Tutorial Ecole MACS:
(Discrete-time) Switched systems

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Ensemble, construisons l'avenir

Outline of the tutorial

Aims of the tutorial

Goals :

- Have an overview about switched systems.
- Consider discrete-time linear autonomous switched systems.
- Understand the main properties of switched systems.
- Be familiar with stability and stabilization of switched systems.

Outline of the tutorial

What are switched systems ?

- Definition and link with hybrid systems
- Illustrations and motivation

About stability

Stability results for discrete-time switched systems (solving P1)

Stability results with constrained switching law (solving P2)

Stabilization results for discrete-time switched systems (solving P3)

Definition of switched systems

Definition :

Switched systems are the association of a finite set of dynamical systems (modes) and a switching law $\sigma(\cdot)$ that indicates at each time which mode is active.

Let $\mathcal{I} = \{1; \cdots ; N\}$, where $N \in \mathbb{N}$ is the number of modes.

Continuous-time

$$\dot{x}(t) = f_{\sigma(t)}(x(t), u(t), t), \quad \forall t \in \mathbb{R}^+,$$

where

- $x(t) \in \mathbb{R}^n$ is the state,
- $u(t)$ the input.
- σ the switching law

$$\sigma : \mathbb{R} \rightarrow \mathcal{I}.$$

Discrete-time

$$x_{k+1} = f_{\sigma(k)}(x_k, u_k, k), \quad \forall k \in \mathbb{N}, \qquad (1)$$

where

- $x_k \in \mathbb{R}^n$ is the state,
- u_k the input.
- σ the switching law

$$\sigma : \mathbb{N} \rightarrow \mathcal{I}.$$

Assumptions for the switching law

Several assumptions :

- $\sigma(\cdot)$ is arbitrary.
  $\sigma(\cdot)$ is seen as a perturbation. The results should be true for all the switching laws. The generation of the signal $k \mapsto \sigma(k)$ could be very difficult to take into account.
- $\sigma(\cdot)$ is state dependent.
  Here we have $\sigma(k) = g(x_k)$.
- $\sigma(\cdot)$ is time dependent or has time constraints.
  This is for instance the case when $\sigma(\cdot)$ is periodic, or has a time constraint such a dwell time.
- $\sigma(\cdot)$ is a control input.
  The issue here is to design the switching law $\sigma(\cdot)$.

Particular case of hybrid systems

Hybrid system :
Heterogenous interaction between continuous and discrete dynamics :

$$\left\{ \begin{array}{ll} \text{If } z(t) \in \mathcal{C}, & \dot{z}(t) \in F(z(t), u(t)), \text{ (flow map)} \\ \text{If } z(t) \in \mathcal{D}, & z(t^+) \in G(z(t), u(t)), \text{ (jump map).} \end{array} \right. \tag{2}$$

For continuous-time switched systems, we have :

$$\mathcal{C} = \mathcal{D} = \mathbb{R}^n \times \mathcal{I}, \quad z(t) = \begin{pmatrix} x(t) \\ \sigma(t) \end{pmatrix} \in \mathbb{R}^{n+1}, \tag{3}$$

$$F(z(t)) = \begin{pmatrix} \{f_i(x(t), u(t))\}_{i \in \mathcal{I}} \\ 0 \end{pmatrix}; \quad G(z(t)) = \begin{pmatrix} x(t) \\ \mathcal{I} \end{pmatrix}. \tag{4}$$

Illustrations

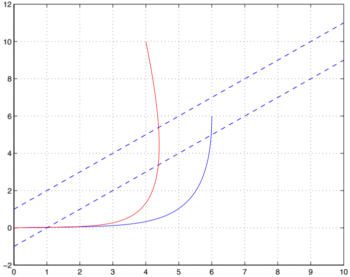
Saturated systems :

Let $x(t) \in \mathbb{R}^2$, with

$$\dot{x}(t) = \begin{bmatrix} -1 & 1 \\ 0 & -5 \end{bmatrix} x(t) + \begin{bmatrix} 0.2 \\ 1 \end{bmatrix} \mathrm{sat}\left(\begin{bmatrix} 1 & -1 \end{bmatrix} x(t) \right).$$

with

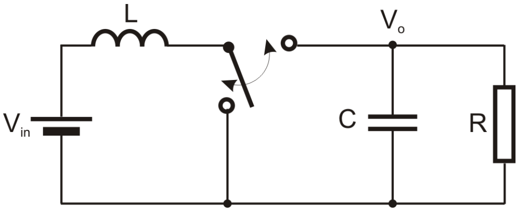
$$\mathrm{sat}(u) = \begin{cases} -1 \text{ if } u < -1, \\ +1 \text{ if } u > +1, \\ u \text{ if } -1 \leq u \leq +1. \end{cases}$$



Illustrations

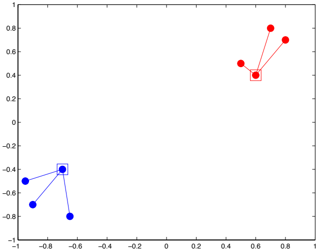
Boost converter :

$$
\begin{aligned}
C\frac{\mathrm{d}v_o}{\mathrm{d}t} &= (2-\sigma)i_L - \frac{1}{R}v_o, \qquad \sigma(t) \in \{1;2\} \\
L\frac{\mathrm{d}i_L}{\mathrm{d}t} &= v_{in} - (2-\sigma)v_o.
\end{aligned}
$$

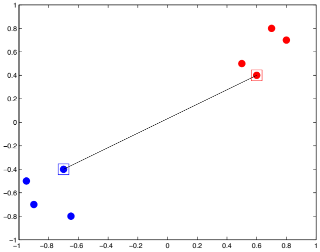


Illustrations

Multiagent systems :
The new position of each agent i is a mean of the position of agents, who are in the current neighborhood (depending on time k). Existence of a consensus $\lim_{k\to+\infty} x_k^{(i)} = x^*$?



