Controller Order-reduction. Application to active suspension control

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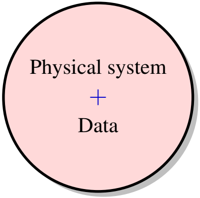
November, 28th 2014

Outlines

Outlines

Problem position



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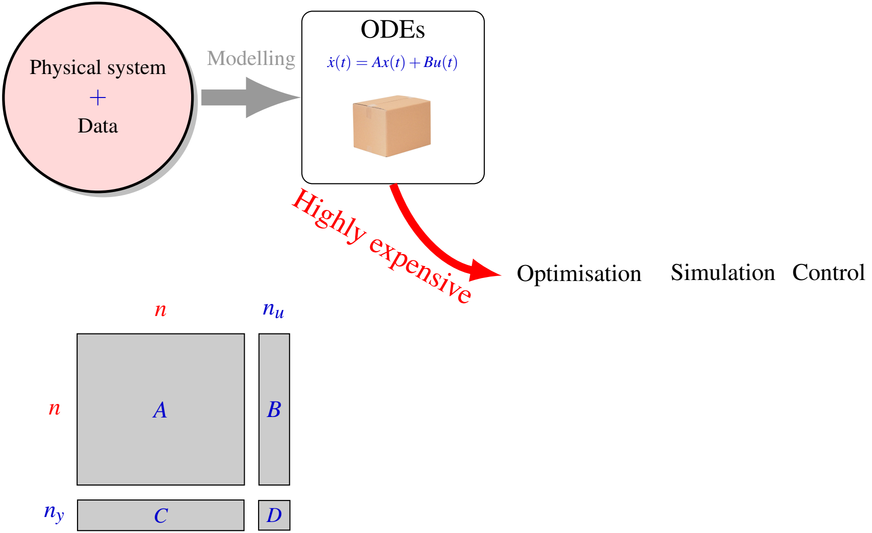

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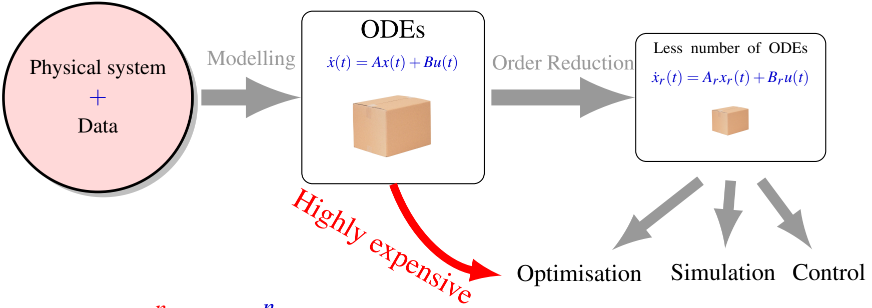
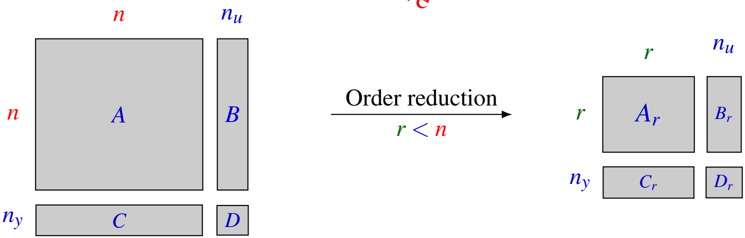
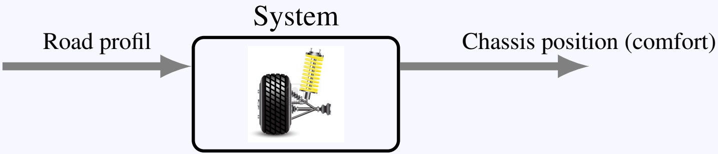




Context and objective

Aim of the work



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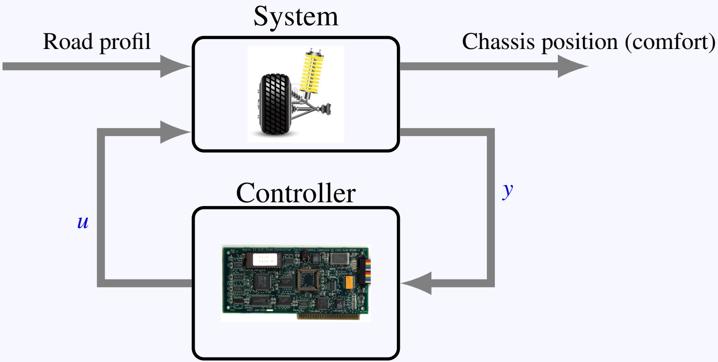

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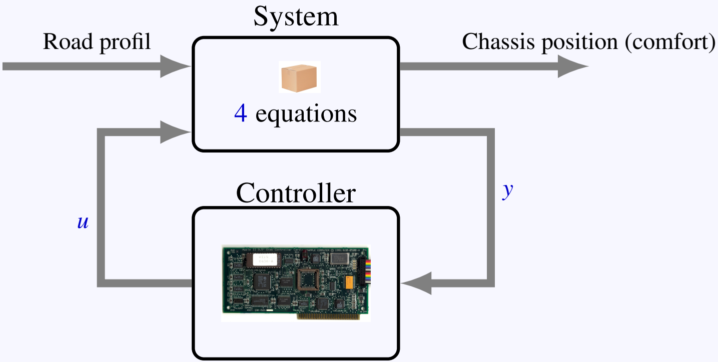

