Systèmes commutés de type Lur'e et propriété de consistance

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UMR 7039

Outline

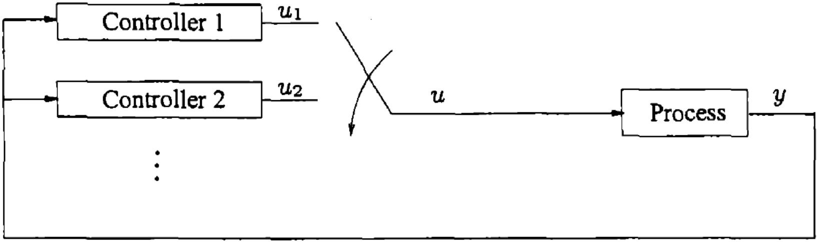
Motivation

Consistency in the switched linear systems case

Consistency in the Lur'e switched systems case

Conclusion

Motivation



One process

- Time varying systems
- Uncertain systems
- Non-linear systems

Multiple controller

- Robustness
- Performance

Switched linear systems

Let us consider the following switched linear systems

$$x_{k+1} = A_{\sigma(k)} x_k, \qquad (1)$$

where:

- $\sigma : \mathbb{N} \to \mathcal{I}_N$ is the switching function, with $\mathcal{I}_N = \{1; \cdots ; N\}$.
- A_i are real matrices with appropriate dimensions and $i \in \mathcal{I}_N$.

Goal 1

Synthesize a state feedback switching control strategy ensuring that the origin $x = 0$ is globally asymptotically stable.

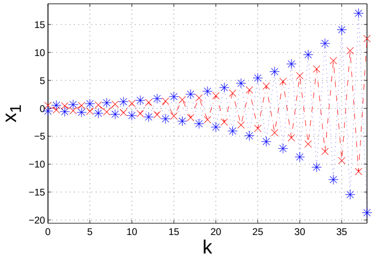
Example 1

Let us consider matrices

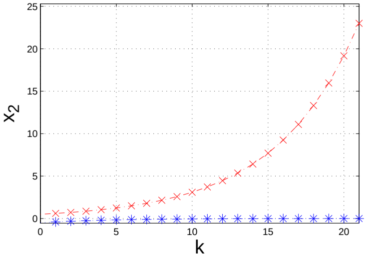
$$A_1 = \begin{bmatrix} -1.1 & 0 \\ 1 & 0.4 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 1.3 \end{bmatrix}, \; x_0 = \begin{pmatrix} -0.5 \\ 0.5 \end{pmatrix}.$$

Both system are unstable

$x_{k+1} = A_1 x_k$

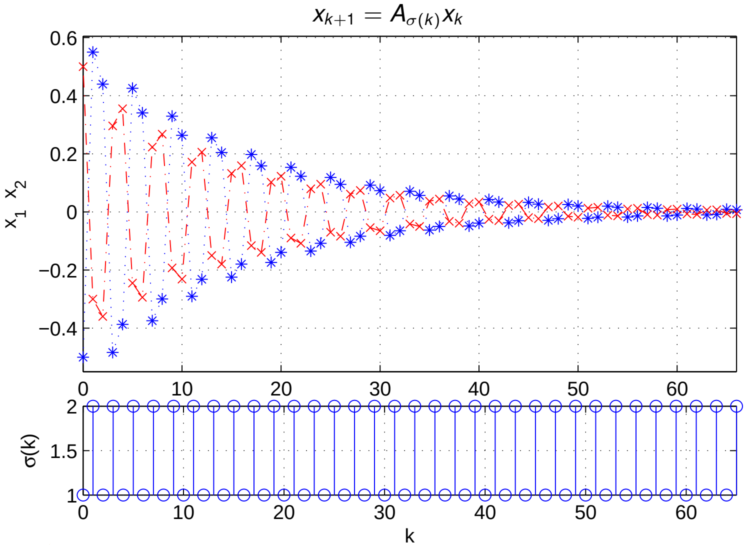


$x_{k+1} = A_2 x_k$



Example 1

$A_1 A_2$ stable $\Rightarrow$ stabilization by using successively mode 1 and 2.



Tools for min-switching

- Let us introduced the Metzler matrices :

$$\mathcal{M} = \left\{ \Pi \in \mathbb{R}^{N\times N}, \forall i \in \mathcal{I}_N,\ \pi_{ii} \geq 0,\ \sum_{\ell \in \mathcal{I}_N} \pi_{\ell i} = 1 \right\}. \tag{2}$$

- Lyapunov function considered

$$V_{\min} : \left\{ \begin{array}{ccc} \mathbb{R}^n & \rightarrow & \mathbb{R}, \\ x_k & \mapsto & \min_{i \in \mathcal{I}_N} V_i(x_k), \end{array} \right. \tag{3}$$

with $\forall i \in \mathcal{I}_N,\ V_i(x_k) = x_k' P_i x_k$ where $P_i \in \mathbb{R}^{n\times n}$ and $P_i = P_i' > 0$.