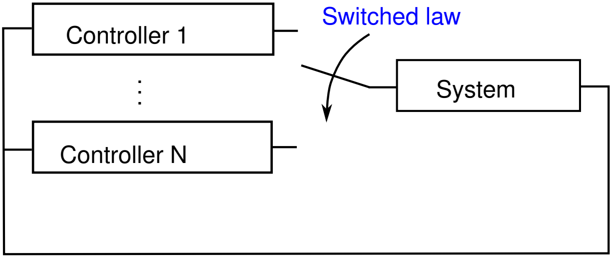
$$x_{k+1} = A_1 x_k$$



$$x_{k+1} = A_2 x_k$$

Illustrations

Switching controllers :

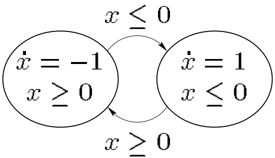


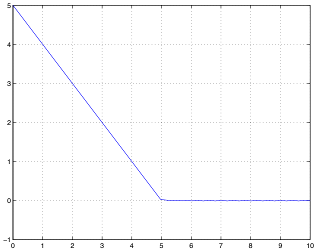
Illustrations

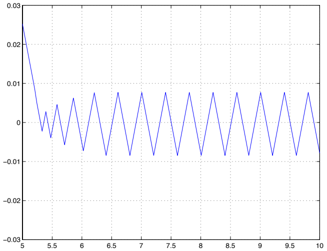
Sliding modes :

Let $x(t) \in \mathbb{R}$, with

$$\dot{x}(t) = -\text{sign}(x(t)) = \begin{cases} -1 \text{ if } x(t) > 0, \\ +1 \text{ if } x(t) < 0, \\ \text{undefined if x(t)=0.} \end{cases}$$







Difficulties : Well-posed solution ? Possible presence of Zeno phenomenon.

Typical examples of embedded systems

Framework of the talk

Consider discrete-time switched systems :

- Avoid well-posedness of solutions (different kinds of solutions : Filipov solution etc),
- Avoid Zeno phenomenon,
- Simplicity and richness of this class of systems.

Assume also for this talk :

- The modes are time invariant,
- The modes are autonomous (or already in their closed-loop form).

To sum up, we consider in the following (with distinct assumptions on $\sigma(\cdot)$) :

$$x_{k+1} = A_{\sigma(k)} x_k. \tag{5}$$

Outline of the tutorial

What are switched systems ?

About stability

- Definitions
- Stability of time invariant discrete-time linear systems
- Properties/Complexity of switched systems
- Main problems

Stability results for discrete-time switched systems (solving P1)

Stability results with constrained switching law (solving P2)

Stabilization results for discrete-time switched systems (solving P3)

Definitions relative to stability

The definitions are relative to an equilibrium point. Here we assume that the equilibrium point is the origin $x^* = 0$. In addition, the following definitions are valid for linear switched systems, for which there does not exist finite time escape.

Global asymptotic stability (GAS) : ensure that

$$\lim_{k \mapsto +\infty} x_k = 0, \quad \forall (x_0, \sigma(0)) \in \mathbb{R}^n \times \mathcal{I}. \tag{6}$$

Global uniform asymptotic stability (GUAS) : ensure that

$$\lim_{k \mapsto +\infty} x_k = 0, \quad \forall (x_0, \sigma(0)) \in \mathbb{R}^n \times \mathcal{I}, \quad \forall \sigma : \mathbb{N} \mapsto \mathcal{I}. \tag{7}$$

The term uniform means uniformly in $\sigma(\cdot)$.

Geometric approach

We recall stability results for the time invariant discrete-time linear system :

$$x_{k+1} = Ax_k, \quad \forall k \in \mathbb{N}. \tag{8}$$

The solution is given by

$$x_k = A^k x_0, \quad \forall k \in \mathbb{N}.$$

Theorem : The system (8) is GAS if and only if

$$\rho(A) = \max_{i \in} |\lambda_i(A)| < 1. \tag{9}$$

Lyapunov function approach

Theorem : Consider the system $x_{k+1} = Ax_k$ and $V : \mathbb{R}^n \to \mathbb{R}$, such that

- $V(x) \to +\infty$ as $\|x\| \to +\infty$. (radially unbounded).
- $V(0) = 0$ and $V(x) > 0$ if $x \neq 0$. (positive definite).
- $V(Ax) - V(x) < 0, \forall x \neq 0$. (decreasing)

Then the origin $x^* = 0$ is GAS.

The function V is called a Lyapunov function and is an extended energy of the system, which should decrease to zero along all trajectories.

Converse theorem

Theorem : If the origin $x^* = 0$ is GAS for the system $x_{k+1} = Ax_k$, then there exists a Lyapunov function $V(\cdot)$.

In such a case, the difficulty is to obtain the expression of the Lyapunov function $V(\cdot)$.

