Global sensitivity analysis for the boundary control of an open channel

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joint work with
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Journées Automatique 2015

Bienvenue à Grenoble pour les premières journées de l'Automatique

Parametrized PDE

Let $\mu = \{\mu_1, \ldots, \mu_p\}$ a random variable, modeling uncertainty parameters, a perturbed parameter, an unknown dynamics...

Let $a(\xi, \mu) = 0$ be a Partial Differential Equation parametrized by μ whose solution is ξ

Let y be the to-be-controlled output: deterministic function: $y = y(\xi)$

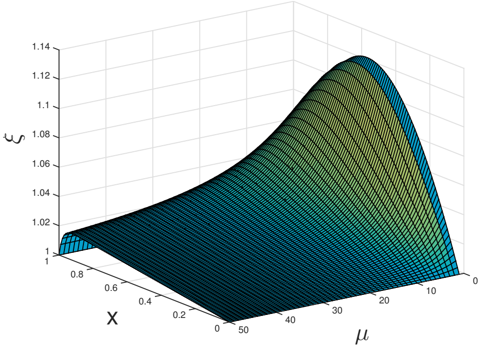
Natural question: What is the impact of random variable μ on y?

Toy-model

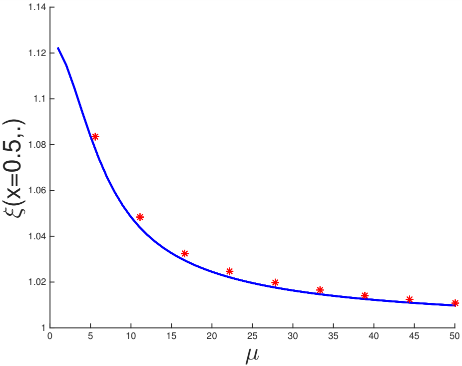
Consider

$$-\xi_{xx} + \mu\xi_x = 1$$
$$\xi(0) = \xi(1) = 1$$

Assume $\mu \sim \mathcal{U}(0, 50)$, where $\mathcal{U}(0, 50)$ is the uniform distribution on $(0, 50)$.
Define the *to-be-controlled output*: $f(\mu) = \xi(0.5, \mu)$.
Statistical property under interest: mean value $\mathbb{E}(y(\mu))$



Plot of $\xi(x, \mu)$



Plot of $\xi(1/2, \mu)$

How to compute $\mathbb{E}(\mu) = \mathbb{E}(\xi(0.5, \mu))$?

Bad idea Monte-Carlo estimator $\mathbb{E}(\mu) \approx \frac{1}{N} \sum_{i=1}^{N} y(\mu^i)$ with large N

Good idea If you compute a sample of e.g. 10 solutions, the 11th should be "close" to it $\xi(., \mu^{11}) \approx \sum_{i=1}^{10} \lambda^i \xi(., \mu^i)$
and thus by computing the *to-be-controlled output*, we should find an estimation of $\mathbb{E}(\mu)$

Goal: Global sensitivity analysis in a boundary control problem.

- the nonlinear **Shallow Water equations**:
  balance laws with effect of bottom slope and the slope's friction;
- the **boundary actions** are defined as the position of both spillways located at the extremities of the reach.

In [Coron, Bastin, d'Andréa-Novel; 08] & [Dos Santos, CP; 08], the authors designed stabilizing **boundary ouput feedback controllers**, with an **exponential convergence to the equilibrium** of water level and water flow.

Issue: we want to determine which factors (bottom slope, slope's friction, ...) are influent in this process. Je

Probabilistic methods

The problem is related to probabilistic methods for control system design as considered by e.g., R. Tempo. Given μ a vector of random variables, define

$$f(\mu) = \begin{cases} +\infty & \text{if } G_\mu \text{ is unstable} \\ \|G_\mu\|_\infty & \text{otherwise} \end{cases}$$

Consider the **reliability estimation** *problem. Given* $\gamma > 0$, *estimate* $Pr\{f(\mu) \leq \gamma\}$

Consider the **performance level estimation** *problem. Given* $\varepsilon > 0$, *estimate* γ *such that* $Pr\{f(\mu) \leq \gamma\} \geq 1 - \varepsilon$

[Calafiore, Dabbene, Tempo, Automatica; 2011]
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1. Modeling by means of Shallow Water equations
   when considering boundary actions, slope, friction...
2. Global sensitivity analysis (GSA)
   of uncertain physical parameters
   for boundary control stability of an open channel
3. Numerical results
   First and total order Sobol indices

   What are the most influent parameters on the output,
   when closing the loop?
   What about alone and combined impacts on the stability?