Closed-loop stability

Theorem 1 (Lyapunov-Metzler inequalities) If there exist matrices $P_i > 0, \forall i \in \mathcal{I}_N$ and $\Pi \in \mathcal{M}$ such that

$$A_i'(P)_{p,i}A_i - P_i < 0, \tag{4}$$

where

$$(P)_{p,i} = \sum_{\ell \in \mathcal{I}_N} \pi_{\ell i} P_\ell, \tag{5}$$

then the state feedback switching strategy $\sigma(k) = \arg \min_{i \in \mathcal{I}_N} x_k' P_i x_k$ ensures that the origin x = 0 is globally asymptotically stable.

Closed-loop stability

Elements of proof

- By post-multiplying by $x_k \neq 0$ and pre-multiplying by x_k',

$$x_{k+1}'(P)_{p,i}x_{k+1} - x_k' P_i x_k < 0 \tag{6}$$

- at time k,

$$i = \arg\min_{j \in \mathcal{I}_N} x_k' P_j x_k \tag{7}$$

- the minimum scalar value of convex polytopes is reached on one of the vertices

$$V_{\min}(x_{k+1}) = \min_{j \in \mathcal{I}_N} x_{k+1}' P_j x_{k+1} \tag{8}$$

$$\min_{\substack{\sum_{j \in \mathcal{I}_N} \lambda_j = 1 \\ \lambda_j \in \mathbb{R}^+;}} \sum_{j \in \mathcal{I}_N} \lambda_j x_{k+1}' P_j x_{k+1}. \tag{9}$$

Each column of the Metzler matrix $\Pi \in \mathcal{M}$ is in the unit simplex, then

$$V_{\min}(x_{k+1}) \leq x_{k+1}'(P)_{p,i}x_{k+1}. \tag{10}$$

$\Rightarrow$ global asymptotic stability holds with

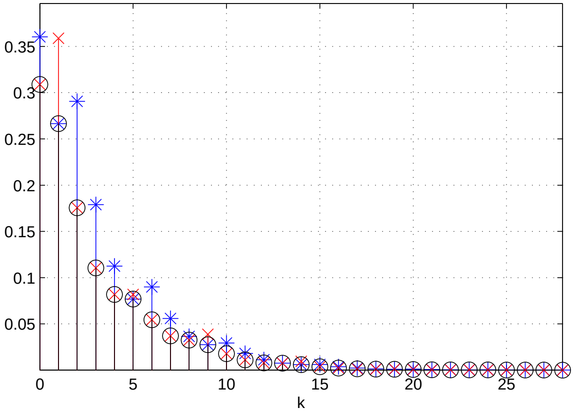
$$V_{\min}(x_{k+1}) - V_{\min}(x_k) < 0, \quad \forall x_k \neq 0. \tag{11}$$

Return on Example 1: state-partition

Theorem 1 provides a solution with

$$\Pi = \begin{bmatrix} 0.3 & 0.7 \\ 0.7 & 0.3 \end{bmatrix}, P_1 = \begin{bmatrix} 1.7097 & 0.3734 \\ 0.3734 & 0.4786 \end{bmatrix}, \ P_2 = \begin{bmatrix} 1.1978 & 0.6398 \\ 0.6398 & 1.3173 \end{bmatrix}.$$

$V_{\min}(x_k),\ V_1(x_k),\ V_2(x_k)$



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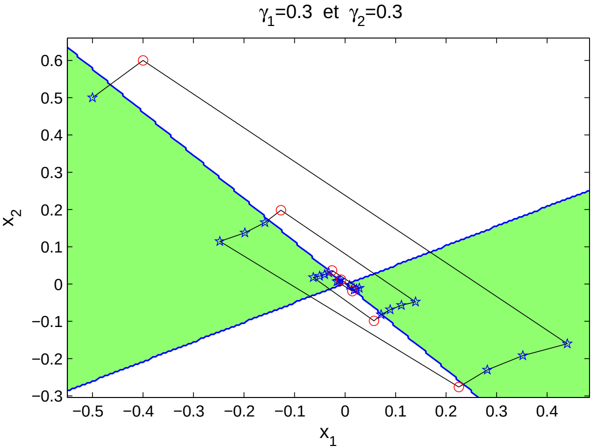


Discussion

- Which switching strategy can be used to stabilize such a linear switched system ?
- If one mode is stable, is there interest to use the other modes with a switching strategy ?
- If all modes are stable, is there exist an interest of using a switching strategy ?

Discussion

- Which switching strategy can be used to stabilize such a linear switched system ?
- If one mode is stable, is there interest to use the other modes with a switching strategy ?
- If all modes are stable, is there exist an interest of using a switching strategy ?

