n_u + n_u(n_u + n_c)$

Extract *N samples of the uncertain matrices according to the probability distribution*

Solve

$$\gamma_{sc}^2(s) = \min_{\{\gamma^2, Q, Y, U, X\}} \gamma^2, \quad \textit{subject to } Q = Q^T > 0,\ U > 0 \textit{ diagonal},$$

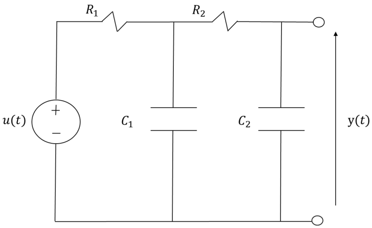
$$\mathrm{He}\begin{bmatrix} A_{cl}^{(i)} Q & B_{cl,d}^{(i)} U + B_{cl,v}^{(i)} X + Y^T & B_{cl,w}^{(i)} & 0 \\ C_{cl,u}^{(i)} Q & D_{cl,ud}^{(i)} U + D_{cl,uv}^{(i)} X - U & D_{cl,uw}^{(i)} & 0 \\ 0 & 0 & -I/2 & 0 \\ C_{cl,z}^{(i)} Q & D_{cl,zd}^{(i)} U + D_{cl,zv}^{(i)} X & D_{cl,zw}^{(i)} & -\frac{\gamma^2}{2} I \end{bmatrix} < 0, \quad \begin{bmatrix} Q & Y_{[k]}^T \\ Y_{[k]} & \bar{u}_k^2 / s^2 \end{bmatrix} \geq 0,$$
$$\forall k = 1, \ldots, n_u \qquad \forall i = 1, \ldots, N$$

If *the above LMIs are feasible, select the static anti-windup gain* $D_{aw} = XU^{-1}$

Then, *for each* $\|w\|_2 < s$, *the zero initial state solution of the closed loop satisfies* $Pr(\|z\|_2 > \gamma_{sc}(s)\, \|w\|_2) \leq \varepsilon$, *with probability no smaller than* $1 - \beta$.

Analysis conditions can also be easily formulated

Illustrative example: a double RC passive network



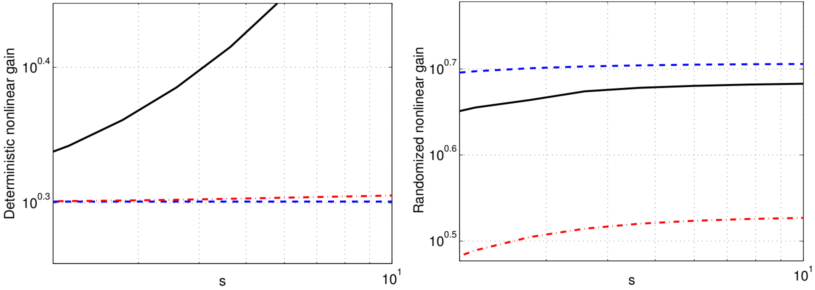
- Uncertain parameters with (known) Gaussian distribution

parameter	mean	standard deviation
R_1	310 Ω	$\pm$ 10 %
R_2	10 Ω	$\pm$ 10 %
C_1	0.01 F	$\pm$ 10 %
C_2	0.01 F	$\pm$ 10 %

- Input generator voltage constrained:
  $u(t) \in [-\bar{u}, \bar{u}] = [-1\,Volt, 1\,Volt]$
- Design parameters are $\varepsilon = 0.05, \beta = 10^{-6}$, $s = 1$
  $\Rightarrow N = 1323$ for analysis and $N = 1482$ for synthesis

Deterministic and Randomized nonlinear $\mathcal{L}_2$ gains

- The probabilistic robust compensator shows better performance (left curves)
- The nominal behavior slightly deteriorated (right curves)



Without anti-windup (black solid), with nominal anti-windup (blue dashed) and with robust anti-windup (red dashed-dotted)

Design variables and certificates

- The scenario approach to AW provides a new viewpoint to robust AW design *allowing us to address hard nonconvex synthesis problems*
- However, it is still very conservative, because we are looking for a **common quadratic Lyapunov function** $x^T Q^{-1} x$ for all $q \in \mathbb{Q}$ that is for "common certificates" of stability and performance
- We would like to have "parameter dependent certificates" because **non-common Lyapunov functions** are known to lead to greatly reduced conservatism
- Indeed, a (much) less conservative solution can be obtained by looking for design variables $\theta = \{\gamma^2, U, X\}$ such that, for each $q \in \mathbb{Q}$, there exist certificates $\xi = \{Q, Y\} = \{Q(q), Y(q)\}$ satisfying the stability/performance LMIs
- This approach is new to within the randomized world. We denote it **design with certificates**