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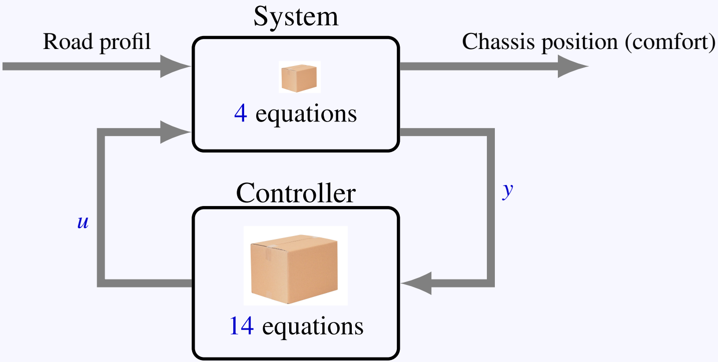

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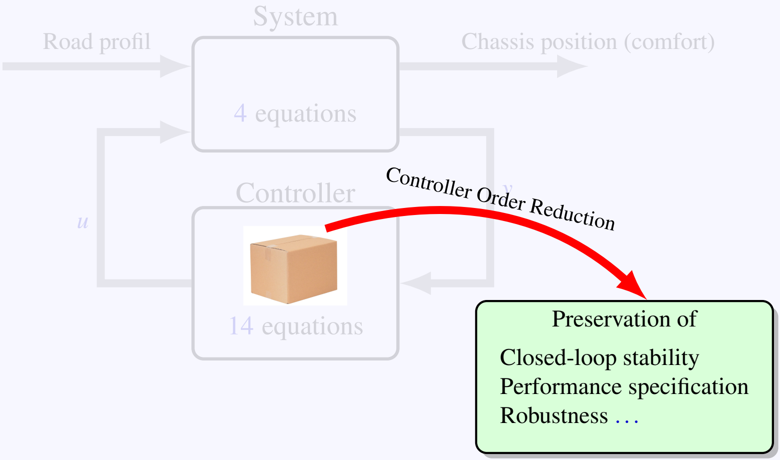


3 different approaches . . .

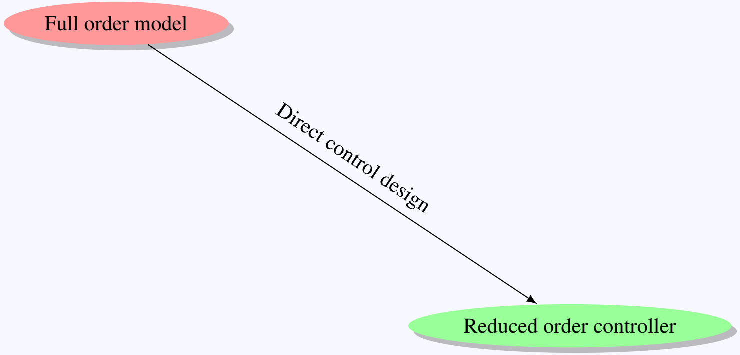
How to obtain reduced-order controller:

Full order model

Reduced order controller

3 different approaches . . .

How to obtain reduced-order controller:



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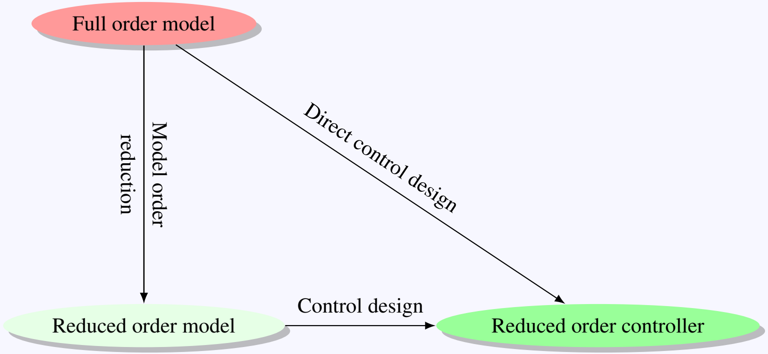

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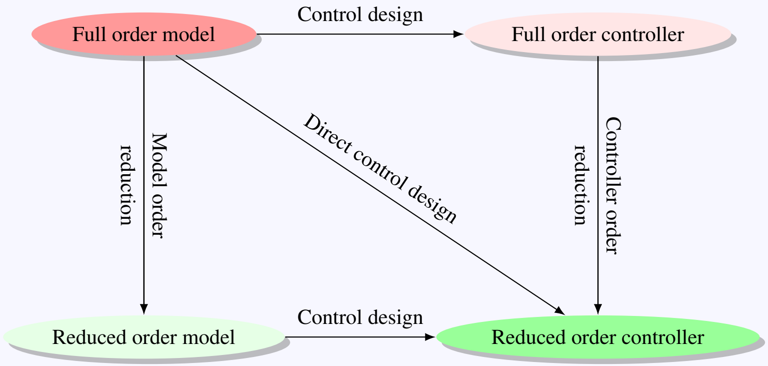

Outlines

Several methods to reduce . . .