How to compare all these strategies ?
A linear switched system which is globally asymptotically stable admits a finite quadratic cost.

Closed-loop performance

Let us, now, consider the following switched linear systems

$$\begin{cases} x_{k+1} = A_{\sigma(k)} x_k, \\ z_k = E_{\sigma(k)} x_k, \end{cases} \quad (12)$$

where $z_k \in \mathbb{R}^r$ is the output. We focus on the quadratic performance, obtained by applying the switching law $\sigma(\cdot)$ for a trajectory starting from x_0,

$$\mathcal{J}_\sigma(x_0) = \sum_{k \in \mathbb{N}} z_k' z_k = \sum_{k \in \mathbb{N}} x_k' Q_{\sigma(k)} x_k, \quad (13)$$

with $Q_{\sigma(k)} = E_{\sigma(k)}' E_{\sigma(k)}$.

Goal 2

Synthesize a state feedback switching control strategy ensuring that the origin $x = 0$ is globally asymptotically stable and improving the performance.

Closed-loop performance

Theorem 2 If there exist matrices $P_i > 0, \forall i \in \mathcal{I}_N$ and $\Pi \in \mathcal{M}$ solution of the optimization problem

$$\min_{P_i,\Pi} \left(\min_{i\in\mathcal{I}_N} \mathrm{trace}(P_i) \right), \tag{14}$$

subject to

$$A_i'(P)_{p,i}A_i - P_i + E_i'E_i < 0, \tag{15}$$

where

$$(P)_{p,i} = \sum_{\ell\in\mathcal{I}_N} \pi_{\ell i} P_\ell, \tag{16}$$

then the state feedback switching strategy $\sigma(k) = \arg\min_{i\in\mathcal{I}_N} x_k' P_i x_k$, called *min-switching* strategy, ensures that the origin x = 0 is globally asymptotically stable and

$$\mathcal{J}_\sigma(x_0) \leq \min_{i\in\mathcal{I}_N} x_0' P_i x_0 = V_{\min}(x_0). \tag{17}$$

Closed-loop performance

Elements of proof

- the proof follows the same lines of the previous one. By post-multiplying by $x_k \neq 0$ and pre-multiplying by x_k',

$$V_{\min}(x_{k+1}) - V_{\min}(x_k) < -z_k' z_k \leq 0, \quad \forall x_k \neq 0, \tag{18}$$

$\Rightarrow$ system globally asymptotically stable.

- by summing on $k \in \mathbb{N}$

$$\lim_{k \to +\infty} V_{\min}(x_k) - V_{\min}(x_0) \leq -\sum_{k \in \mathbb{N}} z_k' z_k, \tag{19}$$

$\Rightarrow$

$$V_{\min}(x_0) \geq \mathcal{J}_\sigma(x_0), \tag{20}$$

Sub-optimal solution: BMI $\rightarrow$ LMI

Inequality (15) is a BMI: we propose a sub-optimal solution via LMI, with a parametrized Metzler matrix.

Corollary 1 The results of the previous theorem remains valid, if there exist $\forall i \in \mathcal{I}_N$ matrices $P_i > 0$ and scalars $\gamma_i \in [0; 1[$, solution of

$$\min_{P_i, \gamma_i} \left(\min_{i \in \mathcal{I}_N} \mathrm{trace}(P_i) \right), \tag{21}$$

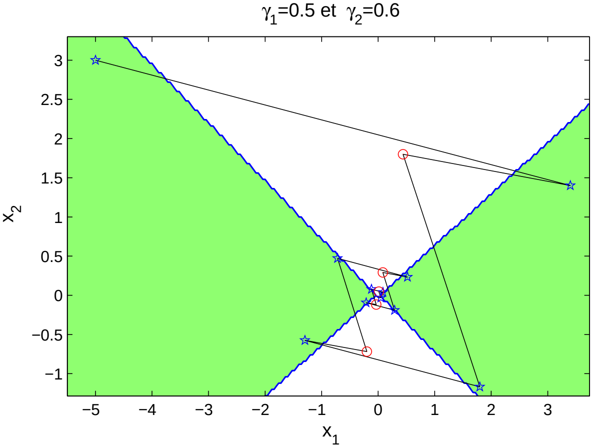
subject to

$$A_i' \left(\gamma_i P_i + (1 - \gamma_i) P_j\right) A_i - P_i + E_i' E_i < 0. \tag{22}$$

New aspect on example 2

$$A_1 = \begin{bmatrix} 0 & 1 \\ -0.2 & 0.6 \end{bmatrix}, \ A_2 = \begin{bmatrix} -0.2 & -0.8 \\ 0.2 & 0.8 \end{bmatrix}, \ E_1' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ E_2' = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ x_0 = \begin{pmatrix} -5 \\ 3 \end{pmatrix}'.$$