Robust static AW synthesis – common certificates (recall)

Given a scalar $s > 0$, if the LMI problem

$$\hat{\gamma}^2(s) = \min_{\{\gamma^2, U, X, Q, Y\}} \gamma \text{ subject to } U > 0 \text{ diagonal},$$

$$Q = Q^T > 0$$

$$\mathrm{He}\begin{bmatrix} A_{cl}(q)Q & B_{cl,d}(q)U + B_{cl,v}(q)X + Y^T & B_{cl,w}(q) & 0 \\ C_{cl,u}(q)Q & D_{cl,ud}(q)U + D_{cl,uv}(q)X - U & D_{cl,uw}(q) & 0 \\ 0 & 0 & -I/2 & 0 \\ C_{cl,z}(q)Q & D_{cl,zd}(q)U + D_{cl,zv}(q)X & D_{cl,zw}(q) & -\frac{\gamma^2}{2} I \end{bmatrix} \prec 0, \ \forall q \in \mathbb{Q}$$

$$\begin{bmatrix} Q & {Y_{[k]}}^T \\ Y_{[k]} & \bar{u}_k^2/s^2 \end{bmatrix} \succeq 0, \quad k = 1, \ldots, n_u$$

is feasible, then, selecting the static AW gain as

$$D_{aw} = XU^{-1}$$

all properties **[Stab]**, **[Reach]**, and **[$\mathcal{L}_2$Perf]** hold robustly

Robust static AW synthesis – non-common certificates

Given a scalar $s > 0$, if the LMI problem

$$\hat{\gamma}^2(s) = \min_{\{\gamma^2, U, X\}} \gamma \text{ subject to } U > 0 \text{ diagonal,}$$

for each $q \in \mathbb{Q}$ **there exist** $\{Q_q, Y_q\}$ **such that**

$$Q_q = Q_q{}^T > 0$$

$$\mathrm{He}\begin{bmatrix} A_{cl}(q)Q_q & B_{cl,d}(q)U + B_{cl,v}(q)X + Y_q^T & B_{cl,w}(q) & 0 \\ C_{cl,u}(q)Q_q & D_{cl,ud}(q)U + D_{cl,uv}(q)X - U & D_{cl,uw}(q) & 0 \\ 0 & 0 & -I/2 & 0 \\ C_{cl,z}(q)Q_q & D_{cl,zd}(q)U + D_{cl,zv}(q)X & D_{cl,zw}(q) & -\frac{\gamma^2}{2}I \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} Q_q & Y_{q[k]}{}^T \\ Y_{q[k]} & \bar{u}_k^2/s^2 \end{bmatrix} \succeq 0, \quad k = 1, \ldots, n_u$$

is feasible, then, selecting the static AW gain as

$$D_{aw} = XU^{-1}$$

all properties **[Stab]**, **[Reach]**, and **[$\mathcal{L}_2$Perf]** hold robustly

Robust optimization with certificates

Problem (Robust optimization with certificates Oishi [2006])

$$\min_{\theta} c^T\theta \text{ subject to} \qquad \text{(RwC)}$$

$$\theta \in \mathcal{S}(q), \text{ for all } q \in \mathbb{Q},$$

where the set $\mathcal{S}(q)$ is defined as

$$\mathcal{S}(q) \doteq \{\theta \in \mathbb{R}^{n_\theta} \text{ such that there exists } \xi \text{ satisfying } f(\theta, \xi, q) \leq 0\}.$$

- The idea of constructing certificates based on random samples was originally introduced by Oishi [2006], in the context of randomized ellipsoid method
- Essentially, one is allowed to use "parameter-dependent" certificates $\xi = \xi(q)$ (e.g., parameter-dependent Lyapunov functions)
- The **scenario with certificates** approach allows to find a solution *without* explicitly assuming the form of the dependence

Approximate RwC based on multisample extraction

Scenario with certificates: contrary to the scenario problem, now a new certificate variable ξ_i is used for each sample $q^{(i)}$, $i = 1, \ldots, N$