Approximation methods

Linear Systems

SVD methods

- -Balanced truncation (BT)
- -Positive real BT
- -Frequency weighted BT

Modal methods

- -Eigenvalues, residues
- -Dominant modes

Hankel Approximation

Krylov methods

- -Arnoldi/Lanczos
- -Rational interpolation

Non-linear systems

- -Petrov-Galerkin projections based
- -Proper orthogonal decomposition

Balanced Truncation BT

Principle

Preserve the r largest invariant parameters α_i

Approach

- Find new basis (called balanced)
  - Define two dual symmetric and positive definite matrices $\mathcal{P}$ and $\mathcal{Q}$.
  - Search the basis where $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{Q}}$ are balanced (diagonal and equal):

$$\tilde{\mathcal{P}} = T^{-1}\mathcal{P}T^{-\top} = \tilde{\mathcal{Q}} = T^{\top}\mathcal{Q}T = \mathrm{diag}(\alpha_1, \ldots, \alpha_n) \text{ with } \alpha_i \geq \alpha_{i+1} > 0$$

- Reduce the order by truncation: conserve the first rth α_i.
- Find an upper bound for the H_∞-error norm: $\|G - G_r\|_\infty$

BT methods

1. Lyapunov Balanced Truncation LBT (Hankel singular values σ_i)
2. Positive Real Balanced Truncation PRBT (positive real singular values δ_i)
3. Frequency Weighted Balanced Truncation FWBT (weighted singular values κ_i)

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Balanced Truncation BT

Example: Lyapunov

Preserve the r largest singular values σ_i

Approach

- Find a new balanced basis:
  - Compute the reachability and observability gramians W_o and W_a,by solving the two Lyapunov equations.

$$\begin{cases} AW_a + W_aA^\top + BB^\top = 0 \\ A^\top W_o + W_oA + C^\top C = 0 \end{cases}$$

  - Search the basis where $\tilde{W}_a$ and $\tilde{W}_o$ are balanced (diagonal and equal):

$$\tilde{W}_a = T^{-1}W_aT^{-\top} = \tilde{W}_o = T^\top W_oT = \text{diag}(\sigma_1, \ldots, \sigma_r, \ldots \sigma_n) \text{ with } \sigma_i \geq \sigma_{i+1} > 0$$

- Reduce the order by truncation: conserve the first rth σ_i.
- $\|G(s) - G_r(s)\|_\infty < 2(\sigma_{r+1} + \cdots + \sigma_n)$

Balanced Truncation BT

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- Find a new balanced basis:
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Frequency Weighted Balanced Truncation Method

FWBT Procedure

- Define the weights $V(s)$ and $W(s)$ such that

$$V(s) \overset{s}{=} \left(\begin{array}{c|c} A_v & B_v \\ \hline C_v & D_v \end{array}\right) \text{ and } W(s) \overset{s}{=} \left(\begin{array}{c|c} A_w & B_w \\ \hline C_w & D_w \end{array}\right)$$

- Define the augmented system
  - $G(s)V(s) = C_{\mathrm{i}}(sI - A_{\mathrm{i}})^{-1}B_{\mathrm{i}} + D_{\mathrm{i}}$
  - $W(s)G(s) = C_{\mathrm{o}}(sI - A_{\mathrm{o}})^{-1}B_{\mathrm{o}} + D_{\mathrm{o}}$

such that:

$$\left(\begin{array}{c|c} A_{\mathrm{i}} & B_{\mathrm{i}} \\ \hline C_{\mathrm{i}} & D_{\mathrm{i}} \end{array}\right) = \left(\begin{array}{c|c} \begin{bmatrix} A & BC_v \\ 0 & A_v \end{bmatrix} & \begin{bmatrix} BD_v \\ B_v \end{bmatrix} \\ \hline \begin{bmatrix} C & DC_v \end{bmatrix} & DD_v \end{array}\right)$$

and

$$\left(\begin{array}{c|c} A_{\mathrm{o}} & B_{\mathrm{o}} \\ \hline C_{\mathrm{o}} & D_{\mathrm{o}} \end{array}\right) = \left(\begin{array}{c|c} \begin{bmatrix} A_w & B_wC \\ 0 & A \end{bmatrix} & \begin{bmatrix} B_wD \\ B \end{bmatrix} \\ \hline \begin{bmatrix} C_w & C_wC \end{bmatrix} & D_wD \end{array}\right).$$

Frequency Weighted Balanced Truncation Method

FWBT Procedure

- Compute $\mathcal{P}$ and $\mathcal{Q}$,the solutions of the two Lyapunov equations.
$$\begin{cases} A_{\mathrm{i}}\mathcal{P}_{\mathrm{i}} + \mathcal{P}_{\mathrm{i}}A_{\mathrm{i}}^{\top} + B_{\mathrm{i}}B_{\mathrm{i}}^{\top} = 0 \\ A_{\mathrm{o}}^{\top}\mathcal{Q}_{\mathrm{o}} + \mathcal{Q}_{\mathrm{o}}A_{\mathrm{o}} + C_o^{\top}C_o = 0 \end{cases}$$

- Balance $\mathcal{P}_{\mathrm{i}}$ and $\mathcal{Q}_{\mathrm{o}}$: $\tilde{\mathcal{P}}_{\mathrm{i}} = \tilde{\mathcal{Q}}_{\mathrm{o}} = \mathrm{diag}(\kappa_1 I_{m_1}, \ldots, \kappa_q I_{m_q})$
  κ_i: weighted singular values of $G(s)$.

