Aperiodic sampling and event-triggered control

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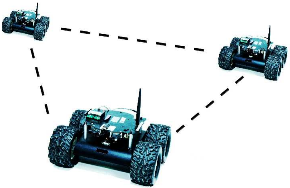
Cyber-physical systems

- **General context** : integration of computational elements in the physical world
- **Object of study** : control of physical systems considering computational and communication implementation/constraints
- **Cybernetic systems** : embedded computers, communication networks, distributed sensors, etc - *discrete models*
- **Physical processes** - *continuous models*
- **Hybrid Dynamics** $\Rightarrow$ new challenges in Control Theory

General context : interaction between computational elements and physical systems



PHYSICAL PROCESS

Sensor data

Control data

```
while (1)
{
        k++;

        y[k] := read (sensor_data);

        if  abs ( y[k] ) <= delta  then
         {
           u[k]:= K*y[k];

           update_control (u[k]);

         }
}
```

Control algorithm
(discret)

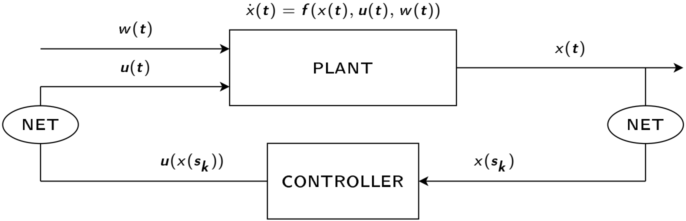
AUTOMATON

```
1000011   1010100   1000010
1001111   1001000   1000101
1001110   1000101   1000001
1010100   1001111   1010101
1010010   1010010   1010100
1001111   1011001   1011001
1001100             1000110
          1011001   1010101
          1010011   1001100
```

$$\dot{x} = f(x, u)$$
$$u = K(x)$$

Mathematical Model
(continuous)

Networked Control Systems (NCS)



- Sampling instants $\{s_k\}_{k\in\mathbb{N}}$, $s_{k+1} = s_k + h_k$
- Fluctuations of the transmission (sampling) step

$$h_k = s_{k+1} - s_k \in [\underline{h}, \overline{h}]$$

Challenges in NCS

$$\left.\begin{array}{l}\textbf{Processor} : \text{limited calculation power} \\ \textbf{Network} : \text{finite bandwidth} \\ \textbf{Sampler} : \text{minimum responding time}\end{array}\right\} \Rightarrow \text{finite number of samples per time unit}$$

How fast SHOULD we sample ? $\leftrightarrow$ How fast CAN we sample ?

Challenges in NCS

Sampler clock : jitter
Network : packet dropouts
Scheduling : interaction between algorithms
Real-time computing : microprocessor latency

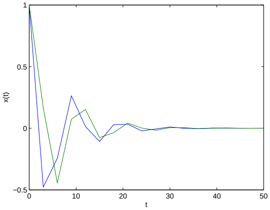
$\Rightarrow$ sampling is not necessarily periodic

Possible destabilizing effect !

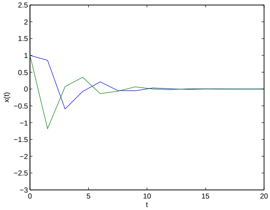
Effects of sampling variation (Zhang, 2001)

$$\dot{x}(t) = Ax(t) + Bu(t), \;\; u(t) = Kx(s_k)$$

Constant sampling step



$h_k = T_1 = 3s, \forall\; k \in \mathbb{N}$:
STABLE



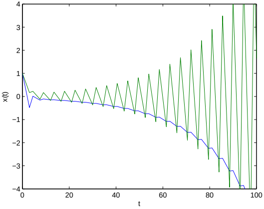
$h_k = T_2 = 1.5s, \forall\; k \in \mathbb{N}$:
STABLE

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad K = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Effects of sampling variation (Zhang, 2001)

$$\dot{x}(t) = Ax(t) + Bu(t), \;\; u(t) = Kx(s_k)$$

Periodic sampling sequence



$$\{h_k\}_{k \in \mathbb{N}} = \{3s, 1.5s, 3s, 1.5s, \cdots\}$$

STABLE + STABLE ⇒ UNSTABLE !

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad K = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

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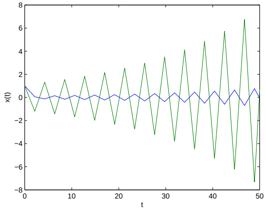
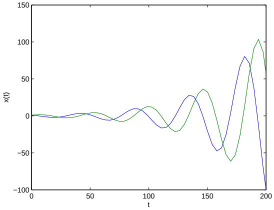


$T = T_1 = 2.13s$: UNSTABLE



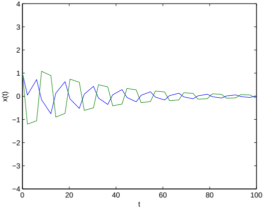
$T = T_2 = 3.95s$: UNSTABLE

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad K = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Effects of sampling variation (Zhang, 2001)

$$\dot{x}(t) = Ax(t) + Bu(t), \;\; u(t) = Kx(s_k)$$

Periodic sampling sequence



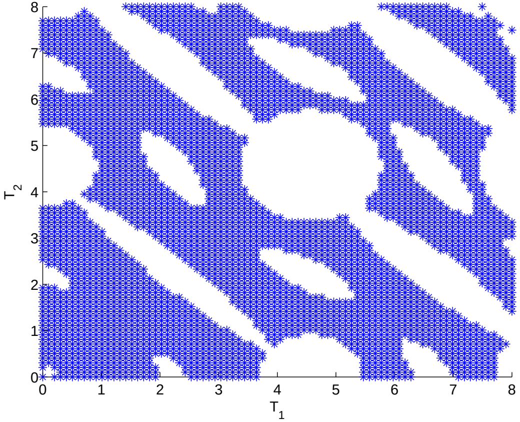
$$T = 3s \rightarrow 2.13s \rightarrow 3.95s \rightarrow 2.13s \rightarrow \cdots$$

UNSTABLE + UNSTABLE ⇒ STABLE!

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad K = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Effects of sampling variation (Zhang, 2001)

Stability domain (allowable sampling intervals, in blue) for a periodic sampling sequence $T_1 \to T_2 \to T_1 \to T_2 \to \cdots$



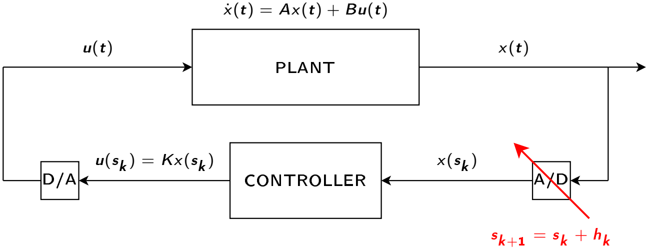
Question

How to reduce the computational (processor and/or network) load while ensuring the system stability ?

Research directions

2 main directions :

- Robust stability analysis with respect to time-varying sampling

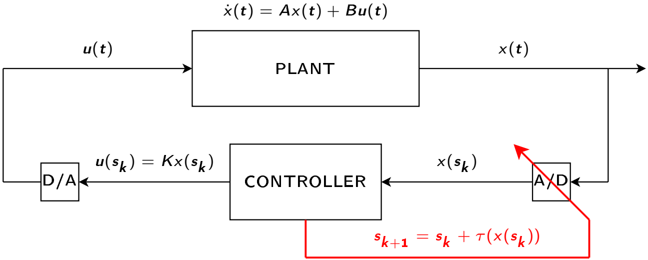


Sampling interval $h_k \in [\underline{h}, \overline{h}]$

Research directions

2 main directions :

- Dynamic control of sampling

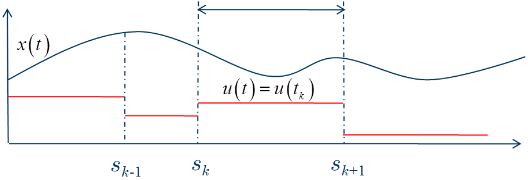


Sampling map $\tau : \mathbb{R}^n \to \mathbb{R}_+$

Robust stability analysis with respect to time-varying sampling

- Discrete-time and Convex Embedding
- Time-delay approach
- Hybrid (Impulsive) System modelling approach
- I/O approach

Discrete-time and Convex Embedding



- Basic references
  - Molchanov, Bauer - IEEE TAC 1999
  - Hetel, Daafouz, Iung - IEEE TAC 2006
  - Fujioka - IEEE TAC 2009
  - Cloosterman et. al - Automatica 2010
  - Hetel, Kruszewski, Perruquetti, Richard - IEEE TAC 2011
- Continuous-time model

$$\dot{x} = Ax + BKx(s_k), \forall t \in [s_k, s_{k+1}), \; h_k = s_{k+1} - s_k \in [\underline{h}, \overline{h}]$$