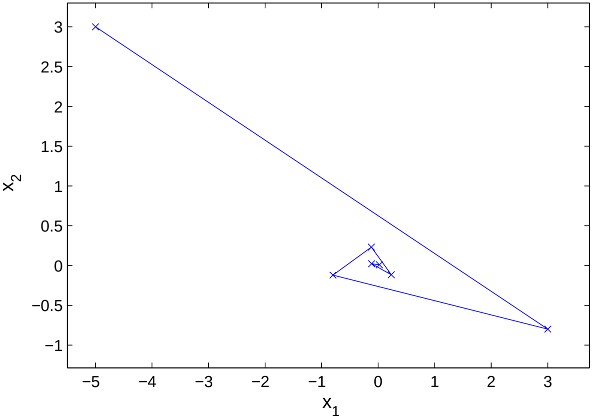
Switching strategy $V_{\min}(x_0) = 21.44$, $\mathcal{J}_\sigma(x_0) = 13.22$



New aspect on example 2

$$A_1 = \begin{bmatrix} 0 & 1 \\ -0.2 & 0.6 \end{bmatrix},\ A_2 = \begin{bmatrix} -0.2 & -0.8 \\ 0.2 & 0.8 \end{bmatrix},\ E_1' = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ E_2' = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\ x_0 = \begin{pmatrix} -5 \\ 3 \end{pmatrix}'.$$

Mode 1, $\mathcal{J}_1(x_0) = 34.72$



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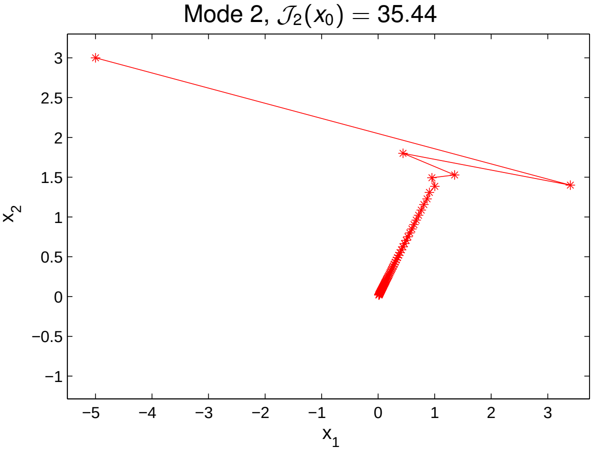

New aspect on example 2

Illustration of Theorem 2:

$$\mathcal{J}_\sigma(x_0) = 13.22 \leq V_{\min}(x_0) = 21.44$$

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$$\mathcal{J}_\sigma(x_0) = 13.22 \leq V_{\min}(x_0) = 21.44$$

Observation:

$$\mathcal{J}_\sigma(x_0) \leq V_{\min}(x_0) \leq \mathcal{J}_1(x_0) \leq \mathcal{J}_2(x_0) \tag{23}$$

$$\frac{\mathcal{J}_1(x_0) - V_{\min}(x_0)}{\mathcal{J}_1(x_0)} = 38\% \tag{24}$$

$$\frac{\mathcal{J}_1(x_0) - \mathcal{J}_\sigma(x_0)}{\mathcal{J}_1(x_0)} = 62\% \tag{25}$$

New aspect on example 2

Illustration of Theorem 2:

$$\mathcal{J}_\sigma(x_0) = 13.22 \leq V_{\min}(x_0) = 21.44$$

Observation:

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$$\frac{\mathcal{J}_1(x_0) - \mathcal{J}_\sigma(x_0)}{\mathcal{J}_1(x_0)} = 62\% \quad (25)$$

Can we formalize this observation ?

Is it always true ?

Consistency

Definition 1

Consider the class of switched discrete-time linear systems, where $\sigma : \mathbb{N} \to \mathcal{I}_N$ is the switching law. A particular switching strategy $\sigma_s(\cdot)$ is consistent, with respect to the performance $\mathcal{J}_\sigma(\cdot)$, if it improves the performance when compared to the performances of each isolated subsystem supposed to be asymptotically stable.

$$\mathcal{J}_{\sigma_s}(x_0) \leq \min_{i \in \mathcal{I}_N} \mathcal{J}_{\sigma=i}(x_0). \tag{26}$$

Theorem 3

The state feedback switching strategy $\sigma_s(k) = \arg\min_{i \in \mathcal{I}_N} x_k' P_i x_k$, given by Theorem 2, is consistent.