Stability for linear systems with Lyapunov functions

Theorem : the following statements are equivalent :

1. The linear system $x_{k+1} = Ax_k$ is GAS.
2. There is a quadratic Lyapunov function

$$V(x) = x^T Px, \tag{10}$$

where P is a positive definite matrix $P > 0_n$ such that the following Lyapunov inequality (Linear Matrix Inequality LMI) is satisfied :

$$A^T PA - P < 0. \tag{11}$$

3. There is a quadratic Lyapunov function

$$V(x) = x^T Px, \tag{12}$$

where P is the positive definite matrix $P > 0_n$ associated with any $Q > 0$ such that the following Lyapunov equation is satisfied.

$$A^T PA - P = -Q. \tag{13}$$

Sketch of proof

3) $\Rightarrow$ 2) . Trivial

$$A^T PA - P = -Q < 0.$$

2) $\Rightarrow$ 1) If the inequality $A^T PA - P < 0$ has a positive definite solution $P > 0_n$, then there exists sufficient small $1 > \epsilon > 0$ such that

$$A^T PA - P < -\epsilon P < 0.$$

Then, by considering $V(x) = x^T Px$, and $x_k \neq 0$,

$$V(x_{k+1}) - V(x_k) = x_k^T (A^T PA - P) x_k < -\epsilon x_k^T P x_k < 0,$$

which implies, with $\lambda_{\min}(P)\|x\|^2 \leq x^T Px \leq \lambda_{\max}(P)\|x\|^2$, that

$$x_k^T P x_k \leq (1-\epsilon)^k V(x_0); \quad \|x_k\|^2 \leq \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \|x_0\|^2 (1-\epsilon)^k.$$

1) $\Rightarrow$ 3) If the system $x_{k+1} = Ax_k$ is GAS, then the Grammian associated with the pair (Q, A), with any $Q > 0$ is well-defined (the sum converges).

$$\sum_{k \in \mathbb{N}} \left(A^T\right)^k QA^k,$$

and is a solution of the Lyapunov equation. To end the proof, we have only to prove that $P > 0$.

Main difficulty concerning the stability

The stability of a switched system is not intuitive

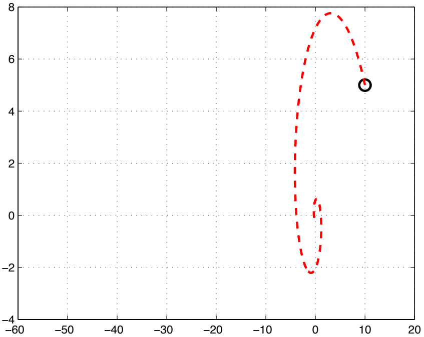
$$x_{k+1} = A_{\sigma(k)} x_k, \quad x(0) = x_0; \quad x_k \in \mathbb{R}^2 \tag{14}$$

where $\sigma : \mathbb{N} \to \{1, 2\}$ is the switching rule, which imposes the active mode.

$$A_1 = \begin{bmatrix} 0.9960 & -0.0100 \\ 0.0100 & 0.9960 \end{bmatrix}; \quad A_2 = \begin{bmatrix} 0.9960 & -0.1992 \\ 0.0005 & 0.9960 \end{bmatrix}; \tag{15}$$

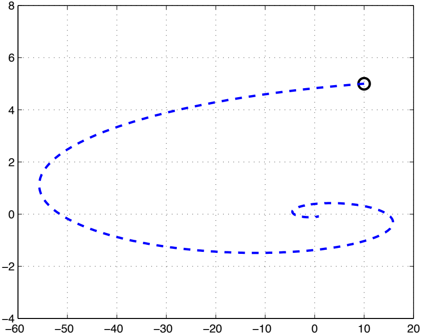
A_1 and A_2 have the same eigenvalues $\lambda_{\pm} = 0.9960 \pm 0.0100i$ and are stable (Schur : $\|\lambda_{\pm}\| < 1$) .

Stability of mode 1



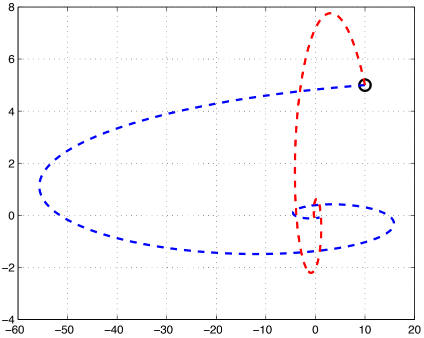
$$\sigma(k) = 1, \quad \forall k \in \mathbb{N}.$$

Stability of mode 2



$$\sigma(k) = 2, \quad \forall k \in \mathbb{N}.$$

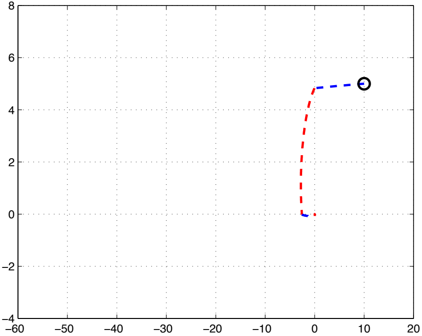
Comparing modal trajectories



$\sigma(k) = 1,$

$\sigma(k) = 2, \quad \forall k \in \mathbb{N}.$

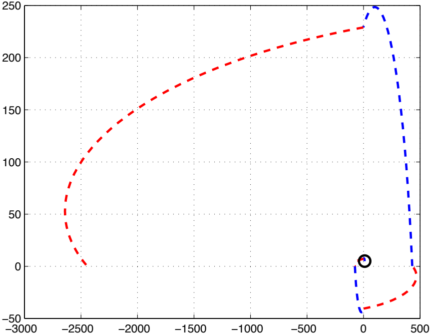
Is the switched system stable for all switching laws ? (I)



$\sigma(k) = 1$ if $x_{k,(1)} x_{k,(2)} \leq 0$
$\sigma(k) = 2$ if $x_{k,(1)} x_{k,(2)} > 0$

Stable

Is the switched system stable for all switching laws ? (II)



$$\sigma(k) = 1 \text{ if } x_{k,(1)}x_{k,(2)} > 0$$
$$\sigma(k) = 2 \text{ if } x_{k,(1)}x_{k,(2)} \leq 0$$

Unstable

Main problems

See [LM99].