- Discrete-time model (LPV system)

$$x_{k+1} = \Lambda(h_k)x_k, \quad \Lambda(h) = e^{Ah} + \int_0^h e^{As} ds BK$$

Discrete-time and Convex Embedding

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- For quadratic Lyapunov functions

$$V(x) = x^T P x, \; P = P^T \succ 0$$

- Stability condition : $V(x_{k+1}) - V(x_k) < 0, \; \forall x \neq 0$

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$$x^T \left(\Lambda^T(h) P \Lambda(h) - P \right) x < 0, x \neq 0$$

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- Parametric set of Linear Matrix Inequalities (LMIs)

$$\Lambda^T(\tau) P \Lambda(\tau) - P \prec 0, \; \tau \in [\underline{h}, \overline{h}]$$

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Infinite number of Lyapunov inequalities

Discrete-time and Convex Embedding

- Parametric set of Linear Matrix Inequalities (LMIs)

$$\Lambda^T(\tau)P\Lambda(\tau) - P \prec 0, \ \tau \in [\underline{h}, \overline{h}]$$

Infinite number of Lyapunov inequalities

- Tractable conditions using a convex embedding :

$$\Lambda(\tau) = e^{A\tau} + \int_0^{\tau} e^{As} ds BK \in co\{L_1, L_2, \ldots, L_N\}, \ \tau \in [\underline{h}, \overline{h}]$$

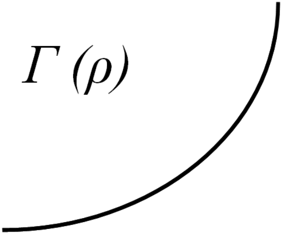
- Finite number of LMI stability conditions for polytopic systems (Daafouz, Bernussou)

$$L_i^T P L_i - P \prec 0, \ i \in \{1, \ldots, N\}$$

Exponential uncertainty

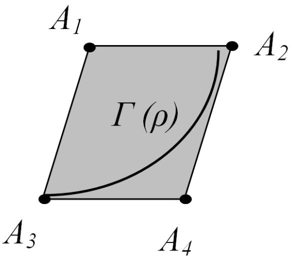
$$\Lambda(\tau) = e^{\tau A} + \int_0^{\tau} e^{sA} ds BK = I + \int_0^{\tau} e^{sA} ds \, (A + BK)$$

$$\Gamma(\rho) = \int_0^{\rho} e^{As} ds, \;\; \underline{h} < \rho < \overline{h}$$



curve in the space of $\mathbb{R}^{n \times n}$ matrices

Exponential uncertainty - Polytopic Embedding



$$\exists \mu_i > 0,\ \forall i = 1, \ldots, N,\ \sum_{i=1}^{N} \mu_i = 1$$

$$\Gamma(\rho) = \int_0^{\rho} e^{As} ds = \sum_{i=1}^{N} \mu_i(\rho) A_i$$

Taylor series :

- (Hetel, Daafouz, Iung, TAC 2006)

Jordan Forms :

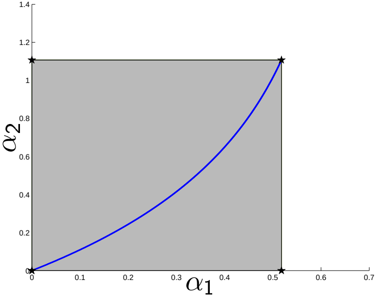
- (Cloosterman, et. al, TAC 2009),
- (Olaru, Niculescu, IFAC World Congress 2007),

Cayley-Hamilton :

- (Gielen, et al. Automatica, 2010)

Jordan normal form

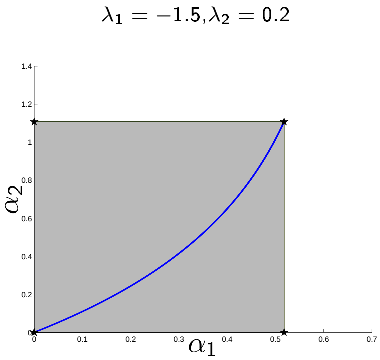
$\lambda_1 = -1.5, \lambda_2 = 0.2$



For $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$$\int_0^\tau e^{As} ds =$$

Jordan normal form



For $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

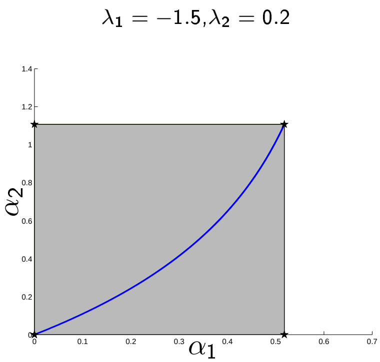
$$\int_0^\tau e^{As}ds = \begin{pmatrix} \alpha_1(\tau) & 0 \\ 0 & \alpha_2(\tau) \end{pmatrix}$$

with

$$\alpha_i(\tau) = \int_0^\tau e^{\lambda_i s}ds = \frac{1}{\lambda_i}(e^{\lambda_i \tau} - 1)$$

vertex for $\tau \in [\underline{h}, \overline{h}]$?

Jordan normal form



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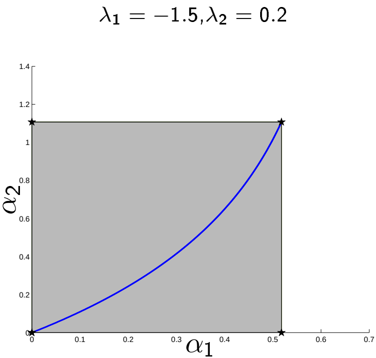
vertex for $\tau \in [\underline{h}, \overline{h}]$? given by max / min $\alpha_i(\tau)$

Jordan normal form



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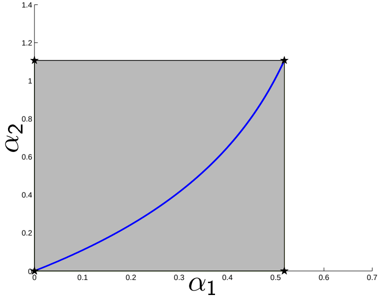
vertex for $\tau \in [\underline{h}, \overline{h}]$? given by max / min $\alpha_i(\tau)$

$$A_1 = \begin{pmatrix} \frac{1}{\lambda_1}(e^{\lambda_1 \underline{h}} - 1) & 0 \\ 0 & \frac{1}{\lambda_2}(e^{\lambda_2 \underline{h}} - 1) \end{pmatrix}, \; A_2 = \begin{pmatrix} \frac{1}{\lambda_1}(e^{\lambda_1 \underline{h}} - 1) & 0 \\ 0 & \frac{1}{\lambda_2}(e^{\lambda_2 \overline{h}} - 1) \end{pmatrix}$$

$$A_3 = \begin{pmatrix} \frac{1}{\lambda_1}(e^{\lambda_1 \overline{h}} - 1) & 0 \\ 0 & \frac{1}{\lambda_2}(e^{\lambda_2 \underline{h}} - 1) \end{pmatrix}, \; A_4 = \begin{pmatrix} \frac{1}{\lambda_1}(e^{\lambda_1 \overline{h}} - 1) & 0 \\ 0 & \frac{1}{\lambda_2}(e^{\lambda_2 \overline{h}} - 1) \end{pmatrix}$$

Jordan normal form

$\lambda_1 = -1.5, \lambda_2 = 0.2$



For $A = T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} T^{-1}$

$$\int_0^\tau e^{As} ds = T \begin{pmatrix} \alpha_1(\tau) & 0 \\ 0 & \alpha_2(\tau) \end{pmatrix} T^{-1}$$

with

$$\alpha_i(\tau) = \int_0^\tau e^{\lambda_i s} ds = \frac{1}{\lambda_i}(e^{\lambda_i \tau} - 1)$$

vertex for $\tau \in [\underline{h}, \overline{h}]$? given by max / min $\alpha_i(\tau)$

$$A_1 = T \begin{pmatrix} \frac{1}{\lambda_i}(e^{\lambda_i \underline{h}} - 1) & 0 \\ 0 & \frac{1}{\lambda_i}(e^{\lambda_i \underline{h}} - 1) \end{pmatrix} T^{-1}, \ \ldots$$

Jordan normal form

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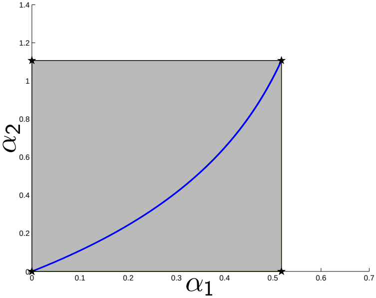


$$\int_0^\tau e^{As} ds = \sum_{i=1}^{N} \mu_i A_i, \forall \tau \in [\underline{h}, \overline{h}]$$

Transition matrix :

$$\Lambda(\tau) = I + \int_0^\tau e^{As} ds (A + BK) = \sum_{i=1}^{N} \mu_i L_i$$