- Truncate the new realisation in the balanced basis

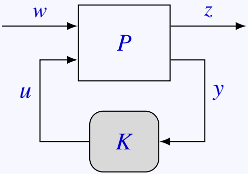
$$G(s) \stackrel{s}{=} \left(\begin{array}{c|c} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} & \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ \hline \begin{bmatrix} C_1 & C_2 \end{bmatrix} & D \end{array} \right) \longrightarrow G_r(s) \stackrel{s}{=} \left(\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array} \right)$$

- Weighted error: $\|W(G-G_r)V\|_\infty \leq 2\sum_{i=r+1}^{n} f(\kappa_i)$

Outlines

H_∞-control

H_∞-standard form



Goal

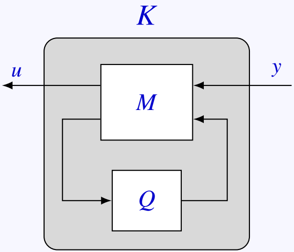
- Attenuate the transfer from w to z:

$$T_{zw} = \mathcal{F}_\ell(P,K) \;=\; P_{yw}(s)K(s)(I - P_{uy}K(s))^{-1}P_{uz}(s) + P_{zw}$$

- Worst case: $\|T_{zw}\|_\infty < \gamma$

H_∞-control

General solution



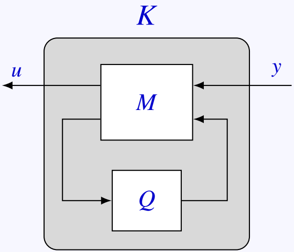
$$K = \mathcal{F}_\ell(M,Q) = M_{11} + M_{12}Q(I - M_{22}Q)^{-1}M_{21}$$

with $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \overset{s}{=} \left(\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{array}\right)$ and $\left\{ \begin{array}{ccc} Q & \in & RH_\infty \\ \|Q\|_\infty & < & \gamma \end{array} \right.$

K, Zhou and J.C. Doyle, "Essentials of Robust Control". Prentice Hall International editions, 1998.

H_∞-control

General solution



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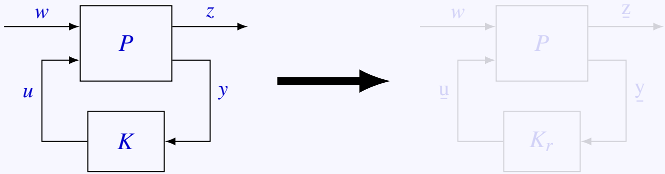
For $Q = 0 \quad \Rightarrow \quad K = M_{11}$.

K, Zhou and J.C. Doyle, "Essentials of Robust Control". Prentice Hall International editions, 1998.

H_∞-controller order reduction

Goal

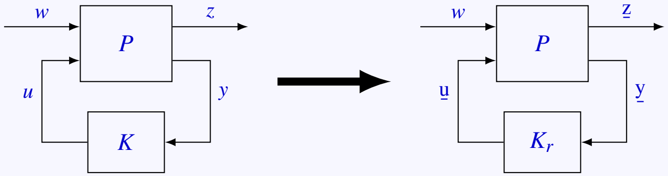


a. Full order controller

b. Reduced order controller

H_∞-controller order reduction

Goal



a. Full order controller

b. Reduced order controller

Solution

Find Q_r such as:

$$\|\Delta_K M_{21}^{-1} M_{22} M_{12}^{-1}\|_\infty < 1 \quad \text{or} \quad \|M_{21}^{-1} M_{22} M_{12}^{-1} \Delta_K\|_\infty < 1$$

with $\Delta_K := K_r - K$ and $K_r = M_{11} + M_{12} Q_r (I - M_{22} Q_r)^{-1} M_{21}$,

H_∞-controller order reduction

Sketch of proof

Approximation error $\Delta_K := K_r - K_c$.

H_∞-controller order reduction

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Approximation error $\Delta_K := K_r - K_c$.

$$\Delta_K = M_{12} Q_r (I - M_{22} Q_r)^{-1} M_{21}$$

H_∞-controller order reduction

Sketch of proof

Approximation error $\Delta_K := K_r - K_c$.

$$\Delta_K = M_{12} Q_r (I - M_{22} Q_r)^{-1} M_{21}$$

Then

$$Q_r = (I + M_{12}^{-1} \Delta_K M_{21}^{-1} M_{22})^{-1} M_{12}^{-1} \Delta_K M_{21}^{-1}$$

H_∞-controller order reduction

Sketch of proof

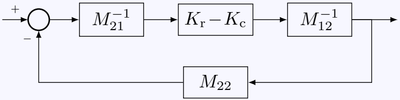
Approximation error $\Delta_K := K_r - K_c$.