Consistency

Elements of proof

- by considering the Metzler matrix $\Pi_\ell \in \mathcal{M}$ with null elements except $\forall j \in \mathcal{I}_N$, $\pi_{\ell j} = 1$, such that $(P)_{p,j} = P_\ell$. We get the following constraint

$$A_i' P_\ell A_i - P_i + E_i' E_i < 0, \ \forall i \in \mathcal{I}_N \tag{27}$$

- By taking matrices $P_i > P_\ell$, $\forall i \in \mathcal{I}_N \setminus \{\ell\}$, the optimization problem is feasible if and only if there exists $P_\ell > 0$ such that

$$\min \left(\mathrm{trace} P_\ell\right), \tag{28}$$

under the constraint,

$$A_\ell' P_\ell A_\ell - P_\ell + E_\ell' E_\ell < 0. \tag{29}$$

- this is equivalent to study one mode independently of the other and the performance associated with is

$$\mathcal{J}_\ell(x_0) = x_0' P_\ell x_0. \tag{30}$$

$\Rightarrow$ the min-switching law is consistent

$$\mathcal{J}_{\sigma_s}(x_0) \leq V_{\min}(x_0) \leq \min_{i \in \mathcal{I}_N} \mathcal{J}_{\sigma=i}(x_0). \tag{31}$$

Conclusion : consistency in the linear framework

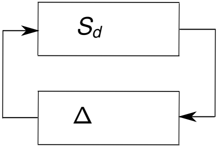
- the *min-switching* strategy is consistent

Non-linear framework: open question !

- no answer in the generic non-linear framework
- what about a particular class of non-linear systems ?

Lur'e switched systems

$$\mathcal{S}_d \;:\; \begin{cases} x_{k+1} = A_{\sigma(k)}x_k + B_{\sigma(k)}\varphi_{\sigma(k)}(y_k), \\ y_k = C_{\sigma(k)}x_k, \end{cases}$$



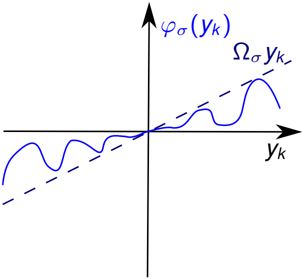


Figure: Condition de secteur

$\varphi_{\sigma(k)} : \mathbb{R}^p \to \mathbb{R}^p$ is assumed to be memoryless, decentralized and verifies a cone bounded sector condition :

$$\varphi'_{\sigma(k)}(y_k)\Lambda_{\sigma(k)}(\varphi_{\sigma(k)}(y_k) - \Omega_{\sigma(k)}y_k) \leq 0$$

with $\forall i \in \mathcal{I}_N$, Ω_i and $\Lambda_i \in \mathbb{R}^{p\times p}$ are positive diagonal matrices.

Interpretation as uncertain systems

- $\varphi(x) = \Delta x, \forall x \in \mathbb{R}$
- structured uncertainties (diagonal)

Closed-loop performance

- We focus on the quadratic performance, obtained by applying the switching law $\sigma(\cdot)$ for a trajectory starting from x_0,

$$\mathcal{J}_\sigma(x_0) = \sum_{k \in \mathbb{N}} x_k' Q_{\sigma(k)} x_k. \tag{32}$$

- Goal 3 Synthesize a state feedback switching control strategy ensuring that the origin $x = 0$ is globally asymptotically stable and improving the performance.
  (= extended Goal 2 for switched Lur'e systems)

Closed-loop performance

Theorem 3 If there exist matrices $P_i > 0, \forall i \in \mathcal{I}_N$ and $\Pi \in \mathcal{M}$ solution of the optimization problem

$$\min_{P_i,\Pi} \left(\min_{i \in \mathcal{I}_N} \mathrm{trace}(P_i) \right), \tag{33}$$

subject to

$$\begin{bmatrix} A_i'(P)_{p,i}A_i - P_i + Q_i & \star \\ B_i'(P)_{p,i}A_i + S_i\Omega_i C_i & B_i'(P)_{p,i}B_i - 2S_i \end{bmatrix} < 0, \tag{34}$$

where

$$(P)_{p,i} = \sum_{\ell \in \mathcal{I}_N} \pi_{\ell i} P_\ell, \tag{35}$$

then the state feedback switching strategy $\sigma(k) = \arg \min_{i \in \mathcal{I}_N} x_k' P_i x_k$ ensures that the origin x = 0 is globally asymptotically stable and

$$\mathcal{J}_\sigma(x_0) \leq \min_{i \in \mathcal{I}_N} x_0' P_i x_0 = V_{\min}(x_0). \tag{36}$$

Closed-loop performance

Elements of proof

- the proof follows the same lines of the one for the linear system. By post-multiplying by $\left(x_k' \ \varphi_{\sigma(k)}'(y_k)\right)'$ and pre-multiplying by its transpose,

$$V_{\min}(x_{k+1}) - V_{\min}(x_k) \leq -x_k' Q_{\sigma(k)} x_k + 2\varphi_{\sigma(k)}'(y_k)\Lambda_{\sigma(k)}(\varphi_{\sigma(k)}(y_k) - \Omega_{\sigma(k)} y_k), \quad (37)$$

then

$$V_{\min}(x_{k+1}) - V_{\min}(x_k) \leq -x_k' Q_{\sigma(k)} x_k, \quad (38)$$

$\Rightarrow$ system globally asymptotically stable.