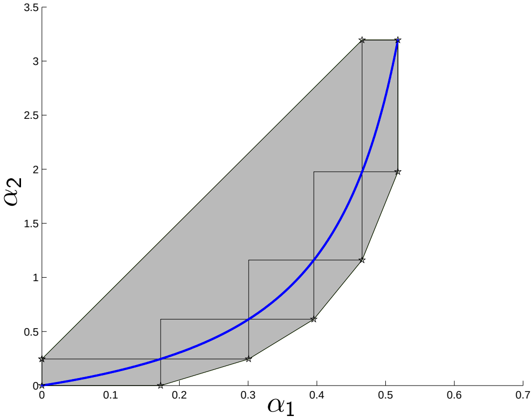
where

$$L_i = I + A_i (A + BK)$$

Finite number of LMI conditions

$$L_i^T P L_i - P \prec 0, \; i \in \{1, \ldots, N\}$$

Discrete-time and Convex Embedding



Finite number of LMI conditions

$$L_i^T P L_i - P \prec 0, \; i \in \{1, \ldots, N\}$$

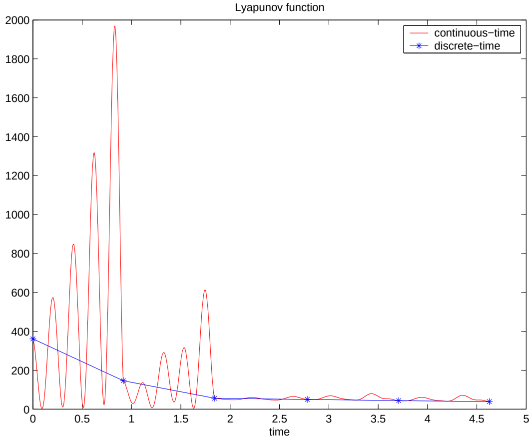
Remarks

- Quadratic stability = sufficient only stability condition
- Lyap. funct. necessary and sufficient for stability (Molchanov and Pyatnitsky ; Hetel et al. TAC 2011) :

$$V(x) = x^T P_{[x]} x, \;\; P_{[x]} = P_{[ax]}, \forall a > 0$$

- Uncertainties in the system matrices $A = A_0 + \Delta A$?
- Linear time-varying systems $A = A(t)$?

Discrete-time and Convex Embedding

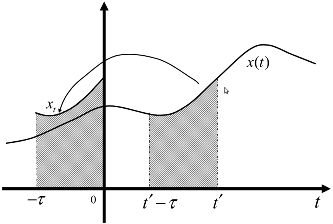


Inter-sampling behaviour ! ! !

Time-delay approaches

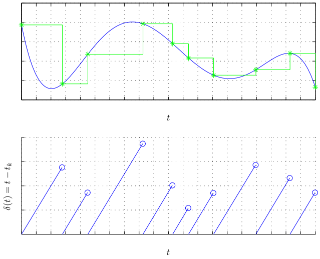
- Basic references
  - Y. Mikheev, V. Sobolev, and E. Fridman. Automation and Remote Control, 1988.
  - A.R. Teel, D. Nesic, and P.V. Kokotovic - IEEE CDC 1998
  - E. Fridman - Automatica, 2010
  - A. Seuret - Automatica, 2012
  - F. Mazenc, M. Malisoff, and T.N. Dinh - Automatica, 2013
  - I. Karafyllis and M. Krstic - IEEE TAC, 2012
- Time-delay system

$$\dot{x} = Ax + A_d x(t - \tau)$$



System state : $x_t(\theta) = x(t + \theta), \ \theta \in [0, -\tau]$

Time-delay approaches



- Continuous-time model

$$\dot{x} = Ax + BKx(s_k), \forall t \in [s_k, s_{k+1}),\ \ h_k = s_{k+1} - s_k \in [0, \overline{h}]$$

- Time delay system : $x(s_k) = x\,(t - (t - s_k))$ is a past value of $x(t)$

$$\dot{x} = Ax + BKx\,(t - \tau(t))$$

- Sawtooth delay

$$\tau = t - s_k,\ \ \dot{\tau}(t) = 1$$

Time-delay approaches

- Time delay system

$$\dot{x} = Ax + BKx\left(t - h(t)\right), \; h \in [0, \overline{h}]$$

- System state :

$$x_t(\theta) = x(t+\theta), \; \theta \in [0, -\overline{h}]$$

- Stability analysis using *Lyapunov-Krasovskii functionals*

$$V(x_t, \dot{x}_t) = x^T(t)Px(t) + \int_{-\underline{h}}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R\dot{x}(s)dsd\theta$$

Time-delay approaches : basic steps

Step 1. *Propose a candidate Lyapunov-Krasovskii functional V*

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Step 2. *Compute the derivative of V.*

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Step 2. *Compute the derivative of V.*

$$\frac{\mathrm{d}}{\mathrm{dt}}V(x_t, \dot{x}_t) \; = \; 2\dot{x}^T(t)Px(t) + \bar{h}\dot{x}^T(t)R\dot{x}(t) - \int_{t-\bar{h}}^{t} \dot{x}^T(s)R\dot{x}(s)\mathrm{d}s$$

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$$\frac{\mathrm{d}}{\mathrm{dt}}V(x_t, \dot{x}_t) = 2\dot{x}^T(t)Px(t) + \bar{h}\dot{x}^T(t)R\dot{x}(t) - \int_{t-\bar{h}}^{t} \dot{x}^T(s)R\dot{x}(s)\mathrm{d}s$$

Step 3. *Over-approximate the integral terms* (here Jensen Inequality)

Time-delay approaches : basic steps

Step 1. *Propose a candidate Lyapunov-Krasovskii functional V*

$$V(x_t, \dot{x}_t) = x^T(t)Px(t) + \int_{-\underline{h}}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R\dot{x}(s)dsd\theta$$

Step 2. *Compute the derivative of V.*

$$\frac{\mathrm{d}}{\mathrm{dt}}V(x_t, \dot{x}_t) \;=\; 2\dot{x}^T(t)Px(t) + \bar{h}\dot{x}^T(t)R\dot{x}(t) - \int_{t-\bar{h}}^{t} \dot{x}^T(s)R\dot{x}(s)\mathrm{d}s$$

Step 3. *Over-approximate the integral terms* (here Jensen Inequality)

$$-\int_{t-\tau}^{t} \dot{x}(s)R\dot{x}(s)\mathrm{d}s \leq -\frac{1}{\tau}\left(x(t) - x(t-\tau)\right)^T R\left(x(t) - x(t-\tau)\right).$$

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$$\Downarrow$$

$$\frac{\mathrm{d}}{\mathrm{dt}} V(x_t, \dot{x}_t) \leq 2\dot{x}^T(t)Px(t) + \bar{h}\dot{x}^T(t)R\dot{x}(t) - \frac{1}{\tau(t)}\left(x(t) - x(t-\tau)\right)^T R\left(x(t) - x(t-\tau)\right)$$

Time-delay approaches : basic steps

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Time-delay approaches : basic steps

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Step 4. *Over-approximate the delay dependent terms.*

Time-delay approaches : basic steps

$$\frac{\mathrm{d}}{\mathrm{dt}} V(x_t, \dot{x}_t) \leq 2\dot{x}^T(t)Px(t) + \bar{h}\dot{x}^T(t)R\dot{x}(t) - \frac{1}{\tau(t)} \left(x(t) - x(t-\tau)\right)^T R \left(x(t) - x(t-\tau)\right)$$

Step 4. *Over-approximate the delay dependent terms.*

$$\frac{\mathrm{d}}{\mathrm{dt}} V(x_t, \dot{x}_t) \leq \begin{bmatrix} x(t) \\ x(t-\tau(t)) \end{bmatrix}^T \Psi(P, R) \begin{bmatrix} x(t) \\ x(t-\tau(t)) \end{bmatrix}$$

where

$$\Psi(P, R) = \begin{bmatrix} A^T P + PA - \frac{1}{\underline{h}} R & PBK + \frac{1}{\underline{h}} R \\ * & -\frac{1}{\underline{h}} R \end{bmatrix} + \bar{h} \begin{bmatrix} A \\ BK \end{bmatrix}^T R \begin{bmatrix} A \\ BK \end{bmatrix}$$

Time-delay approaches : basic steps

$$\frac{\mathrm{d}}{\mathrm{dt}}V(x_t,\dot{x}_t) \leq \; 2\dot{x}^T(t)Px(t)+\bar{h}\dot{x}^T(t)R\dot{x}(t)-\frac{1}{\tau(t)}\left(x(t)-x(t-\tau)\right)^T R\left(x(t)-x(t-\tau)\right)$$

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$$\Psi(P,R)=\begin{bmatrix} A^TP+PA-\frac{1}{\bar{h}}R & PBK+\frac{1}{\bar{h}}R \\ * & -\frac{1}{\bar{h}}R\end{bmatrix}+\bar{h}\begin{bmatrix} A \\ BK\end{bmatrix}^T R\begin{bmatrix} A \\ BK\end{bmatrix}$$

Theorem

Assume that there exists $P \succ 0$ and $R \succ 0$, such that the following linear matrix inequality $\Psi(P,R) \prec 0$ holds . Then, the sampled-data system is asymptotically stable for all arbitrary time-varying sampling sequence $\{s_k\}_{k\in\mathbb{N}}$ with $h_k = t_{s+1} - s_k \leq \bar{h}$.