- P1 Find stability conditions such that the switched system is asymptotically stable for any switching law.
- P2 Given a switching law, determine if the switched system is asymptotically stable.
- P3 Give the switching signal which makes the system asymptotically stable. P3 is called the stabilization problem.

Outline of the tutorial

Geometric approach : the joint spectral radius

The joint spectral radius of a set of matrices $\mathcal{A} = \{A_1, \cdots, A_N\}$, denoted $\rho(\mathcal{A})$ is an extension of the radius of a matrix A (i.e. $\rho(A)$) and gives a necessary and sufficient condition for the stability of the system (5) and solves P1. See [The05].

Remark : the joint spectral radius is the maximal growing rate which may be obtained by using long products of matrices from a given set.

We define

$$\rho(\mathcal{A}) = \lim\sup_{p\to+\infty} \rho_p(\mathcal{A}), \tag{16}$$

where

$$\rho_p(\mathcal{A}) = \sup_{A_{i_1}, A_{i_2}, \cdots, A_{i_p} \in \mathcal{A}} \left\| A_{i_1} A_{i_2} \times \cdots \times A_{i_p} \right\|^{\frac{1}{p}}.$$

Theorem : The switched system (5) is GAS if and only if

$$\rho(\mathcal{A}) < 1. \tag{17}$$

Main difficulty : this is difficult in the generic case to practically compute the joint spectral radius. Several approximations are provided in the literature.

The common Lyapunov function approach

Theorem If all the modes share a common Lyapunov function, then the switched system is GUAS.

Theorem If the switched system is GUAS, then all the modes share a common Lyapunov function.

Remark : be careful, there is no assumption concerning the class of the Lyapunov function. Especially, this Lyapunov function is not necessary on the form $V(x) = x^T P x$ as it will be seen in the following. This existence result does not help roughly speaking about how to find this Lyapunov function. In addition, there exists a common Lyapunov function on the form $V(x) = x^T P(x) x$, where $P(\lambda x) = P(x)$, $\forall \lambda \neq 0$ (homogeneous of degree zero).

The common Lyapunov function approach : sufficient conditions

The previous theorem suggests to look for a common quadratic Lyapunov function in the class $V(x) = x^T Px$.

Theorem : Consider the discrete-time linear switched system (5). If there exists a matrix $P \in \mathbb{R}^{n\times n}$ such that

$$P > 0_n \tag{18}$$

and

$$A_i^T PA_i - P < 0, \qquad \forall i \in \mathcal{I}, \tag{19}$$

then the system (5) admits the common quadratic Lyapunov function $V(x)$ and is GUAS.

Remark : the system (5) may be GUAS without feasible LMI (19).

The common Lyapunov function approach : unfeasibility test

To complete the previous remark, we have the following theorem.

Theorem : If there exist positive definite matrices $R_i \in \mathbb{R}^{n\times n}$, $R_i > 0_n$ such that

$$\sum_{i\in\mathcal{I}} A_i R_i A_i^T - R_i > 0_n, \tag{20}$$

then there does not exist $P > 0_n$ such that

$$A_i^T P A_i - P < 0, \qquad \forall i \in \mathcal{I}, \tag{21}$$

Proof : If there exist R_i $(\in \mathcal{I})$ such that Inequalities (20) hold, then for every $P > 0_0$,

$$0 < \mathrm{Tr}\left[P\left(\sum_{i\in\mathcal{I}} A_i R_i A_i^T - R_i\right)\right] = \mathrm{Tr}\left[R_i\left(A_i^T P A_i - P\right)\right],$$

then there exists $i_0 \in \mathcal{I}$ such that $A_{i_0}^T P A_{i_0} - P > 0$.

Multiple Lyapunov functions

Definition : We consider functions of the form

$$V(\sigma(k), x_k) = V_{\sigma(k)}(x_k) = x_k^T P(\sigma(k), x_k) x_k. \tag{22}$$

Theorem : If there exist P_i, $i \in \mathcal{I}$ such that $P_i > 0$ and

$$A_i^T P_j A_i - P_i < 0, \qquad \forall (i,j) \in \mathcal{I}^2, \tag{23}$$

then the discrete-time switched system (5) is GUAS.

Sketch of proof : By chosing $i = \sigma(k)$ and $j = \sigma(k+1)$, we have $V_{\sigma(k+1)}(x_{k+1}) - V_{\sigma(k)}(x_k) < 0$, $\forall x_k \neq 0$. A common Lyapunov function is

$$V_{\max}(x) = \max_{i \in \mathcal{I}} x^T P_i x. \tag{24}$$

Outline of the tutorial

What are switched systems ?