Problem (Scenario with certificates Formentin et al. [2014])

$$\min_{\theta, \xi_1, \ldots, \xi_N} c^T \theta \text{ subject to: } \tag{SwC}$$

$$f(\theta, \xi_i, q^{(i)}) \leq 0, \ \forall i = 1, \ldots, N$$

Theorem (Scenario with certificates Formentin et al. [2014])

If for any multisample extraction the SwC problem is feasible and attains a unique optimal solution θ_{swc}, then, given an accuracy level $\varepsilon \in (0,1)$, the solution θ_{swc} satisfies

$$\Pr\{\text{Viol}(\theta_{swc}) > \varepsilon\} \leq \text{B}(N, \varepsilon, n_\theta)$$

A sequential algorithm for SwC is also presented in Formentin et al. [2014]

Static AW synthesis based on scenario (recall)

Theorem (Robust static AW synthesis using scenario Formentin et al. [2013])

Fix *a positive value* $s \geq \|w\|_2$, $\varepsilon \in (0,1)$, $\beta \in (0,1)$, *and select* N *satisfying*

$$\mathrm{B}(N, \varepsilon, n_\theta) \leq \beta,$$

with $n_\theta = 1 + n(n+1)/2 + nn_u + n_u + n_u(n_u + n_c)$

Extract *N samples of the uncertain matrices according to the probability distribution*

Solve

$$\gamma_{sc}^2(s) = \min_{\{\gamma^2, Q, Y, U, X\}} \gamma^2, \quad \textit{subject to } Q = Q^T > 0, \; U > 0 \textit{ diagonal},$$

$$\mathrm{He}\begin{bmatrix} A_{cl}^{(i)} Q & B_{cl,d}^{(i)} U + B_{cl,v}^{(i)} X + Y^T & B_{cl,w}^{(i)} & 0 \\ C_{cl,u}^{(i)} Q & D_{cl,ud}^{(i)} U + D_{cl,uv}^{(i)} X - U & D_{cl,uw}^{(i)} & 0 \\ 0 & 0 & -I/2 & 0 \\ C_{cl,z}^{(i)} Q & D_{cl,zd}^{(i)} U + D_{cl,zv}^{(i)} X & D_{cl,zw}^{(i)} & -\frac{\gamma^2}{2} I \end{bmatrix} < 0, \quad \begin{bmatrix} Q & Y_{[k]}^T \\ Y_{[k]} & \bar{u}_k^2 / s^2 \end{bmatrix} \geq 0, \quad \begin{matrix} \forall k = 1, \ldots, n_u \\ \forall i = 1, \ldots, N \end{matrix}$$

If *the above LMIs are feasible, select the static anti-windup gain* $D_{aw} = XU^{-1}$

Then, *for each* $\|w\|_2 < s$, *the zero initial state solution of the closed loop satisfies* $Pr(\|z\|_2 > \gamma_{sc}(s) \|w\|_2) \leq \varepsilon$, *with probability no smaller than* $1 - \beta$.

Analysis conditions can also be easily formulated

Static AW synthesis based on scenario with certificates

Theorem (Robust static AW using scenario with certificates Formentin et al. [2014])

Fix *a positive value* $s \geq \|w\|_2$, $\varepsilon \in (0,1)$, $\beta \in (0,1)$, *and select* N *satisfying*

$$\mathrm{B}(N, \varepsilon, n_\theta) \leq \beta,$$

with $n_\theta = 1 + \cancel{n(n+1)/2 + nn_u} + n_u + n_u(n_u + n_c)$

Extract N *samples of the uncertain matrices according to the probability distribution*

Solve

$$\gamma_{sc}^2(s) = \min_{\{\gamma^2, U, X\}, \{Q_i, Y_i\}} \gamma^2, \quad \text{subject to } Q_i = Q_i{}^T > 0,\ U > 0 \text{ diagonal},$$

$$\mathrm{He}\begin{bmatrix} A_{cl}^{(i)} Q_i & B_{cl,d}^{(i)} U + B_{cl,v}^{(i)} X + Y_i^T & B_{cl,w}^{(i)} & 0 \\ C_{cl,u}^{(i)} Q_i & D_{cl,ud}^{(i)} U + D_{cl,uv}^{(i)} X - U & D_{cl,uw}^{(i)} & 0 \\ 0 & 0 & -I/2 & 0 \\ C_{cl,z}^{(i)} Q_i & D_{cl,zd}^{(i)} U + D_{cl,zv}^{(i)} X & D_{cl,zw}^{(i)} & -\frac{\gamma^2}{2} I \end{bmatrix} < 0, \quad \begin{bmatrix} Q_i & Y_{i[k]}^T \\ Y_{i[k]} & \bar{u}_k^2 / s^2 \end{bmatrix} \geq 0,$$