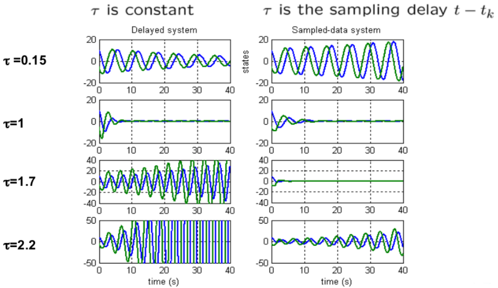
Remarks

Sampled-data Systems vs. Time-delay systems

Are these two classes of systems equivalent?

Consider the following example

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} x(t_k)$$



Further improvement

- Take into account the derivative of the delay $\dot{\tau}(t) = 1$
- Other choices of Lyapunov-Krasovskii functionals

$$V(t, x(t), \dot{x}_t) = x^T(t)Px(t) + (h_k - \tau(t)) \int_{t-\tau(t)}^{t} \dot{x}^T(s)R\dot{x}(s)ds$$

- Less conservative over-approximations of integral terms

$$\int_{t-\tau}^{t} \dot{x}(s)R\dot{x}(s)\mathrm{d}s$$

- Can be adapted to deal with uncertain system matrices

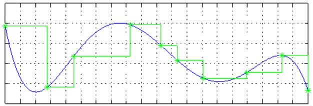
Hybrid system approach

- Basic references
  - G. E. Dullerud and S. Lall, Systems and Control Letters, 1999
  - L. Hu, et. al, TSMC, 2003
  - D. Nesic, A. Teel - IEEE TAC 2004
  - P. Naghshtabrizi, J.-P. Hespanha, and A.-R. Teel. Systems and Control Letters, 2008
- Impulsive model

$$\begin{cases} \dot{\xi}(t) = \bar{A}\xi(t), & t \neq s_k, \forall k \in \mathbb{N}, , \\ \xi(t) = J\xi(t^-) & t = s_k, \forall k \in \mathbb{N}. \end{cases}$$

$$\dot{v} = -g, \ t \neq s_k \quad v(s_k) = -v(s_k^-)$$

Hybrid system approach



- Sampled-data system

$$\dot{x} = Ax + BKx(s_k)$$

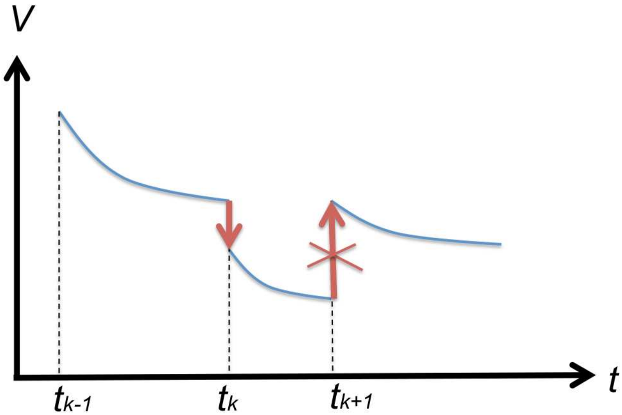
- Augmented state

$$\xi(t) = [x^T(t), z^T(t)]^T, \; z(t) = x(s_k)$$

- Impulsive model

$$\begin{cases} \dot{\xi}(t) = \begin{bmatrix} A & BK \\ 0 & 0 \end{bmatrix} \xi(t), & t \neq s_k, \forall k \in \mathbb{N}, , \\ \xi(s_k) = \begin{bmatrix} x(s_k^-) \\ x(s_k^-) \end{bmatrix}, & t = s_k, \forall k \in \mathbb{N}. \end{cases}$$

Stability of impulsive systems



$$\dot{V}(\xi(t)) < 0, \quad \forall t \neq t_k, \quad \xi \neq 0$$
$$V(\xi(t_k)) \leq V(\xi(t)), \quad t = t_k^-$$

Hybrid system approach

- Impulsive model

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- Stability conditions using time (clock) dependent Lyapunov function with discontinuities at the impulse times

$$V(\tau,\xi) = \xi^T P(\tau)\xi,\ \ P(\tau) = P^T(\tau) \succ 0,\ \ \tau(t) = t - s_k \in [0, \overline{h}]$$

- Stability conditions

$$\dot{V}(\tau,\xi) < 0, \forall t \neq s_k, \xi \neq 0$$

$$V(0,\xi) \leq V(\tau(t),\xi), t = s_k^-$$

Hybrid system approach

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- Parametric LMI conditions

Hybrid system approach

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$$\bar{A}^T P(\tau) + P(\tau)\bar{A} + \dot{P}(\tau) \prec 0, \;\; \tau \in [0, \overline{h}],$$

Hybrid system approach

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(Sun & Khargonekar ; Toivonen)

Hybrid system approach

- Parametric LMI conditions

$$\bar{A}^T P(\tau) + P(\tau)\bar{A} + \dot{P}(\tau) \prec 0,\ \tau \in [0, \overline{h}],\quad J^T P(0) J - P(\tau) \prec 0,\ \tau \in [\underline{h}, \overline{h}].$$

- Examples :
  - polynomial (linear) function

$$P(\tau) = P_1 + (P_2 - P_1)\frac{\tau}{\overline{h}}$$

  - exponential

$$P(\tau) = e^{-\gamma\tau} P_0$$

  - inspired by Lyapunov-Krasovskii functionals

$$P(\tau) = \int_{-\tau}^{0} (\overline{h} + s)\bar{A}^T e^{\bar{A}^T s} R e^{\bar{A}s} \bar{A} ds$$

where

$$\tau(t) = t - s_k \in [0, \overline{h}]$$

Hybrid system approach

- Parametric LMI conditions

$$\bar{A}^T P(\tau) + P(\tau)\bar{A} + \dot{P}(\tau) \prec 0,\ \tau \in [0, \overline{h}],\quad J^T P(0) J - P(\tau) \prec 0,\ \tau \in [\underline{h}, \overline{h}].$$

- Example :
  - For linear function

$$P(\tau) = P_1 + (P_2 - P_1)\frac{\tau}{\overline{h}}$$

  - LMI condition

Hybrid system approach

- Parametric LMI conditions

$$\bar{A}^T P(\tau) + P(\tau)\bar{A} + \dot{P}(\tau) \prec 0,\ \tau \in [0, \overline{h}], \quad J^T P(0) J - P(\tau) \prec 0,\ \tau \in [\underline{h}, \overline{h}].$$

- Example :
  - For linear function

$$P(\tau) = P_1 + (P_2 - P_1)\,\frac{\tau}{\overline{h}}$$

  - LMI condition

$$\bar{A}^T P_1 + P_1 \bar{A} + \frac{P_2 - P_1}{\overline{h}} \prec 0,$$
$$\bar{A}^T P_2 + P_2 \bar{A} + \frac{P_2 - P_1}{\overline{h}} \prec 0,$$

Hybrid system approach

- Parametric LMI conditions

$$\bar{A}^T P(\tau) + P(\tau)\bar{A} + \dot{P}(\tau) \prec 0,\ \tau \in [0, \overline{h}],\quad J^T P(0) J - P(\tau) \prec 0,\ \tau \in [\underline{h}, \overline{h}].$$

- Example :
  - For linear function

$$P(\tau) = P_1 + (P_2 - P_1)\,\frac{\tau}{\overline{h}}$$

  - LMI condition

$$\begin{aligned}
&\bar{A}^T P_1 + P_1 \bar{A} + \frac{P_2 - P_1}{\overline{h}} \prec 0,\\
&\bar{A}^T P_2 + P_2 \bar{A} + \frac{P_2 - P_1}{\overline{h}} \prec 0,\\
&J^T P_1 J \prec P_2,\\
&J^T P_1 J \prec P_1 + (P_2 - P_1)\,\underline{h}/\overline{h}.
\end{aligned}$$

Hybrid system approach : relations with discrete-time approach

(a) $\exists\ V_d(\xi_k) = \xi_k^T L \xi_k$ $(L \succ 0)$ such that :

$$V_d(\xi_{k+1}) < V_d(\xi_k) : \quad \left(e^{\bar{A}\tau} J\right)^T L \left(e^{\bar{A}\tau} J\right) - L \prec 0,\ \forall\, \tau \in [\underline{h}, \overline{h}],$$

$$\equiv$$

(b) $\exists\ V(\tau, \xi) = \xi^T P(\tau) \xi$ $(P(\tau) = P^T(\tau) \succ 0)$ such that :

$$\dot{V}(\tau, \xi) \leq 0 : \ \bar{A}^T P(\tau) + P(\tau)\bar{A} + \dot{P}(\tau) \preceq 0,\ \forall\, \tau \in [0, \overline{h}]$$

$$V(0, \xi) < V(\tau, \xi) : \ J^T P(0) J - P(\tau) + \epsilon I \preceq 0,\ \forall\, \tau \in [\underline{h}, \overline{h}].$$

(Briat, Automatica 2013)

see also the looped functionals in (Seuret, Automatica 2012)

Why do we look for τ dependent Lyapunov functions ?