$$\Delta_K = M_{12}Q_r(I - M_{22}Q_r)^{-1}M_{21}$$

Then

$$Q_r = (I + M_{12}^{-1}\Delta_K M_{21}^{-1}M_{22})^{-1}M_{12}^{-1}\Delta_K M_{21}^{-1}$$

Q_r may be represented under the form:



By using small gain theorem: $\|\Delta_K M_{21}^{-1}M_{22}M_{12}^{-1}\|_\infty < 1$ or $\|M_{21}^{-1}M_{22}M_{12}^{-1}\Delta_K\|_\infty < 1$

H_∞ controller order reduction

Reduction overview

K_r is a stabilizing controller (i.e: $\|\mathcal{F}_\ell(K_r,P)\|_\infty < \gamma$) if:

H_∞ controller order reduction

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K_r is a stabilizing controller (i.e: $\|\mathcal{F}_\ell(K_r,P)\|_\infty < \gamma$) if:

$$\|(K_r - K)M_{21}^{-1}M_{22}M_{12}^{-1}\|_\infty < 1$$

H_∞ controller order reduction

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Frequency weighted balanced truncation

$V = M_{21}^{-1}M_{22}M_{12}^{-1}$ and $W = I$

H_∞ controller order reduction

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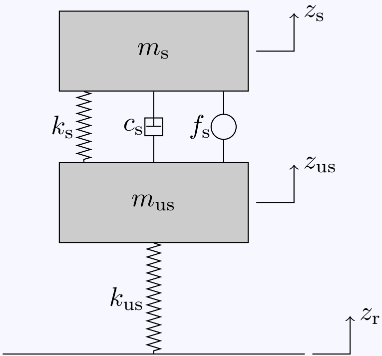
$V = M_{21}^{-1}M_{22}M_{12}^{-1}$ and $W = I$

$$\|W(K_r - K)V\|_\infty < 1$$

Outlines

Application: H_∞ active suspension control

Quarter vehicle model



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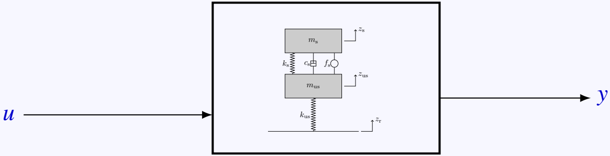

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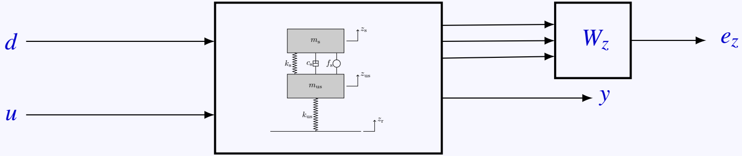

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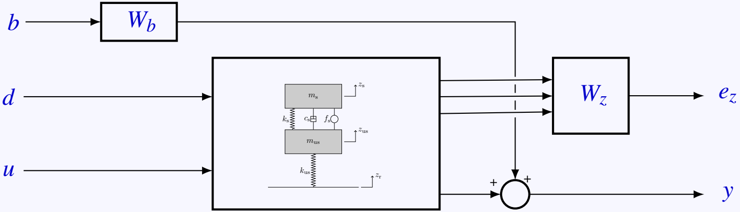

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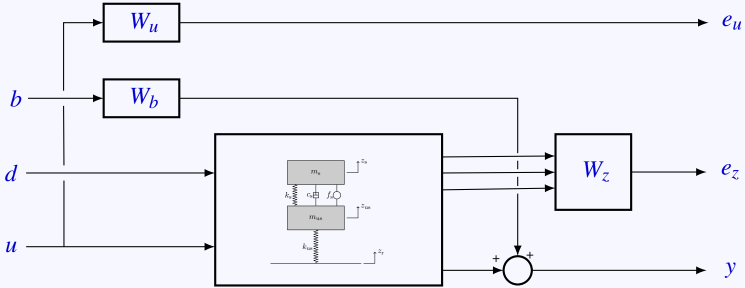

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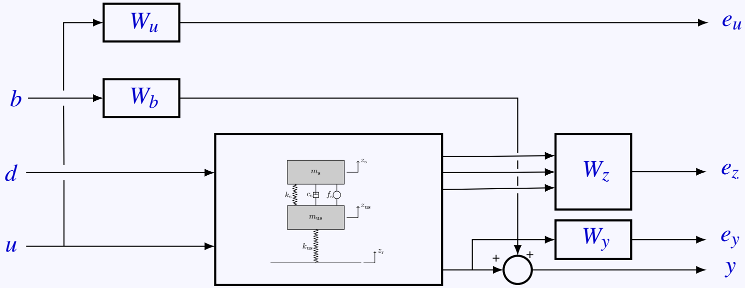

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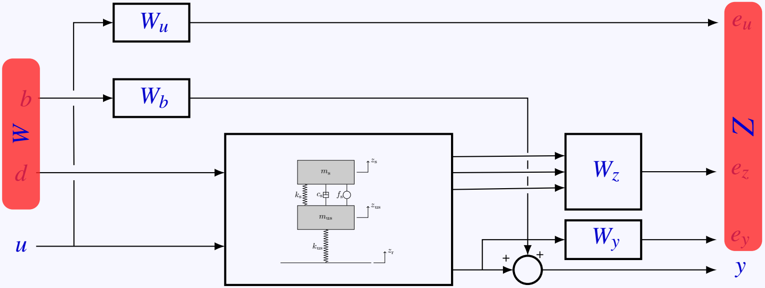