   Use of a model reduction
4. Conclusion and perspectives

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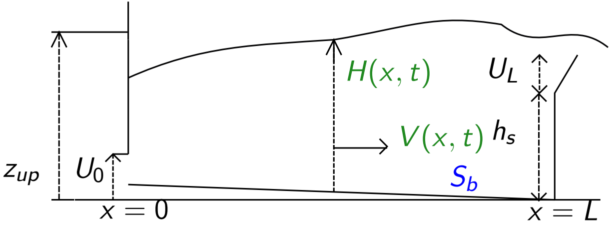
Use of a model reduction

4 Conclusion and perspectives

One considers the classical Shallow Water equations for describing the flow dynamics inside an open-channel, $\forall (x,t) \in [0,L] \times \mathbb{R}_+$,

$$\partial_t \begin{pmatrix} H \\ V \end{pmatrix} + \begin{pmatrix} V & H \\ g & V \end{pmatrix} \partial_x \begin{pmatrix} H \\ V \end{pmatrix} + \begin{pmatrix} 0 \\ g(S_f - S_b) \end{pmatrix} = 0 \,,$$



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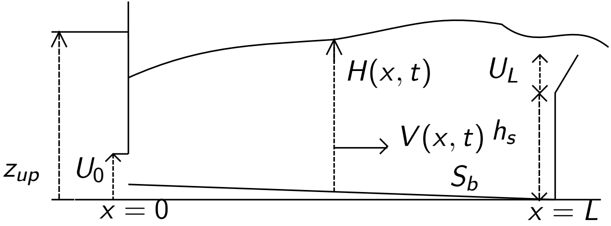
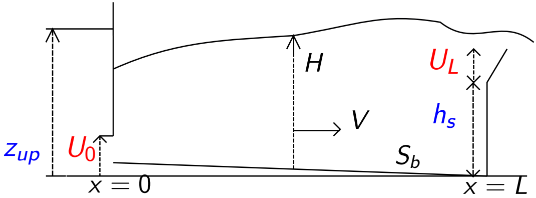


The friction slope is given $S_f = C \dfrac{V^2}{H}$.
Different models are possible (see in particular [Bastin, Coron, d'Andréa-Novel; 09] & [Dos Santos, CP; 08]).
C is a constant friction coefficient



Denoting the channel width by B and the water flow by $Q = BHV$, the controls are the position U_0 and U_L of both spillways:

- a submerged underflow gate:

$$Q(0,t) = U_0 B p_0 \sqrt{2g(z_{up} - H(0,t))}, \qquad (1)$$

- a submerged overflow gate:

$$H(L,t) = \left(\frac{Q^2(L,t)}{2gB^2 {p_L}^2}\right)^{1/3} + h_s + U_L, \qquad (2)$$

where p_0, p_L are water flow coefficients of the gates.

Linearization around an equilibrium

The steady state equilibrium of Shallow Water equations satisfies:
$V^\star = \left(\frac{S_b Q^\star}{BC}\right)^{1/3}, \quad H^\star = \frac{Q^\star}{BV^\star}.$

We first define

$$v(x,t) = V(x,t) - V^\star$$
$$h(x,t) = H(x,t) - H^\star$$

We then introduce the classical characteristic coordinates:

$$\xi_1(x,t) = v(x,t) + h(x,t)\sqrt{\frac{g}{H^\star}}$$
$$\xi_2(x,t) = v(x,t) - h(x,t)\sqrt{\frac{g}{H^\star}}$$

and the characteristic velocities:

$$\lambda_1 = V^\star + \sqrt{gH^\star}$$
$$-\lambda_2 = V^\star - \sqrt{gH^\star}$$

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The linearized Shallow Water equations is

$$\partial_t \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix} \partial_x \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} \gamma & \delta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \tag{3}$$

with

$$\gamma = g\,C \frac{(V^\star)^2}{H^\star} \left(\frac{1}{V^\star} - \frac{1}{2\sqrt{gH^\star}} \right) , \quad \delta = g\,C \frac{(V^\star)^2}{H^\star} \left(\frac{1}{V^\star} + \frac{1}{2\sqrt{gH^\star}} \right) .$$