More general hybrid models (Goebel, Sanfelice, Teel)

$$\dot{z} = F_z(z), \quad z \in C,$$
$$z^+ = J_z(z), \quad z \in D,$$

(state triggered jumps)

More general hybrid models (Goebel, Sanfelice, Teel)

Sampled-data system

$$\dot{x} = Ax + BKx(s_k), \forall t \in [s_k, s_{k+1}),\ \ h_k = s_{k+1} - s_k \in [0, \overline{h}]$$

Hybrid model

$$\left\{ \begin{array}{l} \left. \begin{array}{rcl} \dot{x} & = & Ax + BK\hat{x} \\ \dot{\hat{x}} & = & 0 \\ \dot{\tau} & = & 1 \end{array} \right\} \qquad \tau \in [0, \overline{h}], \\ \\ \left. \begin{array}{rcl} x^+ & = & x \\ \hat{x}^+ & = & x \\ \tau^+ & = & 0 \end{array} \right\} \qquad \tau \in [\underline{h}, \overline{h}]. \end{array} \right.$$

(includes the dynamic of the clock τ)

More general hybrid models (Goebel, Sanfelice, Teel)

Sampled-data system

$$\dot{x} = Ax + BKx(s_k), \forall t \in [s_k, s_{k+1}),\ h_k = s_{k+1} - s_k \in [0, \overline{h}]$$

Hybrid model with $z = (x^T \quad \hat{x}^T \quad \tau)^T = (\xi^T \quad \tau)^T$

$$F_z(z) = \begin{pmatrix} Ax + BK\hat{x} \\ 0 \\ 1 \end{pmatrix},\ J_z(z) = \begin{pmatrix} x \\ x \\ 0 \end{pmatrix}$$

$$C = \left\{ z \in \mathbb{R}^{n_z} : \tau \in [0, \overline{h}] \right\}$$

and

$$D = \left\{ z \in \mathbb{R}^{n_z} : \tau \in [\underline{h}, \overline{h}] \right\}.$$

Stability of the set $\mathcal{A} = \{z^T = (x^T, \hat{x}^T, \tau) \in \mathbb{R}^{n_z} :\ (x, \hat{x}) = (0, 0)\}$?

Necessary and sufficient stability conditions (Cai,Goebel and Teel) :

A set $\mathcal{A}$ is asymptotically stable

$$\dot{z} = F_z(z), \quad z \in C,$$
$$z^+ = J_z(z), \quad z \in D,$$

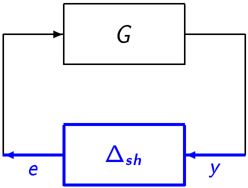
If and only if there exists a $\mathcal{C}^\infty$ function $\tilde{V}(z)$ such that

$$\frac{\partial \tilde{V}}{\partial z} F_z(z) < 0 \text{ for all } z \in C \setminus \mathcal{A},$$
$$\tilde{V}\left(J_z(z)\right) - \tilde{V}(z) < 0 \text{ for all } z \in D \setminus \mathcal{A}.$$

i.e. a $\mathcal{C}^\infty$ function $V(\tau, \xi)$ for the impulsive model.

Input/Output stability approaches

- Basic references
  - L. Mirkin - IEEE TAC 2007
  - H. Fujioka - Automatica 2009
  - Y.C. Kao - ACC 2014
  - H. Omran et al. - Automatica 2014
  - H. Omran et al. - ECC 2013



Input/Output stability approaches

- LTI sampled-data system
$$\dot{x} = Ax + BKx(s_k)$$
- Sampling error : $e(t) = x(s_k) - x(t)$

$$\dot{x}(t) = [\underbrace{A + BK}_{A_{cl}}]x(t) + \underbrace{BK}_{B_{cl}}\underbrace{(x(s_k) - x(t))}_{e(t)}$$

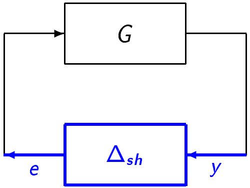
The system can be represented by the interconnection of :

$$G := \begin{cases} \dot{x}(t) = & A_{cl}x(t) + B_{cl}e(t) \\ y(t) = & \dot{x}(t) \end{cases}$$

with the operator $\Delta_{sh} : y \to e$ defined by :

$$e(t) = -\int_{s_k}^{t} y(s)ds := (\Delta_{sh}y)(t), \quad \forall t \in [s_k, s_{k+1})$$

Input/Output stability approaches



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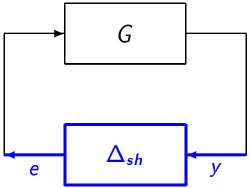
- Stability conditions using the properties of Δ_{sh}
- e.g. finite L_2 gain (Mirkin, 2007)

$$\|\Delta_{sh}\|_{2,2} = \sup_{y \neq 0} \frac{\|e\|_{L_2}}{\|y\|_{L_2}} \leq \delta_0 := \frac{2}{\pi}\overline{h}$$

Input/Output stability approaches



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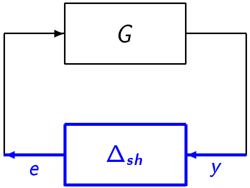
- Small Gain Theorem : the interconnection is $\mathcal{L}_2$ stable if

Input/Output stability approaches



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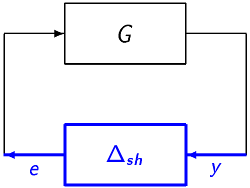
- Small Gain Theorem : the interconnection is $\mathcal{L}_2$ stable if

$$\|\mathbf{G}\|_{2,2}\|\Delta_{sh}\|_{2,2} < 1$$

Input/Output stability approaches



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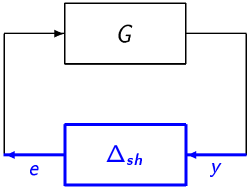
- Frequency domain condition

Input/Output stability approaches



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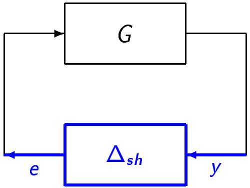
$$\|\mathbf{G}\|_{2,2}\|\Delta_{sh}\|_{2,2} < 1$$

- Frequency domain condition

$$\|\mathbf{G}\|_{2,2} = \|\mathbf{G}\|_{\infty} := \sup_{\omega\in\mathbb{R}} \bar{\sigma}\left(\hat{\mathbf{G}}(j\omega)\right) < \frac{\pi}{2\overline{h}},$$

Input/Output stability approaches



$$G := \begin{cases} \dot{x}(t) = & A_{cl}x(t) + B_{cl}e(t) \\ y(t) = & \dot{x}(t) \end{cases}$$

$$e(t) = -\int_{s_k}^{t} y(s)ds := (\Delta_{sh}y)(t).$$

- Stability conditions using the properties of Δ_{sh}
- e.g. finite L_2 gain (Mirkin, 2007)

$$\|\Delta_{sh}\|_{2,2} = \sup_{y\neq 0} \frac{\|e\|_{L_2}}{\|y\|_{L_2}} \leq \delta_0 := \frac{2}{\pi}\overline{h}$$

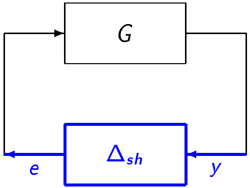
- Small Gain Theorem : the interconnection is $\mathcal{L}_2$ stable if

$$\|\mathbf{G}\|_{2,2}\|\Delta_{sh}\|_{2,2} < 1$$

- Frequency domain condition

$$\|\mathbf{G}\|_{2,2} = \|\mathbf{G}\|_{\infty} := \sup_{\omega\in\mathbb{R}} \bar{\sigma}\left(\hat{\mathbf{G}}(j\omega)\right) < \frac{\pi}{2\overline{h}}, \quad \hat{\mathbf{G}}(s) = s(sI - A_{cl})^{-1}B_{cl}.$$

Input/Output stability approaches



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$$e(t) = -\int_{s_k}^{t} y(s)ds := (\Delta_{sh}y)(t).$$

- Scaled Small Gain condition

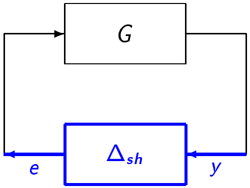
$$\exists\, M \in \mathbb{R}^{n\times n},\ \ M \succ 0 \quad \text{such that } \|M\hat{\mathbf{G}}(s)M^{-1}\|_\infty < \frac{\pi}{2\bar{h}}.$$

- LMI formulation

$$\begin{bmatrix} XA_{cl} + A_{cl}^T X & \frac{2}{\pi}\bar{h}XBK & A_{cl}^T Y \\ * & -Y & \frac{2}{\pi}\bar{h}K^T B^T Y \\ * & * & -Y \end{bmatrix} \prec 0$$

to be solved for $X, Y \succ 0$ (obtained with $Y = M^2$).

Input/Output stability approaches



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$$e(t) = -\int_{s_k}^{t} y(s)ds := (\Delta_{sh}y)(t).$$

Integral Quadratic Constraints (IQC)

$$\int_0^\infty \begin{bmatrix} y(\theta) \\ e(\theta) \end{bmatrix}^T \Pi \begin{bmatrix} y(\theta) \\ e(\theta) \end{bmatrix} d\theta \geq 0$$

for all $y \in \mathcal{L}_2^n[0,\infty)$ and $e = \Delta_{sh}y$.