About stability

Stability results for discrete-time switched systems (solving P1)

Stability results with constrained switching law (solving P2)

- A periodic switching law
- Dwell time constraint

Stabilization results for discrete-time switched systems (solving P3)

Periodic switching law

A K-periodic switching law is defined by $\sigma : \mathbb{N} \to \mathcal{I}$ such that

$$\sigma(k+K) = \sigma(k), \quad k \in \mathbb{N}. \tag{25}$$

We define the monodromy matrix as

$$\Phi_k = A_{\sigma(k+K-1)} A_{\sigma(k+K-2)} \times \cdots \times A_{\sigma(k)}. \tag{26}$$

Theorem : the eigenvalues of the monodromy matrix Φ_k are called characteristic multipliers and are independent of i. The system (5) is GUAS if its characteristic multipliers belong strictly to the unit circle.

Then there exists $W > 0_n$ such that $\Phi_k^T W \Phi_k - W < 0$. Moreover there exists a K-periodical Lyapunov function $V(x_k, k) = x_k^T \tilde{P}(k) x_k$,l with $\tilde{P}(k+K) = \tilde{P}(k)$, such that $0 \leq \forall k \leq K-2$:

$$A_{\sigma(k)}^T \tilde{P}_{k+1} A_{\sigma(k)} - \tilde{P}_k = 0_n; \tag{27}$$

$$A_{\sigma(K-1)}^T \tilde{P}_0 A_{\sigma(K-1)} - \tilde{P}_{K-1} < 0_n. \tag{28}$$

Sketch of proof : choose $\tilde{P}_{K-1} = W$, and because $\tilde{P}_0 = \tilde{P}_K$, then

$$\tilde{P}_{K-2} = A_{\sigma(K-2)}^T W A_{\sigma(K-2)}; \; \tilde{P}_{K-3} = A_{\sigma(K-3)}^T A_{\sigma(K-2)}^T W A_{\sigma(K-2)} A_{\sigma(K-3)}; \; \cdots \tag{29}$$

Dwell time constraint

Definition : For an integer $\Delta \in \mathbb{N}^*$, the set of the switching laws satisfying a dwell time at least equal to Δ is defined by

$$\mathcal{D}_\Delta = \big\{\sigma : \mathbb{N} \to \mathcal{I};\ \exists\{\ell_q\}_{q\in\mathbb{N}}, \ell_{q+1} - \ell_q \geq \Delta; \\ \sigma(k) = \sigma(\ell_q), \forall \ell_q \leq k < \ell_{q+1}; \sigma(\ell_q) \neq \sigma(\ell_{q+1})\big\}.$$

Theorem : (See [GC06]) If there exist P_i $(i \in \mathcal{I})$ such that

$$A_i^T P_i A_i - P_i < 0_n, \quad \forall i \in \mathcal{I} \tag{30}$$

$$(A_i^T)^\Delta P_j A_i^\Delta - P_i < 0_n, \quad \forall (i,j) \in \mathcal{I}^2,\ i \neq j, \tag{31}$$

then the system (5) is GAS for any switching law $\sigma \in \mathcal{D}_\Delta$.

Outline of the tutorial

What are switched systems ?

About stability

Stability results for discrete-time switched systems (solving P1)

Stability results with constrained switching law (solving P2)

Stabilization results for discrete-time switched systems (solving P3)
- Lyapunov–Metzler inequalities
- Geometric approach
- LMI sufficient condition
- Periodic stabilizability

Stabilization of linear discrete-time switched systems

The problem P3 is to design a switching law that stabilizes the system (5).

Assumption : A_i $(\forall i \in \mathcal{I})$ are not Schur.

This assumption is to avoid a trivial solution : if there exists i_0 such that A_{i_0} is Schur, then $\sigma(k) = i_0$ globally asymptotically stabilizes the system.

Lyapunov-Metzler BMI conditions : sufficient conditions

Let consider the $\mathcal{M}_d$ the set of the Metzler matrices in discrete-time, that is the matrices whose elements are nonnegative and $\sum_{j\in\mathcal{I}} \pi_{ji} = 1$.
Theorem (see [GC06]. If there exist $P_i > 0$ ($i \in \mathcal{I}$) and $\pi \in \mathcal{M}_d$ such that

$$A_i^T \left(\sum_{j\in\mathcal{I}} \pi_{ji} P_j \right) A_i - P_i < 0, \quad \forall i \in \mathcal{I} \tag{32}$$

holds, then the switched system is globally asymptotically stabilizable with the min-switching strategy

$$\sigma(k) \in \arg\min_{i\in\mathcal{I}} x_k^T P_i x_k. \tag{33}$$

The inequality (32) is a Bilinear Matrix Inequality (BMI). The condition implies that the homogeneous function induced by $\bigcup_{i\in\mathcal{I}} \mathcal{E}(P_i)$ (where $\mathcal{E}(P) = \{x \in \mathbb{R}^n,\ x^T P x \leq 1\}$) is a control Lyapunov function.

Sketch of proof

Lyapunov function considered

$$V_{\min}: \left\{ \begin{array}{ccc} \mathbb{R}^n & \to & \mathbb{R}, \\ x_k & \mapsto & \min_{i \in \mathcal{I}} x_k^T P_i x_k, \end{array} \right. \tag{34}$$

Notation : $(P)_{p,i} = \sum_{\ell \in \mathcal{I}} \pi_{\ell i} P_\ell$.