- by summing on $k \in \mathbb{N}$

$$\lim_{k \to +\infty} V_{\min}(x_k) - V_{\min}(x_0) \leq -\sum_{k \in \mathbb{N}} x_k' Q_{\sigma(k)} x_k, \quad (39)$$

$\Rightarrow$

$$V_{\min}(x_0) \geq \mathcal{J}_\sigma(x_0), \quad (40)$$

Sub-optimal solution: BMI $\rightarrow$ LMI

- Theorem 3 $\Rightarrow$ BMI constraints,
- A sub-optimal solution $\Rightarrow$ LMI constraints,

Corollary 2 The results of the previous theorem remains valid, if there exist $\forall i \in \mathcal{I}_N$ matrices $P_i > 0$ and scalars $\gamma_i \in [0; 1[$, solution of

$$\min_{P_i,\gamma_i} \left(\min_{i \in \mathcal{I}_N} \mathrm{trace}(P_i) \right), \tag{41}$$

subject to

$$\begin{bmatrix} A_i' \left(\gamma_i P_i + (1-\gamma_i) P_j\right) A_i - P_i + Q_i & \star \\ B_i' \left(\gamma_i P_i + (1-\gamma_i) P_j\right) A_i + S_i \Omega_i C_i & B_i' \left(\gamma_i P_i + (1-\gamma_i) P_j\right) B_i - 2 S_i \end{bmatrix} < 0. \tag{42}$$

Elements of proof As in linear case, considering the set sub-Metzler matrices $\mathcal{M}^{\mathrm{sub}}$ implies that Corollary 2 is a particular case of Theorem 3.

Example 3

Consider a switched Lur'e system composed by two subsystems defined by the following parameters:

$$A_1 = \begin{bmatrix} 0.9 & 0 \\ 0.4 & -0.72 \end{bmatrix},\ A_2 = \begin{bmatrix} -0.58 & -0.8 \\ 0 & -0.8 \end{bmatrix},\ B_1 = -\begin{bmatrix} 0.5 \\ 0.2 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.2 \\ 0.5 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.6 & 0.24 \end{bmatrix},\ C_2 = \begin{bmatrix} 0.4 & 1.1 \end{bmatrix},\ \varphi_1(y_k) = \frac{\Omega_1 y_k}{2}(1 + \cos(2y_k)),$$

$$\varphi_2(y_k) = \frac{\Omega_2 y_k}{2}(1 - \sin(5.5y_k)),\ \Omega_1 = 0.6,\ \Omega_2 = 1.2,\ x_0 = \begin{pmatrix} -4 \\ 5 \end{pmatrix}.$$

Performances for $Q_i = q_i I_n$ with $i \in \mathcal{I}_2$

q_1	q_2	$\mathcal{J}_{\sigma_s}$	$V_{\min}(x_0)$
1	1	52	82
4	1	76	168
1	4	121	175

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$$\varphi_2(y_k) = \frac{\Omega_2 y_k}{2}(1 - \sin(5.5y_k)), \; \Omega_1 = 0.6, \; \Omega_2 = 1.2, \; x_0 = \begin{pmatrix} -4 \\ 5 \end{pmatrix}.$$

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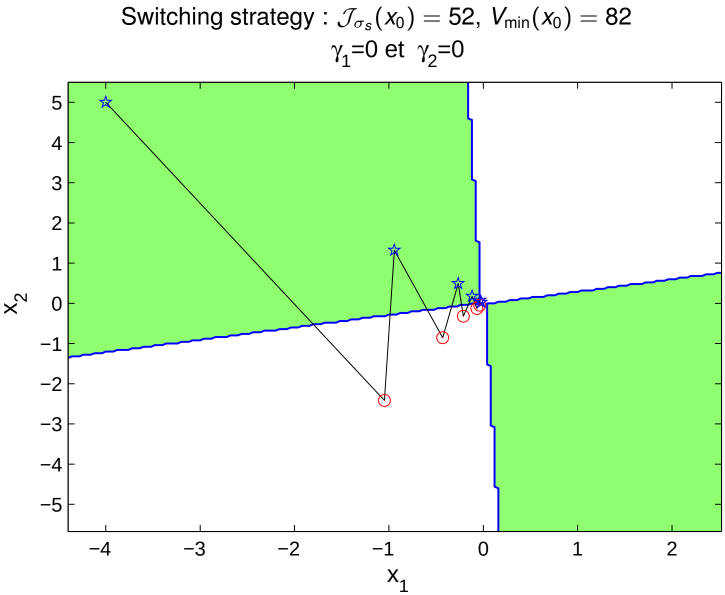
q_1	q_2	$\mathcal{J}_{\sigma_s}$	$V_{\min}(x_0)$
1	1	52	82
4	1	76	168
1	4	121	175