(Megretski & Rantzer, IEEE TAC 1997)

Theorem (IQC Theorem)

Suppose that $A_{cl} = A + BK$ is Hutwitz and assume that

- *there exists a matrix*
$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^T & \Pi_{22} \end{bmatrix}$$
with $\Pi_{11}, \Pi_{12}, \Pi_{22} \in \mathbb{R}^{n \times n}$, $\Pi_{11} \succeq 0$, $\Pi_{22} \preceq 0$, such that the operator Δ_{sh} satisfies the IQC defined by Π;
- *there exists $\epsilon > 0$ such that*
$$\begin{bmatrix} \hat{G}(j\omega) \\ I \end{bmatrix}^* \Pi \begin{bmatrix} \hat{G}(j\omega) \\ I \end{bmatrix} \leq -\epsilon I, \ \forall \ \omega \in \mathbb{R}.$$

Then the interconnection is $\mathcal{L}_2$ stable.

Input/Output stability approaches

$$G := \begin{cases} \dot{x}(t) = & A_{cl}x(t) + B_{cl}e(t) \\ y(t) = & \dot{x}(t) = C_{cl}x + D_{cl}e(t) \end{cases}$$

Equivalent LMI condition (**Kalman-Yakubovich-Popov Lemma**) :

$$\begin{bmatrix} A_{cl}^T P + PA_{cl} & PB_{cl} \\ B_{cl}^T P & 0 \end{bmatrix} + \begin{bmatrix} C_{cl} & D_{cl} \\ 0 & I \end{bmatrix}^T \Pi \begin{bmatrix} C_{cl} & D_{cl} \\ 0 & I \end{bmatrix} < 0$$

to be solved for $P \succ 0$.

Example of IQCs for Δ_{sh}

- Finite L_2 gain (Mirkin, 2007) :

$$\|\Delta_{sh}\|_{2,2} = \sup_{y \neq 0} \frac{\|e\|_{L_2}}{\|y\|_{L_2}} \leq \delta_0 := \frac{2}{\pi}\bar{h}$$

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- Time domain formulation :

$$\int_0^{+\infty} \|(\Delta_{sh} y)(\theta)\|^2 d\theta \leq \delta_0^2 \int_0^{+\infty} \|y(\theta)\|^2 d\theta,$$

for all $y \in \mathcal{L}_2^n[0, \infty)$

- IQC :

$$\int_0^{\infty} \begin{bmatrix} y(\theta) \\ e(\theta) \end{bmatrix}^T \Pi \begin{bmatrix} y(\theta) \\ e(\theta) \end{bmatrix} d\theta \geq 0$$

for all $y \in \mathcal{L}_2^n[0, \infty)$ and $e = \Delta_{sh} y$ with

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Example of IQCs for Δ_{sh}

- (Anti-)Passivity property (Fujioka, Automatica, 2009)

$$\int_0^{+\infty} y^T(\theta)(\Delta_{sh}y)(\theta)d\theta \leq 0,$$

  for all $y \in \mathcal{L}_2^n[0, \infty)$.
- IQC :

$$\int_0^{\infty} \begin{bmatrix} y(\theta) \\ e(\theta) \end{bmatrix}^T \Pi \begin{bmatrix} y(\theta) \\ e(\theta) \end{bmatrix} d\theta \geq 0$$

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Example of IQCs for Δ_{sh}

- IQCs can be combined

$$\int_0^{+\infty} y^T(\theta)(\Delta_{sh}y)(\theta)d\theta \leq 0, \quad \int_0^{+\infty} \|(\Delta_{sh}y)(\theta)\|^2 d\theta \leq \delta_0^2 \int_0^{+\infty} \|y(\theta)\|^2 d\theta,$$

- Resulting IQC :

$$\int_0^{\infty} \begin{bmatrix} y(\theta) \\ e(\theta) \end{bmatrix}^T \Pi \begin{bmatrix} y(\theta) \\ e(\theta) \end{bmatrix} d\theta \geq 0$$

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Remarks :

- Many other performance specifications and nonlinearities can be described as IQCs (saturations, sector-bounded nonlinearities, etc.)

  (Megretski and Rantzer, IEEE TAC 1997)

- For LTV, polytopic and nonlinear systems : re-interpretation in terms of supply functions

  (Omran et al, Automatica 2014 ; ECC 2013)

Dissipativity-based approach

Closed-loop system

$$\dot{x}(t) = f\left(x(t)\right) + g\left(x(t)\right) K\left(x(s_k)\right)$$

$$\dot{x}(t) = \underbrace{f\left(x(t)\right) + g\left(x(t)\right) K\left(x(t)\right)}_{f_n(x(t))} + \underbrace{g\left(x(t)\right)}_{g_n(x(t))} \underbrace{\left(K\left(x(t_k)\right) - K\left(x(t)\right)\right)}_{e(t)}$$

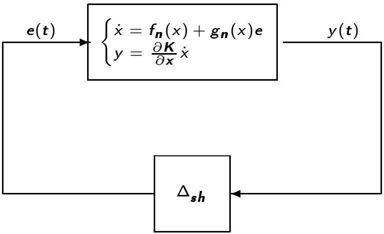
The system can be represented by the interconnection of :

$$\begin{cases} \dot{x}(t) = f_n\left(x(t)\right) + g_n\left(x(t)\right) e(t) \\ y(t) = \frac{\partial K}{\partial x} \dot{x}(t) \end{cases}$$

with the operator $\Delta_{sh} : y \to e$ defined by :

$$e(t) = (\Delta_{sh}\, y)(t) = -\int_{s_k}^{t} y(\tau) d\tau$$

Dissipativity-based approach



For $(\Delta_{sh} y)(t) := -\int_{s_k}^{t} y(s)ds$, derive "supply functions" $\mathbf{S}(y, \Delta_{sh} y)$ such that

$$\int_{s_k}^{t} \mathbf{S}(y, \Delta_{sh} y)ds \leq 0, \forall t \in [s_k, s_{k+1}).$$

Exponential stability condition : $\exists$ "storage function" $V(x)$, $\alpha > 0$ such that

$$\begin{aligned} \dot{V}(x(t)) + \alpha V(x(t)) &\leq \mathbf{S}(y(t), e(t)) \\ \dot{V}(x(t)) + \alpha V(x(t)) &\leq \mathbf{S}(y(t), e(t)) exp(-\alpha \overline{h}) \end{aligned}$$

Properties of the operator

Boundedness property

For all $y \in L_2[s_k, s_{k+1})$ and $0 < X^* = X \in \mathbb{R}^{n\times n}$:

$$\int_{s_k}^{t} (\Delta_{sh} y)^* X (\Delta_{sh} y)\, ds - \delta_0^2 \int_{s_k}^{t} y^* X y\, ds \le 0, \quad \forall t \in [s_k, s_{k+1})$$

(Anti-)Passivity property

For all $y \in L_2[s_k, s_{k+1})$ and $0 \le Y^* = Y \in \mathbb{R}^{n\times n}$:

$$\int_{s_k}^{t} (\Delta_{sh} y)^* Y y\, ds + \int_{s_k}^{t} y^* Y (\Delta_{sh} y)\, ds \le 0, \quad \forall t \in [s_k, s_{k+1})$$

$$\Rightarrow \quad \int_{s_k}^{t} \underbrace{\begin{bmatrix} y \\ e \end{bmatrix}^T \begin{bmatrix} -\delta_0^2 X & Y \\ Y & X \end{bmatrix} \begin{bmatrix} y \\ e \end{bmatrix}}_{\text{supply rate } \mathcal{S}(y,e)} ds \le 0, \quad \forall t \in [s_k, s_{k+1})$$

Robust stability analysis with respect to time-varying sampling

Strong inter-action between approaches

- Time-delay approach
  $\Rightarrow$ less conservative L_2 bound for I/O approach
  $\Rightarrow$ continuous-time version of convex embedding approach
- Hybrid (Impulsive) System modelling approach
  $\Rightarrow$ discontinuous LKF
- Discrete-time and Convex Embedding
  $\Rightarrow$ taking into account the sawtooth form of the delay
- I/O approach
  $\Rightarrow$ new LKF based on Wirtingers inequalities