Elements of proof

- By post-multiplying by $x_k \neq 0$ and pre-multiplying by x_k',

$$x_{k+1}'(P)_{p,i} x_{k+1} - x_k' P_i x_k < 0 \tag{35}$$

- the minimum scalar value of convex polytopes is reached on one of the vertices

$$V_{\min}(x_{k+1}) = \min_{j \in \mathcal{I}} x_{k+1}' P_j x_{k+1} = \min_{\substack{\sum_{j \in \mathcal{I}} \lambda_j = 1 \\ \lambda_j \in \mathbb{R}^+;}} \sum_{j \in \mathcal{I}} \lambda_j x_{k+1}' P_j x_{k+1}. \tag{36}$$

  Each column of the Metzler matrix $\Pi \in \mathcal{M}$ is in the unit simplex, then

$$V_{\min}(x_{k+1}) \leq x_{k+1}'(P)_{p,i} x_{k+1}. \tag{37}$$

$\Rightarrow$ global asymptotic stability holds with

$$V_{\min}(x_{k+1}) - V_{\min}(x_k) < 0, \quad \forall x_k \neq 0. \tag{38}$$

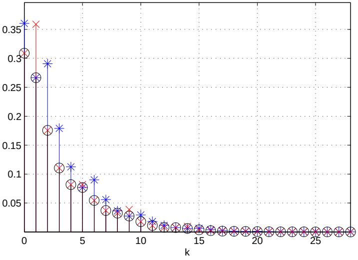
Example of state-partition

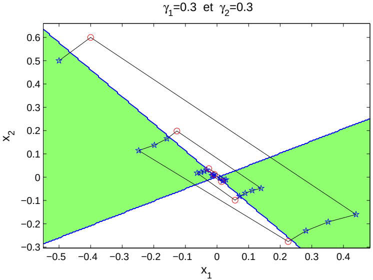
With

$$A_1 = \begin{bmatrix} -1.1 & 0 \\ 1 & 0.4 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 1.3 \end{bmatrix}, \; x_0 = \begin{pmatrix} -0.5 \\ 0.5 \end{pmatrix}$$

we have

$$\Pi = \begin{bmatrix} 0.3 & 0.7 \\ 0.7 & 0.3 \end{bmatrix}, P_1 = \begin{bmatrix} 1.7097 & 0.3734 \\ 0.3734 & 0.4786 \end{bmatrix}, \; P_2 = \begin{bmatrix} 1.1978 & 0.6398 \\ 0.6398 & 1.3173 \end{bmatrix}.$$





A C-set is a compact, convex set containing the origin in its interior.
Definition A set $\Omega \subseteq \mathbb{R}^n$ is a C^*-set if it is compact, star-convex with respect to the origin and $0 \in \text{int}(\Omega)$.

Notice a set is

- convex if $\forall x_0 \in \Omega$ and $\forall x \in \Omega$, then $\alpha x_0 + (1-\alpha)x \in \Omega,\ \forall \alpha \in [0,1]$.
- star-convex if $\exists x_0 \in \Omega$, such that $\forall x \in \Omega$, then $\alpha x_0 + (1-\alpha)x \in \Omega,\ \forall \alpha \in [0,1]$.

Minkowski function of a C^*-set Ω : $\Psi_\Omega(x) = \min_\alpha \{\alpha \in \mathbb{R} : x \in \alpha\Omega\}$.

- Any C-set is a C^*-set.
- Given a C^*-set Ω, we have that $\alpha\Omega$ is a C^*-set and $\alpha\Omega \subseteq \Omega$ for all $\alpha \in [0,1]$.
- $\Psi_\Omega(\cdot)$ is : defined on $\mathbb{R}^n$; homogenous of degree one ; positive definite and radially unbounded. But nonconvex in general !

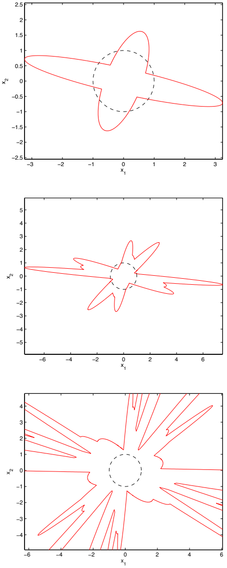
Geometric approach

Algorithm 1 Control λ-contractive C-set for the switched system (5).

- **Initialization** : given the C*-set $\Omega \subseteq \mathbb{R}^n$, define $\Omega_0 = \Omega$ and $k = 0$;
- **Iteration** for $k \geq 0$:

$$\Omega^i_{k+1} = A_i^{-1}\Omega_k, \quad \forall i \in \mathcal{I},$$
$$\Omega_{k+1} = \bigcup_{i\in\mathcal{I}} \Omega^i_{k+1};$$

- **Stop** if $\Omega \subseteq \mathrm{int}\left(\bigcup_{j\in\mathbb{N}_{k+1}} \Omega_j\right)$; denote $\check{N} = k + 1$ and $\check{\Omega} = \bigcup_{j\in\{1;\cdots;\check{N}\}} \Omega_j$.



Geometric approach

Geometrical interpretation :

- the set Ω_k^i is the set of x that can be stirred in Ω in k steps by a switching sequence beginning with $i \in \mathcal{I}$;
- then Ω_k is the set of points that can be driven in Ω in k steps ;
- and hence $\check{\Omega}$ the set of those which can reach Ω in $\check{N}$ or less steps, by an adequate switching law.

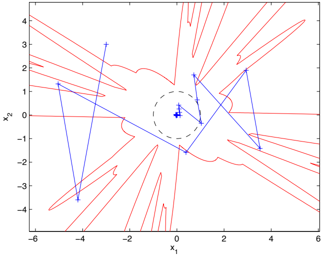
Necessary and sufficient condition for stabilizability.