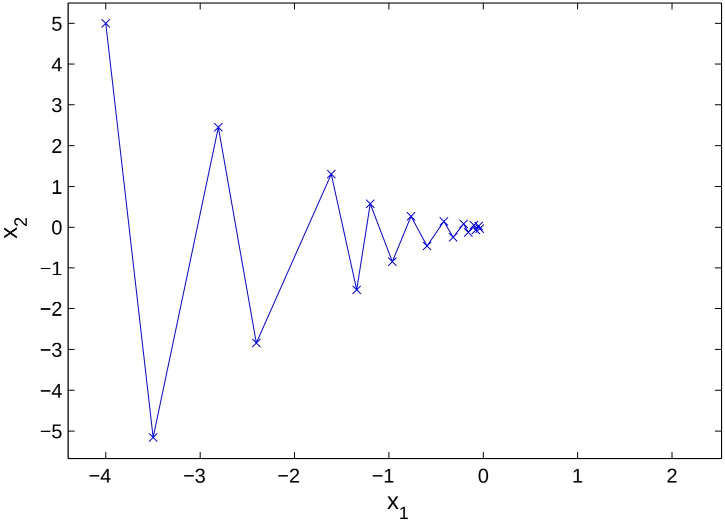
$$\mathcal{J}_{\sigma_s} \leq V_{\min}(x_0).$$

Trajectories obtained for parameters $q_1 = q_2 = 1$



Trajectories obtained for parameters $q_1 = q_2 = 1$

Mode 1 : $\mathcal{J}_{\sigma_s}(x_0) = 121$



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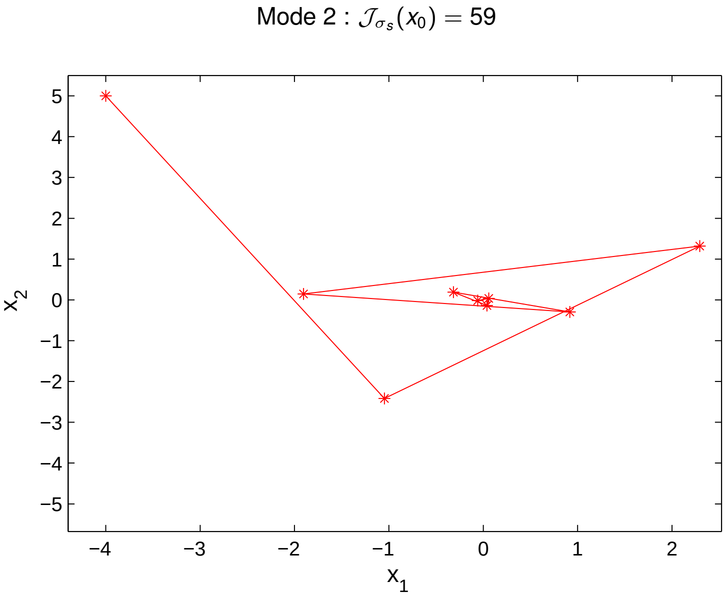

Example 3

Is the min-switching strategy consistent ?

$$\mathcal{J}_{\sigma_s}(x_0) \leq V_{\min}(x_0) \overset{?}{\leq} \min_{i \in \mathcal{I}_2} \mathcal{J}_{\sigma=i}(x_0)$$

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Performances for $Q_i = q_i I_n$ with $i \in \mathcal{I}_2$

q_1	q_2	$\mathcal{J}_{\sigma_s}$	$V_{\min}(x_0)$	$\mathcal{J}_1$	$\mathcal{J}_2$
1	1	52	82	121	59
4	1	76	168	484	59
1	4	121	175	121	238

Example 3

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Performances for $Q_i = q_i I_n$ with $i \in \mathcal{I}_2$

q_1	q_2	$\mathcal{J}_{\sigma_s}$	$V_{\min}(x_0)$	$\mathcal{J}_1$	$\mathcal{J}_2$
1	1	52	82	121	59
4	1	76	168	484	59
1	4	121	175	121	238

The answer is NO!

Discussion

In the Lur'e system framework with $Q_1 = I_2$,

- if $\varphi_1(y_k) = 0$, **linear case** $\Rightarrow$ value of the cost function $\mathcal{J}_1^a(x_0) = 169$,
- if $\varphi_1(y_k) = \Omega_1 y_k$, **linear case** $\Rightarrow$ value of the cost function $\mathcal{J}_1^b(x_0) = 96$,
- Theorem 3 provides the upper bound $\overline{\mathcal{J}_1}(x_0) = 175$.

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$$\mathcal{J}_1^a(x_0) \leq \overline{\mathcal{J}_1}(x_0),$$
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How to adapt the consistency concept for the class of switched Lur'e systems in discrete-time ?