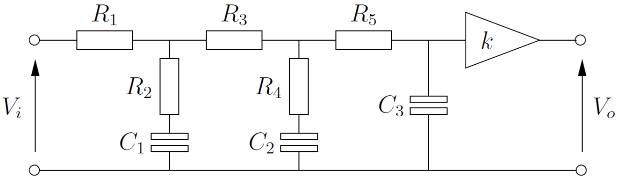
$$\forall k = 1, \ldots, n_u \qquad \forall i = 1, \ldots, N$$

If *the above LMIs are feasible, select the static anti-windup gain* $D_{aw} = XU^{-1}$

Then, *for each* $\|w\|_2 < s$, *the zero initial state solution of the closed loop satisfies* $Pr(\|z\|_2 > \gamma_{sc}(s)\,\|w\|_2) \leq \varepsilon$, *with probability no smaller than* $1 - \beta$.

Analysis conditions can also be easily formulated

Illustrative example: Another (larger) passive network



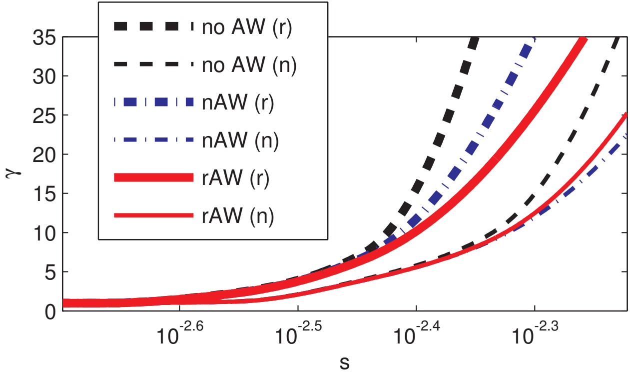
- Uncertain parameters with (known) Gaussian distribution

parameter	mean	std dev	parameter	mean	std dev
R_1	313 Ω	± 10	R_5	10 F	± 10
R_2	20 Ω	± 10	C_1	0.01 F	± 10
R_3	315 Ω	± 10	C_2	0.01 F	± 10
R_4	17 Ω	± 10	c_3	0.01 F	± 10

- Input generator voltage constrained:
  $u(t) = V_i(t) \in [-\bar{u}, \bar{u}] = [-1\,Volt, 1\,Volt]$
- Design parameters are $\varepsilon = 0.01, \beta = 10^{-6}$, $s = 0.003$, $n_\theta = 35$
  $\Rightarrow N = 2270$ (not 7565) for design based on sequential algorithm

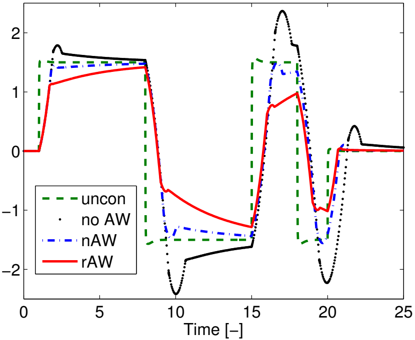
Deterministic and Randomized nonlinear $\mathcal{L}_2$ gains

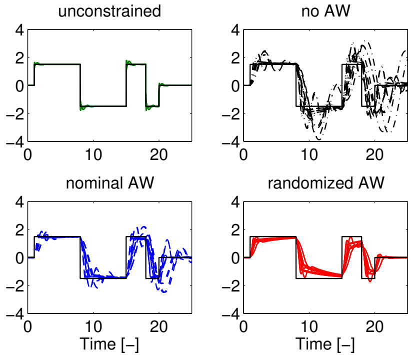
- Robust compensator shows better robust performance (red curves)
- The nominal behavior slightly deteriorated (thin curves)