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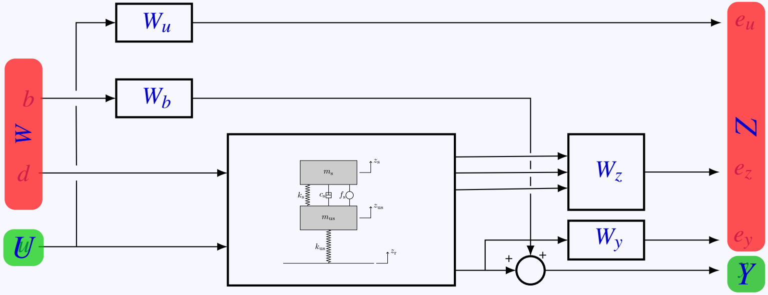

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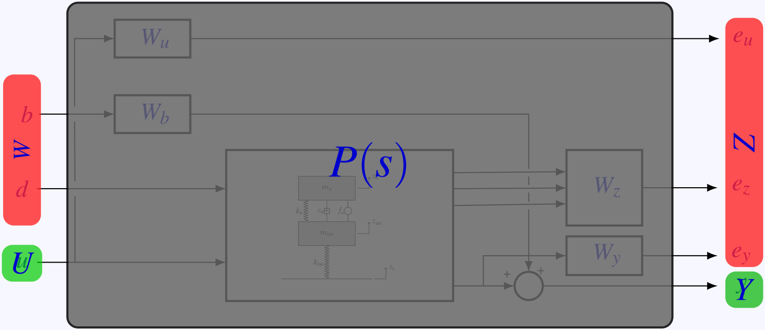

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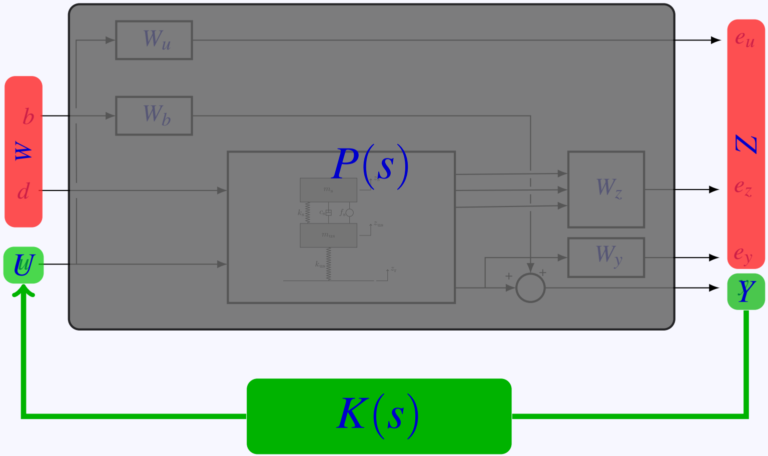
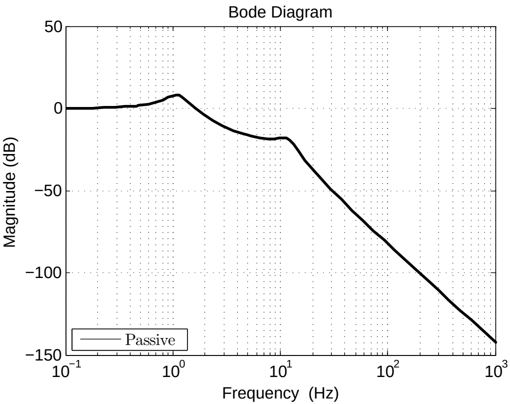


Application: H_∞ active suspension control

Frequency domain analysis: $\frac{Z_s}{Z_r}$ (K: 14 states, K_r : 5 states)



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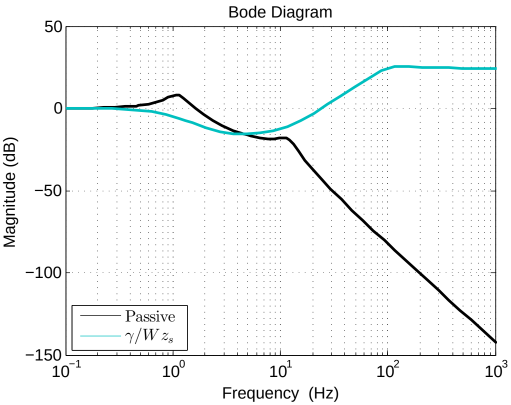

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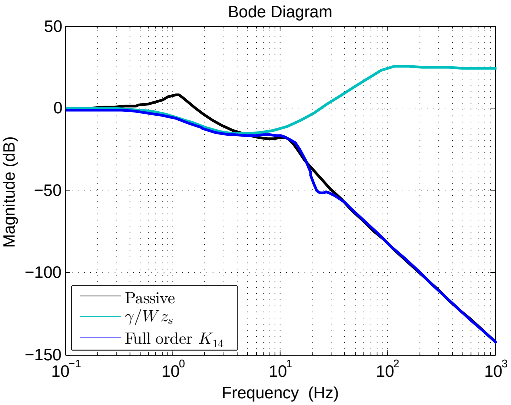