Dynamic control of sampling

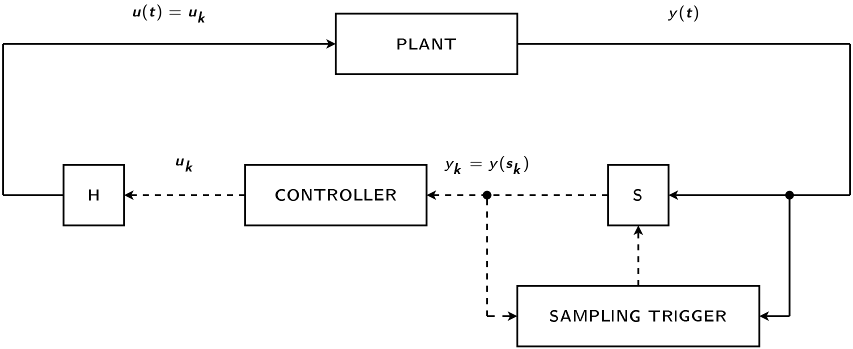
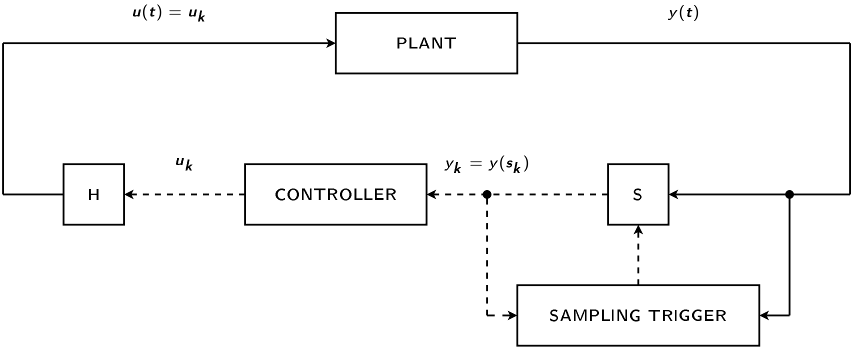


Figure: Generic configuration

- Event-Triggered Control
- Periodic Event-Triggered Control
- Self-Triggered Control (emulation of event-triggered control)
- State-Dependent Sampling (based on partition of the state space)

Dynamic control of sampling



Event-Triggered Control (continuous monitoring, control computation and actuation at event)

- E. Hendricks et. al - ACC, 1994
- K. Astrom and B. Bernhardsson - IFAC World Conf., 1999
- K.-E. Arzen - IFAC World Conf., 1999
- Tabuada - IEEE TAC 2007
- Lunze, Lehmann - Automatica 2010

Example of Event-Triggered Control

- Basic triggering condition

$$\|x(t) - x(t_k)\|^2 \geq \sigma^2 \|x(t)\|^2$$

- Linearized model of inverted pendulum on a cart

$$\dot{x} = Ax + Bu$$

where

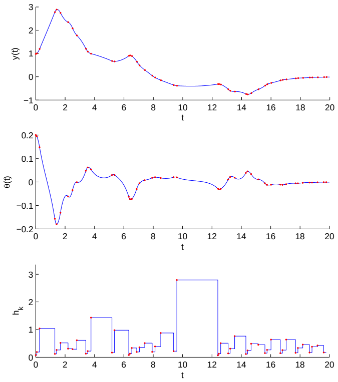
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -mg/M & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & g/l & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 1/M \\ 0 \\ -1/(Ml) \end{bmatrix}.$$

- M - the cart mass, m - the mass of the pendulum, l - the length of the pendulum arm, g - gravitational acceleration.
- system state is the vector $x = [y \quad \dot{y} \quad \theta \quad \dot{\theta}]^T$; y - the cart position, θ - the pendulum angle with respect to the vertical.
- $T_{Schur} \approx 0.35s$

Example of Event-Triggered Control

- Basic triggering condition

$$\|x(t) - x(t_k)\|^2 \geq \sigma^2 \|x(t)\|^2$$

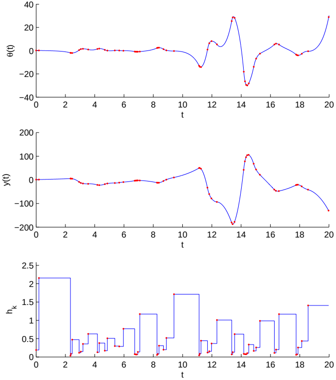


- Simulation with $\sigma = 0.5$; Average sampling $= 0.41s > 0.35s$ (periodic case).

Example of Event-Triggered Control

- Basic triggering condition

$$\|x(t) - x(t_k)\|^2 \geq \sigma^2 \|x(t)\|^2$$



- Simulation with $\sigma = 0.8$!!!.

Systematic ETC design methods (Tabuada, TAC, 2007)

- Linear System

$$\dot{x} = Ax + BKx(s_k), \ \forall t \in [s_k, s_{k+1})$$

- Nominal control loop + perturbation

$$\dot{x}(t) = [\underbrace{A + BK}_{A_{cl}}]x(t) + \underbrace{BK}_{B_{cl}}\underbrace{(x(s_k) - x(t))}_{e(t)}$$

- ISS - Lyapunov function $V(x) = x^T Px$

$$\dot{V}(x) \leq -\alpha\|x\|^2 + \gamma\|e\|^2, \quad \forall \ (x, e)$$

- Consider $\rho \in (0, 1)$

$$\begin{aligned} \dot{V}(x(t)) &\leq -\alpha\|x(t)\|^2 + \gamma\|e(t)\|^2, \\ &\leq -(1-\rho)\alpha\|x(t)\|^2 - \rho\alpha\|x(t)\|^2 + \gamma\|e(t)\|^2, \end{aligned}$$

- Assume now that we enforce $-\rho\alpha\|x(t)\|^2 + \gamma\|e(t)\|^2 \leq 0 \Rightarrow \dot{V}(x(t)) < 0$
- Trigger condition : $\|e(t)\|^2 = \underbrace{\rho\alpha\gamma^{-1}}_{\sigma^2}\|x(t)\|^2$

Remarks

- For LTI systems the exists a minimum inter-execution times
- The average sampling interval depends on the initial condition (not always greater than T_{Schur})
- Other ETC schemes
  - $\|x(t) - x(t_k)\| \geq \delta$
  - $\|x(t) - x(t_k)\| \geq \delta + \sigma \|x(t)\|$
  - ...

Dynamic control of sampling

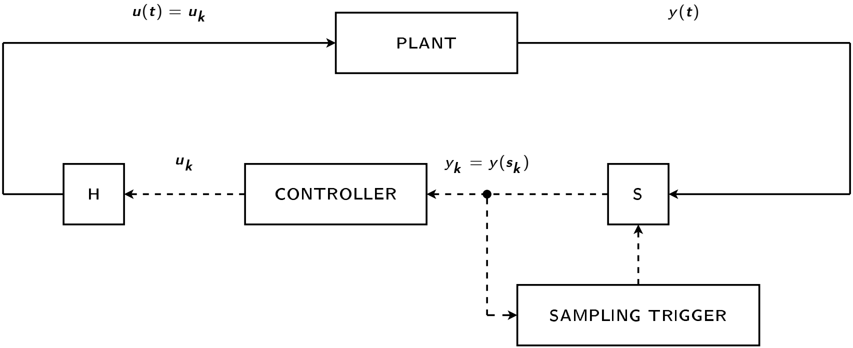


Figure: Generic configuration

Self-Triggered Control (computation of the next time instant sensor / control data is needed)

- Wang, Lemmon - IEEE TAC 2010
- Anta, Tabuada - IEEE TAC 2010
- Mazo-Jr, Anta, Tabuada - Automatica 2010
- Fiter et al. - Nolcos 2013

Fundamental motivations

$$s_{k+1} = s_k + \tau(x_k), \ \tau : \mathbb{R}^n \to \mathbb{R}_+$$

- What is the "best" sampling pattern ?

- Which is the "best" trigger function / sampling map $\tau(x)$?

Self-triggering based on convex embedding (PhD C. Fiter)

Goal : use tools from the *robust control* framework to optimize the design of *sampling maps* !

(Fiter et al. Automatica,2012)

Problem formulation

Consider the LTI system

$$\dot{x}(t) = Ax(t) + BKx(s_k),\ \forall t \in [s_k, s_{k+1}),$$

with sampling intervals defined by

$$s_{k+1} - s_k = \tau(x(s_k)),\ \forall k \in \mathbb{N},$$

with a sampling map $\tau : \mathbb{R}^n \to \mathbb{R}_+$ to be designed.

Problem : Given an LF $V(x) = x^T Px$, design a sampling map $\tau(x)$.