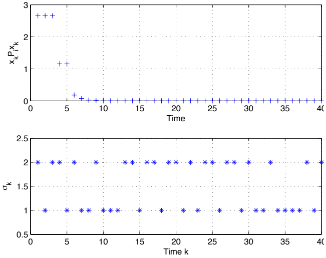
Theorem There exists a control Lyapunov function for the switched system if and only if the Algorithm 1 ends with finite $\check{N}$.

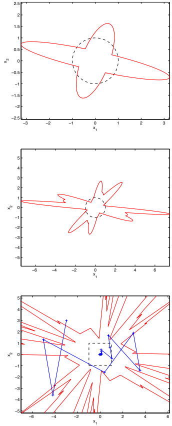
Example 1

Non-Schur switched system with $q = n = 2$.

$$A_1 = \begin{bmatrix} 1.2 & 0 \\ -1 & 0.8 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.6 & -2 \\ 0 & -1.2 \end{bmatrix},$$





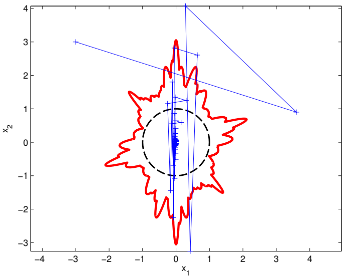


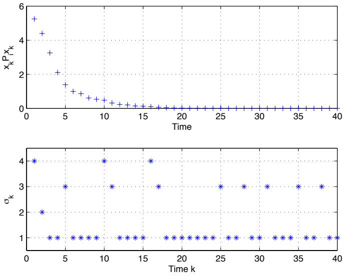
Example 2

System with $q = 4$, $n = 2$ and

$$A_1 = \begin{bmatrix} 1.5 & 0 \\ 0 & -0.8 \end{bmatrix}, \qquad A_2 = 1.1\, R(\tfrac{2\pi}{5})$$
$$A_3 = 1.05\, R(\tfrac{2\pi}{5} - 1), \qquad A_4 = \begin{bmatrix} -1.2 & 0 \\ 1 & 1.3 \end{bmatrix}.$$

The matrices A_i, with $i \in \mathbb{N}_4$, are not Schur. Notice : only one stable eigenvalue !





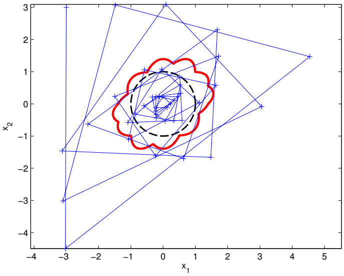
Example 3

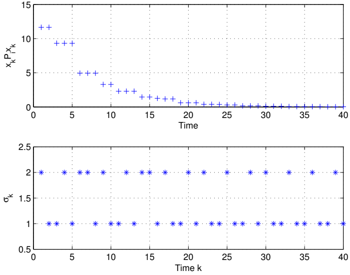
Switched system with

$$A_1 = \begin{bmatrix} 0 & -1.01 \\ 1 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & -1.01 \\ 1 & -0.5 \end{bmatrix}.$$

The technique based on Lyapunov-Metzler inequalities [GC06] has been numerically checked (gridding) and it results not feasible.

Nevertheless...



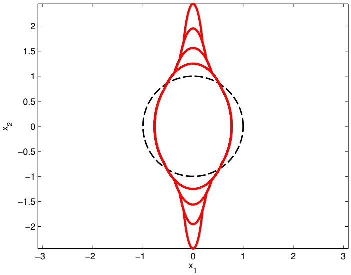


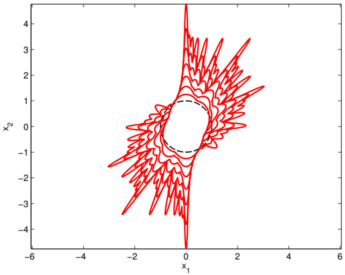
Example 4

Switched system with

$$A_1 = \begin{bmatrix} 1.3 & 0 \\ 0 & 0.9 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1.4 & 0 \\ 0 & 0.8 \end{bmatrix},$$

for $\theta = 0$ (left) and $\theta = \frac{\pi}{5}$ (right).





LMI sufficient condition

Our next aim is to formulate an alternative condition that could be checked efficiently, a convex one.

Theorem The switched system is stabilizable if there exist $N \in \mathbb{N}$ and $\eta \in \mathbb{R}^{\bar{N}}$ such that $\eta \geq 0$, $\sum_{i \in \mathcal{I}} \eta_i = 1$ and

$$\sum_{i \in \mathcal{I}} \eta_i \mathbb{A}_i^T \mathbb{A}_i < I.$$

with $\mathbb{A}_i = \prod_{j=1}^{k} A_{i_j} = A_{i_k} \cdots A_{i_1}$.

The condition is just sufficient (except for particular cases), is it also necessary ?

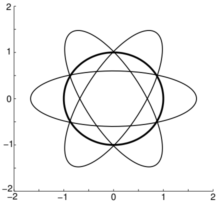
No !

Counterexample

Consider the three modes given by the matrices

$$A_1 = AR(0), \quad A_2 = AR\left(\frac{2\pi}{3}\right), \quad A_3 = AR\left(\frac{-2\pi}{3}\right), \quad \text{with} \quad A = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$$

and $a = 0.6$. The geometric condition holds with $N = 1$.



For every N and every $\mathbb{B}_i$ with $i \in \mathcal{I}$, the related $\mathbb{A}_i$ is such that $\det(\mathbb{A}_i^T\mathbb{A}_i) = 1$ and $\mathrm{Tr}(\mathbb{A}_i^T\mathbb{A}_i) \geq 2$.