Extension of Consistency concept

Definition 2 Consider switched Lur'e systems, a particular switching strategy $\sigma_s(\cdot)$ is consistent, with respect to the performance $\mathcal{J}_{\sigma_s}$, if it improves the upper bound of the performance when compared to the performances of each isolated subsystem.

$$\mathcal{J}_{\sigma_s}(x_0) \leq V_{\min}(x_0) \leq \min_{i \in \mathcal{I}_N} \overline{\mathcal{J}_{\sigma=i}}(x_0), \tag{43}$$

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Theorem 4 The state feedback switching strategy $\sigma(k) = \arg \min_{i \in \mathcal{I}_N} x_k' P_i x_k$, given by Theorem 3 is consistent according Definition 2.

Extension of Consistency concept

Elements of proof

- assume there exists at least one mode ℓ whose Theorem 3 can guarantee the stability
- by considering the Metzler matrix $\Pi_\ell \in \mathcal{M}$ with null elements except $\forall j \in \mathcal{I}_N$, $\pi_{\ell j} = 1$, such that $(P)_{p,j} = P_\ell$. We get the following constraint

$$\begin{bmatrix} A_i' P_\ell A_i - P_i + Q_i & \star \\ B_i' P_\ell A_i + S_i \Omega_i C_i & B_i' P_\ell B_i - 2S_i \end{bmatrix} < 0, \ \forall i \in \mathcal{I}_N \tag{44}$$

- By taking matrices $P_i > P_\ell$ and $S_i > S_\ell$, $\forall i \in \mathcal{I}_N \setminus \{\ell\}$, the optimisation problem is feasible if and only if there exists $P_\ell > 0$ such that

$$\min \left(\mathrm{trace} P_\ell \right), \tag{45}$$

under the constraint,

$$\begin{bmatrix} A_\ell' P_\ell A_\ell - P_\ell + Q_\ell & \star \\ B_\ell' P_\ell A_\ell + S_\ell \Omega_\ell C_\ell & B_\ell' P_\ell B_\ell - 2S_\ell \end{bmatrix} < 0, \ \forall i \in \mathcal{I}_N \tag{46}$$

- this is equivalent to study one mode independently of the other and the performance associated with is

$$\overline{\mathcal{J}_\ell}(x_0) = x_0' P_\ell x_0, \tag{47}$$

- Π_ℓ is a particular case of Metzler matrices, it follows

$$\overline{\mathcal{J}_\ell}(x_0) = \min_{\{\Pi_\ell; P_j,\ j \in \mathcal{I}_N\}} \left(x_0' P_\ell x_0 \right) \geq \min_{\{\Pi; P_j,\ j \in \mathcal{I}_N\}} \left(x_0' P_j x_0 \right) = V_{\min}(x_0) \tag{48}$$

with $\Pi \in \mathcal{M}$ and $\sigma(\cdot)$ given by Theorem 3

$\Rightarrow$ the min-switching law is consistent on the upper bound

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$\Rightarrow$ the min-switching law is consistent on the upper bound

$$\mathcal{J}_{\sigma_s}(x_0) \leq V_{\min}(x_0) \leq \min_{i \in \mathcal{I}_N} \overline{\mathcal{J}_{\sigma=i}}(x_0),$$

Return on Example 3

Consider a switched Lur'e system composed by two subsystems defined by the following parameters:

$$A_1 = \begin{bmatrix} 0.9 & 0 \\ 0.4 & -0.72 \end{bmatrix},\ A_2 = \begin{bmatrix} -0.58 & -0.8 \\ 0 & -0.8 \end{bmatrix},\ B_1 = -\begin{bmatrix} 0.5 \\ 0.2 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.2 \\ 0.5 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.6 & 0.24 \end{bmatrix},\ C_2 = \begin{bmatrix} 0.4 & 1.1 \end{bmatrix}, \varphi_1(y_k) = \frac{\Omega_1 y_k}{2}(1+\cos(2y_k)),$$

$$\varphi_2(y_k) = \frac{\Omega_2 y_k}{2}(1-\sin(5.5y_k)),\ \Omega_1 = 0.6,\ \Omega_2 = 1.2,\ x_0 = \begin{pmatrix} -4 \\ 5 \end{pmatrix}.$$

q_1	q_2	$\mathcal{J}_{\sigma_s}$	$V_{min}(x_0)$	$\overline{\mathcal{J}_1}$	$\overline{\mathcal{J}_2}$	$\mathcal{J}_1$	$\mathcal{J}_2$
1	1	52	96	175	231	121	59
4	1	76	168	782	231	484	59
1	4	121	175	175	927	121	238

Conclusion

- Recall of consistency for linear framework.
- The differences between a consistent switching law in switched Lur'e framework and linear one has been highlighted.
- The consistency concept has been adapted to the class of switched Lur'e systems.
- For the class of switched Lur'e systems, a switching law is consistent on the upper bound of the cost function.

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Thanks for your attention.

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