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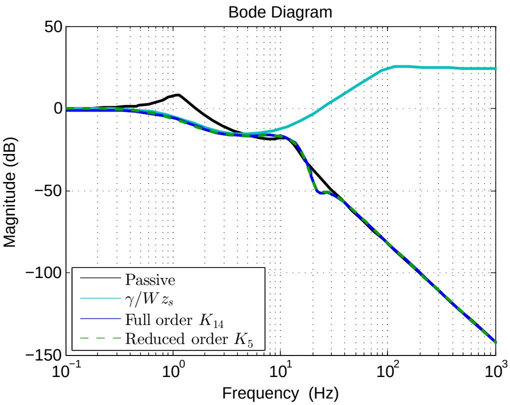

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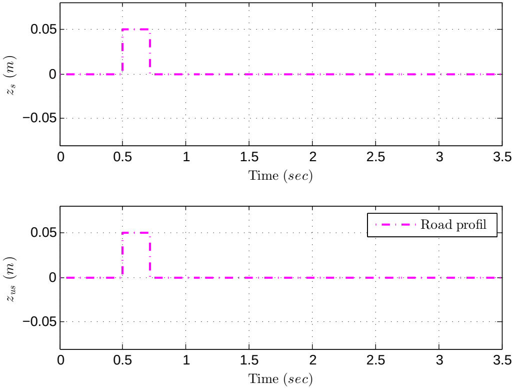
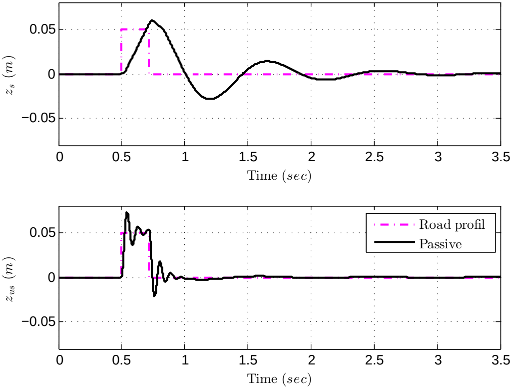


Application: H_∞ active suspension control

Time Domain analysis: (bump of 0.05×0.2m at 30km/h)



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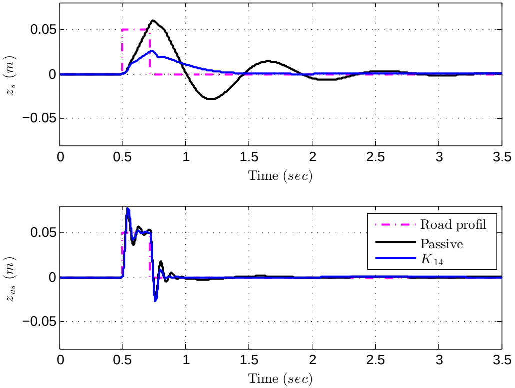

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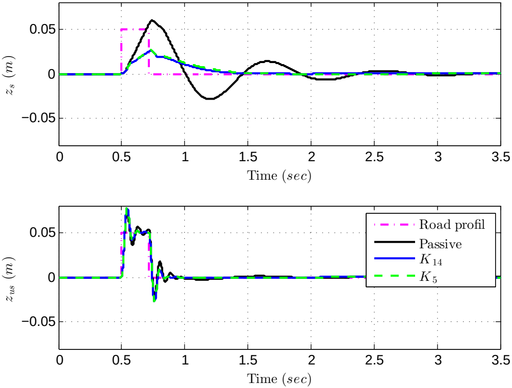
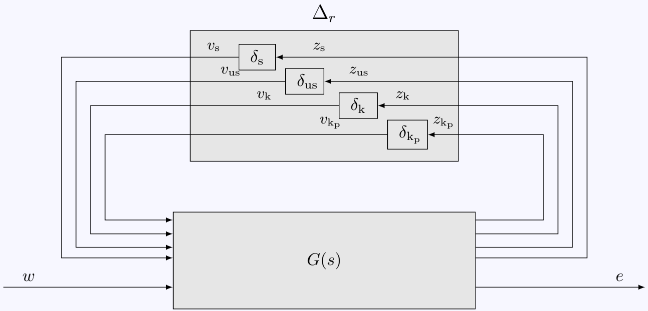


Robustness analysis

Robust Stability

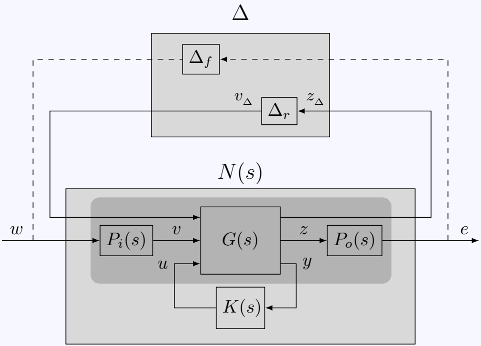


$$\begin{aligned} \overline{k} &= k(1+p_k\delta_k), & \overline{k}_p &= k(1+p_{kp}\delta_{kp}) \\ \overline{m}_s &= k(1+p_s\delta_s), & \overline{m}_{us} &= k(1+p_{us}\delta_{us}) \end{aligned}$$

where $\{p_k,\ p_{kp},\ p_s,\ p_{us}\} \in [-1\ ;1]$, $\{\delta_k,\ \delta_{kp},\ \delta_s,\ \delta_{us}\} \in [0\ ;1]$ and $\Delta_r(s)$ is the uncertain block that contains the model uncertainties.