Notice that, for all the matrices $Q > 0$ in $\mathbb{R}^{2\times 2}$ such that $\det(Q) = 1$, then $\mathrm{Tr}(Q) \geq 2$ and $\mathrm{Tr}(Q) = 2$ if and only if $Q = I$.

Thus, for every subset $K \subseteq \mathcal{I}$, we have that $\sum_{i\in K} \eta_i \mathbb{A}_i^T\mathbb{A}_i < I$, cannot hold, since either $\mathrm{Tr}(\mathbb{A}_i^T\mathbb{A}_i) > 2$ or $\mathbb{A}_i^T\mathbb{A}_i = I$.

Periodic stabilizability

A periodic switching law is given by $\sigma(k+K) = \sigma(k)$.

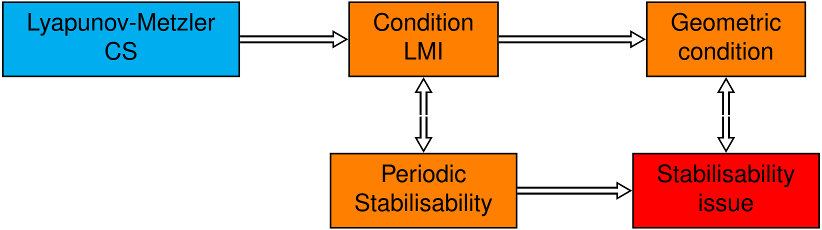
The stabilizability through periodic switching law, i.e. periodic stabilizability, is formalized below.
definition The switched system is periodic stabilizable if there exist a periodic switching law $\sigma : \mathbb{N} \rightarrow \mathcal{I}$, such that the system is stabilizable for all $x \in \mathbb{R}^n$.

Notice that for stabilizability the switching function might be state-dependent, hence a state feedback, whereas for having periodic stabilizability the switching law must be independent on the state.

Is there an equivalence relation between periodic stabilizability and the LMI condition ? The answer is below.

Theorem : A stabilizing periodic switching law for the switched system exists if and only if the LMI condition holds.

Sum up on the stabilizability



Bibliography

P. Bolzern and P. Colaneri.
The periodic Lyapunov equation.
SIAM J. Matrix Anal. Appl., 9(4) :499–512, 1988.

S. Bittanti and P. Colaneri.
Periodic Systems Filtering and Control.
Springer, 2009.

M. S. Branicky.
Multiple Lyapunov functions and other analysis tools for switched and hybrid systems.
IEEE Transactions on Automatic Control, 43(4) :475–482, 1998.

R. A. Decarlo, M. S. Branicky, S. Pettersson, and B. Lennartson.
Perspective and results on the stability and stabilization of hybrid systems.
Proceedings of the IEEE, 88 :1069–1082, 2000.

J. Daafouz, P. Riedinger, and C. Iung.
Stability analysis and control synthesis for switched systems : A switched Lyapunov function approach.
IEEE Transactions on Automatic Control, 47 :1883–1887, 2002.

Bibliography

J. Daafouz, S. Tarbouriech, and M. Sigalotti, editors.
Hybrid Systems with Constraints.
Wiley-ISTE, July 2013.

M. Fiacchini and M. Jungers.
Necessary and sufficient condition for stabilizability of discrete-time linear switched systems : a set-theory approach.
Automatica, regular paper, 50(1) :75–83, January 2014.

M. Fiacchini, M. Jungers, and A. Girard.
On stabilizability conditions for discrete-time switched linear systems.
In *53rd IEEE Conference on Decision and Control*, Los Angeles, California, USA, December 2014.

J. C. Geromel and P. Colaneri.
Stability and stabilization of discrete-time switched systems.
International Journal of Control, 79(7) :719–728, July 2006.

R. Goebel, R. G. Sanfelice, and A. R. Teel.
Hybrid Dynamical Systems : Modeling, Stability, and Robustness.
Princeton University Press, 2012.

Bibliography

L. Gurvits.
Stability of discrete linear inclusion.
Linear Algebra and its Applications, 231 :47–85, 1995.

M. Jungers and J. Daafouz.
Guaranteed cost certification for linear switched system with a dwell time.
IEEE Transactions on Automatic Control, 58(3) :768–772, March 2013.

M. Jungers, C. A. C. Gonzaga, and J. Daafouz.
Min-switching local stabilization for discrete-time switching systems with nonlinear modes.
Nonlinear Analysis : Hybrid Systems, 9 :18–26, August 2013.

D. Liberzon.
Switching in Systems and Control, volume in series Systems and Control : Foundations and Applications.
Birkhäuser, Boston, MA, 2003.

J. Lunze and F. Lamnabhi-Lagarrigue, editors.
Handbook of Hybrid Systems Control : Theory, Tools, Applications.
Cambridge, 2009.

Bibliography

D. Liberzon and A.S. Morse.
Basic problems in stability and design of switched systems.
IEEE Control Systems Magazine, 19 :59–70, 1999.

A. P. Molchanov and Y. S. Pyatnitskiy.
Criteria of asymptotic stability of differential and difference inclusions encounterd in control theory.
Systems & Control Letters, 13 :59–64, 1989.

Z. Sun and S. S. Ge.
Stability Theory of Switched Dynamical Systems.
Springer, 2011.

J. Theys.
Joint Spectral Radius : theory and approximations.
PhD thesis, UCL Belgium, 2005.

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Thank you very much !