Idea of the approach

$$\dot{V}(x(t)) + 2\beta V(x(t)) \leq 0$$

For

$$V(x) = x^T P x \text{ and } \dot{x}(t) = Ax(t) + BKx(s_k)$$

Stability condition :

$$\underbrace{\begin{bmatrix} x(t) \\ x(s_k) \end{bmatrix}^T \begin{bmatrix} A^T P + PA + 2\beta P & PBK \\ * & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x(s_k) \end{bmatrix}}_{\mathcal{C}(x(t), x(s_k))} \leq 0,$$

Event-trigger :

$$\mathcal{C}\left(x(t), x(s_k)\right) = 0$$

Idea of the approach

Use the transition matrix Λ :

$$x(t) = \Lambda(\sigma)x(s_k) = \left(I + \int_0^{\sigma} e^{sA} ds(A + BK) \right) x(s_k), \ \sigma = t - s_k$$

Stability condition :

$$x^T(s_k) \underbrace{\begin{bmatrix} \Lambda(\sigma) \\ I \end{bmatrix}^T \begin{bmatrix} A^T P + PA + 2\beta P & PBK \\ * & 0 \end{bmatrix} \begin{bmatrix} \Lambda(\sigma) \\ I \end{bmatrix}}_{\Pi(\sigma)} x(s_k) \leq 0,$$

For self-triggering control the sampling map $\tau(x)$ must satisfy

$$x^T(s_k)\Pi(\sigma)x(s_k) \leq 0, \ \forall \ \sigma \in [0, \tau(x(s_k))]$$

Remarks on the sampling map

$$x^T \Pi(\sigma) x \leq 0$$

for all $x \in \mathbb{R}^n$ and $\sigma \in [0, \tau(x)]$ with

$$\Pi(\sigma) = \begin{bmatrix} \Lambda(\sigma) \\ I \end{bmatrix}^T \begin{bmatrix} A^T P + PA + 2\beta P & PBK \\ * & 0 \end{bmatrix} \begin{bmatrix} \Lambda(\sigma) \\ I \end{bmatrix},$$

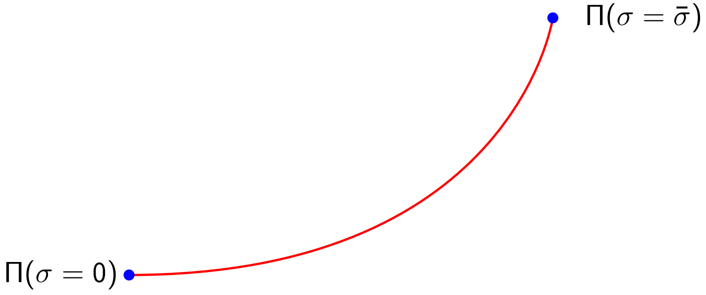
- homogeneity property : $\tau(x) = \tau(\alpha x), \ \alpha > 0$
- $\tau(x)$ - characterized by intersection of conic regions
- state-space region where $\tau(x_k) = \bar{\sigma}$ ensures the decay of $V(x)$

$$\mathcal{R}(\bar{\sigma}) = \left\{ x \in \mathbb{R}^n : x^T \Pi(\sigma) x \leq 0, \forall \sigma \in [0, \bar{\sigma}] \right\}$$

- infinite number of conic regions

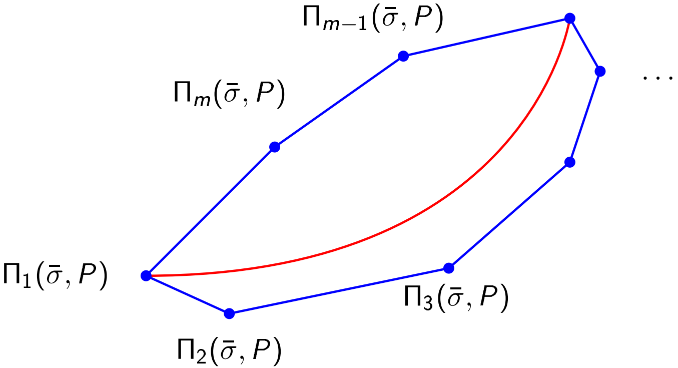
Convex embedding according to time

Embed the matrix function $\Pi(\sigma),\ \sigma \in [0, \bar{\sigma}]$ in a convex polytope



Convex embedding according to time

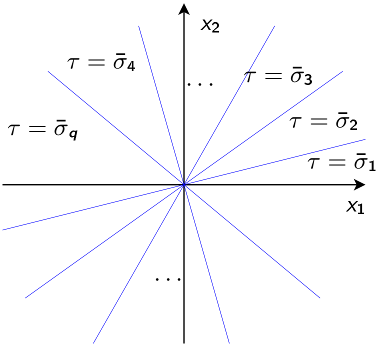
Embed the matrix function $\Pi(\sigma),\ \sigma \in [0, \bar{\sigma}]$ in a convex polytope



Numerical construction based on Taylor series approximation

(Hetel, Daafouz, Iung, IEEE TAC, 2006)

Solution (design of sampling map)



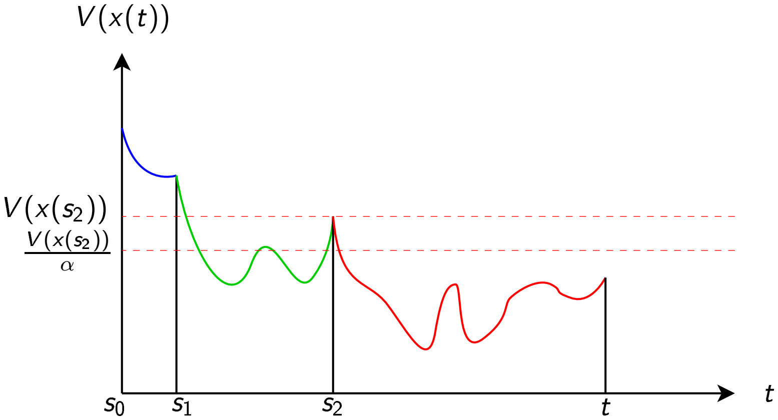
- For given P matrix, approximation of the state-space region where $\tau(x_k) = \bar{\sigma}$ ensures the decay of $V(x)$

$$\tilde{\mathcal{R}}(\bar{\sigma}) = \left\{ x \in \mathbb{R}^n : x^T \Pi_i(P, \bar{\sigma}) x \leq 0, i = 1, \dots, m \right\}$$

- Intersection of **finite** number of conic regions.

Extension based on Lyapunov-Razumikhin functions (submitted to Automatica)

$$\dot{V}(x(t)) + 2\beta V(x(t)) \leq 0 \text{ whenever } V(x(t)) \geq \frac{V(x(s_k))}{\alpha},$$



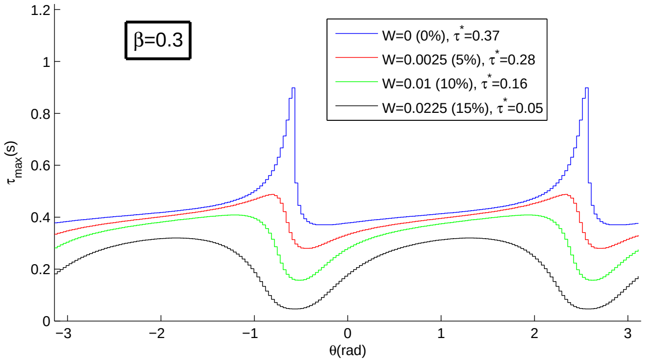
Examples

Consider the system from Tabuada, IEEE TAC 2007 :

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x(t) - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 4 \end{bmatrix} x(t_k) + w(t).$$

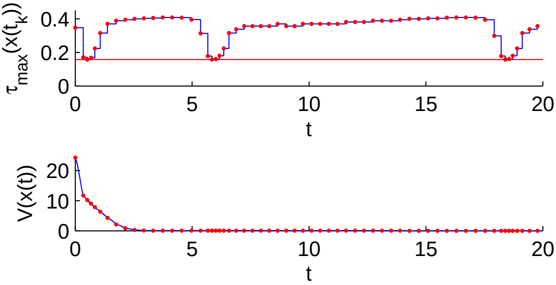
Examples

State-angle dependent sampling map $\tau_{\max}$ for $\beta = 0.3$, and different values of W



Examples

Simulations with $\beta = 0.3$ and $W = 0.01$ $(\|w(t)\|_2 \leq 10\%\|x(s_k)\|_2)$



Examples

Batch Reactor system from Mazo et. al ECC 2009

$$\dot{x}(t) = \begin{bmatrix} 1.38 & -0.20 & 6.71 & -5.67 \\ -0.58 & -4.29 & 0 & 0.67 \\ 1.06 & 4.27 & -6.65 & 5.89 \\ 0.04 & 4.27 & 1.34 & -2.10 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 5.67 & 0 \\ 1.13 & -3.14 \\ 1.13 & 0 \end{bmatrix} u(t),$$

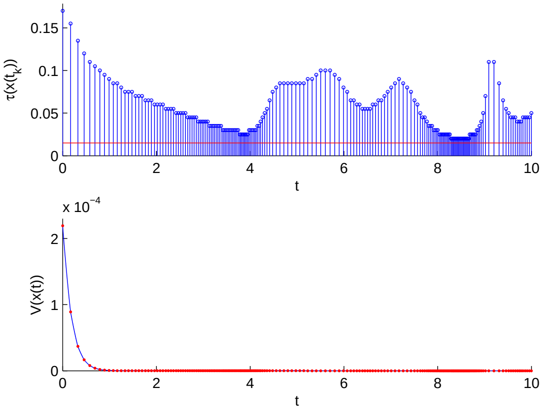
$$u(t) = -\begin{bmatrix} -0.1006 & 0.2469 & 0.0952 & 0.2447 \\ -1.4099 & 0.1966 & -0.0139 & -0.0823 \end{bmatrix} x(t_k).$$

Lower bound of inter-execution time :

- Mazo et. al ECC 2009 : **0.02s**
- Convex embeddings (5th order Taylor approx., 36 subdivisions) : **0.18s**

Examples

Simulations for the Batch Reactor system (with 10% perturbation)



$$\tau^* = 0.015$$

Conclusion

- Present recent approaches for robust stability analylis
- Introduction to event- and self-triggering control

More information : see the tutorial paper in the proceedings of ECC 2014