Robustness analysis

Robust Performance



Δ_f : fictive uncertain block that relays the controlled outputs to the exogenous inputs.

Robustness analysis

Structured singular value: μ

For $N \in \mathbb{C}^{n\times n}, \mu_{\underline{\Delta}}(N)$ is defined as :

$$\mu_{\underline{\Delta}}(N) = \big(\inf_{\Delta \in \underline{\Delta}} \{\overline{\sigma}(\Delta),\ \det(I - \Delta N) = 0\}\big)^{-1}$$

$\underline{\Delta} = \mathrm{diag}[\delta_i I_{r_1}, \ldots, \delta_s I_{r_S}, \Delta_1, \ldots, \Delta_F]$ $\quad \delta_i \in \mathbb{C}, \Delta_j \in \mathbb{C}^{m_j \times m_j}$ $\quad \sum_{i=1}^{S} r_i + \sum_{j=1}^{F} m_j = n,$

Theorem[**Skogestad1996**]

Let $\beta > 0$ and assume that the nominal system $N(s)$ and the perturbations Δ are stable. Then, the feedback system is stable for all allowed perturbations $\Delta(\cdot) \in \mathcal{M}(\underline{\Delta})$ such that $||\Delta||_\infty < 1/\beta$ if and only if:

$$\forall \omega \in \mathbb{R},\ \mu_{\underline{\Delta}}(N_{z_\Delta v_\Delta}(j\omega))\ \leq\ \beta$$

Then, the following rules are verified $\forall \omega$:

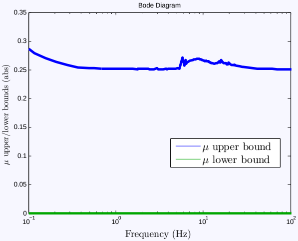
$$\begin{array}{lcll} \text{Robust Stability} & \Longleftrightarrow & \mu_{\underline{\Delta}_r}(N_{z_\Delta v_\Delta}(j\omega)) & < 1 \\ \text{Robust Performance} & \Longleftrightarrow & \mu_{\underline{\Delta}}(N(j\omega)) & < 1 \end{array}$$

$\mathcal{M}(\underline{\Delta}) := \{\Delta(\cdot) \in \mathcal{RH}_\infty : \Delta(s_0) \in \underline{\Delta}$ for all $s_o \in \bar{\mathbb{C}}^+\}$

Robustness analysis

μ-upper and lower bounds

Robust Stability: $\mu_{\underline{\Delta}}(N_{z_\Delta v_\Delta}(j\omega))$



In practice :

$$\mu_{\underline{\Delta}}(N_{z_\Delta v_\Delta}(j\omega)) \in [\mu_{\underline{\Delta}}^{\text{lb}}(N_{z_\Delta v_\Delta}(j\omega));\ \mu_{\underline{\Delta}}^{\text{ub}}(N_{z_\Delta v_\Delta}(j\omega))]$$

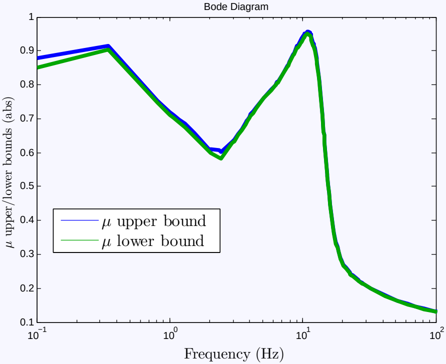
$\mu_{\underline{\Delta}}^{\text{lb}}(N_{z_\Delta v_\Delta}(j\omega))$: μ-lower bound,
$\mu_{\underline{\Delta}}^{\text{ub}}(N_{z_\Delta v_\Delta}(j\omega))$: μ-upper bound.

$m_{\text{s}} = 415 \pm 30\%$ kg, $m_{\text{us}} = 52 \pm 30\%$ kg, $k_{\text{s}} = 22000 \pm 10\%$ N/m, and $k_{\text{us}} = 270000 \pm 10\%$ N/m.

Robustness analysis

μ-upper and lower bounds

Robust Performance: $\mu_{\underline{\Delta}}(N(j\omega))$



$m_s = 415 \pm 10\%$ kg, $m_{us} = 52 \pm 10\%$ kg, $k_s = 22000 \pm 4\%$ N/m, and $k_{us} = 270000 \pm 4\%$ N/m.

Outlines

Conclusions and Perspectives

Conclusions

FWBT techniques applied to the control order reduction

- Closed–loop performance and stability.
- Easy real–time processing.
- Strong conditions on initial models.

Work in progress...

- Robust control (applying FWBT on DK-iteration)
- LPV-H_∞ controller order reduction.
- Applying to control semi-active suspension system.
- Controller order reduction for uncertain systems.

Thank you for your attention! Questions?