Linear hyperbolic system of balance laws in Riemann coordinates.
For suitable controllers, boundary conditions are simple. Indeed:

Proposition 1

Given any constant values k_0 and k_L, defining the controls U_0 and U_L by, for all $t \geq 0$,

$$U_0(t) = \frac{H(0,t)\left(V^\star - \frac{1+k_0}{1-k_0}(H(0,t)-H^\star)\sqrt{\frac{g}{H^\star}}\right)}{p_0\sqrt{2g(z_{up}-H(0,t))}}$$

$$U_L(t) = -\left(\frac{H(L,t)\left(V^\star + \frac{1+k_L}{1-k_l}(H(L,t)-H^\star)\sqrt{\frac{g}{H^\star}}\right)}{\sqrt{2g}\,p_L}\right)^{\frac{2}{3}} + H(L,t) - h_s \ ,$$

the boundary conditions (1) and (2) may be rewritten as

$$\begin{pmatrix} \xi_1(0,t) \\ \xi_2(L,t) \end{pmatrix} = \begin{pmatrix} 0 & k_0 \\ k_L & 0 \end{pmatrix} \begin{pmatrix} \xi_1(L,t) \\ \xi_2(0,t) \end{pmatrix} . \tag{4}$$

Note that U_0 and U_L depend only on water heights at both extremities of the channel.

Applying Lyapunov techniques, we may compute stabilizing controllers:

Proposition 2 ([Bastin, Coron, d'Andréa-Novel, 09])

For any $(k_0, k_L) \in \mathbb{R}$ such that

$$\max\left\{|k_0|\sqrt{\frac{\lambda_1\gamma}{\lambda_2\delta}}, |k_L|\sqrt{\frac{\lambda_2\delta}{\lambda_1\gamma}}\right\} < 1 \ ,$$

defining U_0 and U_L with Proposition 1, the linear hyperbolic system of balance laws is exponentially stable, that is, it holds

$$\begin{aligned}\|(H(\cdot,t),V(\cdot,t)) - (H^\star,V^\star)\|_{L^2((0,L);\mathbb{R}^2)} \\ \leq Me^{-\nu t}\|(H^0,V^0) - (H^\star,V^\star)\|_{L^2((0,L);\mathbb{R}^2)}\end{aligned}$$

where H^0 and V^0 stand for the initial conditions, and M, ν are two positive values.

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Global sensitivity analysis: How is the asymp. stability property impacted by the not well-known physical parameters?

Let $\mu = \left(h_s, B, p_0, p_L, S_b, C, z_{up}, \xi_1^0, \xi_2^0\right)$ be the uncertain parameters,

i.e. random variables around the nominal values $\mu_{nom} = \left(h_{s,nom}, B_{nom}, p_{0,nom}, p_{L,nom}, S_{b,nom}, C_{nom}, z_{up,nom}, \xi^0_{1,nom}, \xi^0_{2,nom}\right)$

Name μ	Nominal value μ_{nom}	$\mathcal{N}(m,\sigma)$: normal distribution $\mathcal{U}(a,b)$: uniform distribution
h_s	$4m$	$\mathcal{N}(4, 0.03)$
B	$80m$	$\mathcal{N}(80, 1.03)$
p_0, p_L	0.65	$\mathcal{N}(0.65, 0.0066)$
S_b	0.0002	$\mathcal{N}\left(2\times10^{-4}, 2.5\times10^{-6}\right)$
C	0.001	$\mathcal{U}\left(9\times10^{-4}, 0.0011\right)$
z_{up}	$10m$	$\mathcal{N}(10, 0.13)$
ξ_1^0	0	initial state component $\mathcal{U}(-0.01, 0.01)$
ξ_2^0	0	initial state component $\mathcal{U}(-0.01, 0.01)$
k_0, k_l	$0.6, 0.7$	known
$Q^\star, g$	$50, 9.81$	known

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Given any constant values k_0 and k_L, defining the controls U_0 and U_L by Proposition 1 with μ_{nom} instead of μ, the boundary conditions (1) and (2) for with the parameters μ are linearized as

$$\left(1 - \frac{\mathcal{B} + \sqrt{g/H^\star}}{2\sqrt{g/H^\star}}\right)\xi_1(0,t) + \frac{\mathcal{B} + \sqrt{g/H^\star}}{2\sqrt{g/H^\star}}\xi_2(0,t) = \mathcal{A}$$

$$-\frac{\mathcal{D} - \sqrt{g/H^\star}}{2\sqrt{g/H^\star}}\xi_1(L,t) + \left(1 + \frac{\mathcal{D} - \sqrt{g/H^\star}}{2\sqrt{g/H^\star}}\right)\xi_2(L,t) = \mathcal{C}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are values which depends linearly on μ.