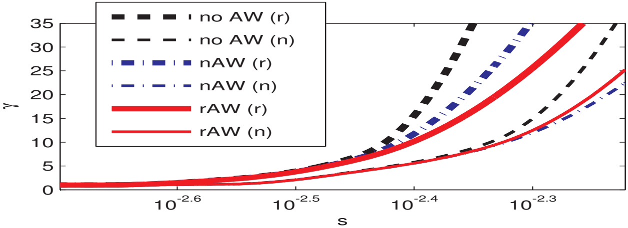


Without anti-windup (black dashed), with nominal anti-windup (blue dashed-dotted) and with robust anti-windup (red solid)

Time responses confirm nonlinear $\mathcal{L}_2$ gain trends







Optimization of the reachable set estimate

Theorem (Robust static AW using scenario with certificates Formentin et al. [2014])

Fix *a positive value* $s \geq \|w\|_2$, $\varepsilon \in (0,1)$, $\beta \in (0,1)$, *and select* N *satisfying*

$$\mathrm{B}(N, \varepsilon, n_\theta) \leq \beta,$$

with $n_\theta = n(n+1)/2 + n_u + n_u(n_u + n_c)$

Extract N *samples of the uncertain matrices according to the probability distribution*

Solve

$$\gamma_{sc}^2(s) = \min_{\{\bar{Q},U,X\},\{Q_i,Y_i\}} \mathrm{trace}(\bar{Q}), \quad \textit{subject to } Q_i = Q_i^T > 0,\ U > 0 \textit{ diagonal},$$

$$\bar{Q} \geq Q_i \qquad \begin{bmatrix} Q_i & Y_{i[k]}^T \\ Y_{i[k]} & \bar{u}_k^2/s^2 \end{bmatrix} \geq 0, \qquad (1)$$

$$\mathrm{He}\begin{bmatrix} A_{cl}^{(i)} Q_i & B_{cl,d}^{(i)} U + B_{cl,v}^{(i)} X + Y_i^T & B_{cl,w}^{(i)} \\ C_{cl,u}^{(i)} Q_i & D_{cl,ud}^{(i)} U + D_{cl,uv}^{(i)} X - U & D_{cl,uw}^{(i)} \\ 0 & 0 & -I/2 \end{bmatrix} < 0, \qquad \forall k = 1, \ldots, n_u \quad \forall i = 1, \ldots, N$$

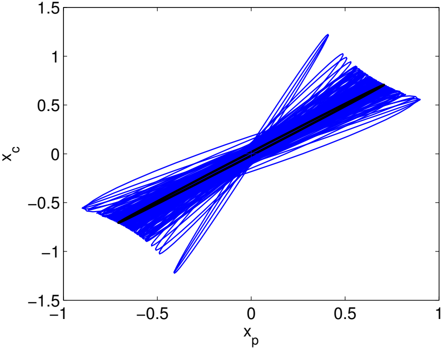
If *the above LMIs are feasible, select the static anti-windup gain* $D_{aw} = XU^{-1}$

Then, *for each* $\|w\|_2 < s$, *the zero initial state solution of the closed loop has probability* $1 - \epsilon$ *of remaining in the ellipsoid* $\mathcal{E}(\bar{Q}, s)$ *with level of confidence no smaller than* $1 - \beta$.

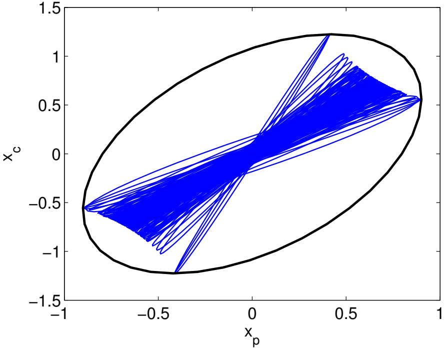
Analysis conditions can also be easily formulated

Reachable sets for simple 2D example

Left is Nominal design:
Clearly unsuitable
Nominal parameters (black),
Perturbed parameters (blue)



Right is Robust design:
Potential behind noncommon Q_i's
Guaranteed region $\mathcal{E}(\bar{Q}, s)$ (black),
A collection of sets $\mathcal{E}(Q_i, s)$ (blue)



Concluding remarks

- Deterministic formulation of the robust static anti-windup design problem is nonconvex
- Scenario approach can be used for **robust static anti-windup** compensator synthesis and for robust stability and performance analysis with **common certificates**
- New tool **scenario with certificates** allows for non-common certificates and results with reduced conservativeness

- Current/future work:
  - transform s into a random variable to deal with uncertain disturbances/references
  - address robust dynamic anti-windup compensation

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