Remark Of course, if $\mu = \mu_{\text{nom}}$, then we recover the same boundary conditions than the previous ones for Riemann coordinates.

To-be-controlled-output

Definition of the *to-be-controlled output* (output of interest):

$$f(\mu) = \sqrt{\int_{t=0}^{T^\star} \int_{x=0}^{L} \xi_1(x,t)^2 + \xi_2(x,t)^2 \, \mathrm{d}x \, \mathrm{d}t}\,,$$

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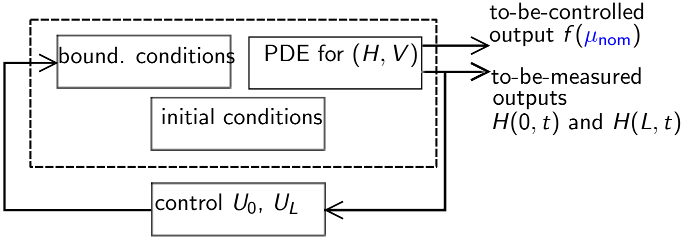

Closed loop when they are no uncertainties

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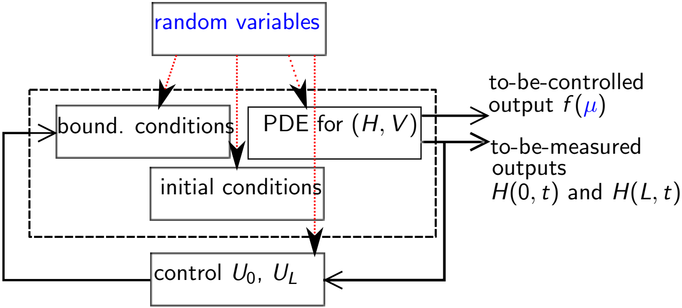

Closed loop in presence of random variables

The *to-be-controlled output* can then be considered as a scalar random variable $Y = f(\mu)$.

The conditional expectation $\mathbb{E}(Y|\mu_i)$ is a random variable which gives the mean of Y over the distributions of the μ_j $(j \neq i)$
Its variance quantifies the influence of μ_i on the dispersion of Y.

The relative influence of μ_i is given by the **first-order Sobol' index**

$$S_{\{i\}} = \frac{\mathrm{Var}\left(\mathbb{E}(f(\mu)|\mu_i)\right)}{\mathrm{Var}(f(\mu))} \in [0, 1]$$

Examples: If $S_{\{i\}} = 1$, then only μ_i has an impact on Y
If $S_{\{i\}} = 0$, then Y does not depend on μ_i.

S_i^{total} measures the influence of μ_i combined with other physical parameters in μ on the output Y

(we may also define second, third... order Sobol' indices)

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How to compute $S_{\{i\}} = \frac{\text{Var}(\mathbb{E}(f(\mu)|\mu_i))}{\text{Var}(f(\mu))}$?

For any $i \in \{1, \ldots, p\}$, let $\mu_i^{j,1}$ and $\mu_i^{j,2}$, $j = 1, \ldots, n$ be two independent samples of size n of the parameter μ_i.

We define:

$$\mu^{j,1} = (\mu_1^{j,1}, \ldots, \mu_{i-1}^{j,1}, \mu_i^{j,1}, \mu_{i+1}^{j,1}, \ldots, \mu_p^{j,1}) \quad j = 1, \ldots, n$$

$$\mu_{\{i\}}^{j,2} = (\mu_1^{j,2}, \ldots, \mu_{i-1}^{j,2}, \mu_i^{j,1}, \mu_{i+1}^{j,2}, \ldots, \mu_p^{j,2}) \quad j = 1, \ldots, n\,, \; i = 1, \ldots, p$$

Let us consider the following quantities

$$Y^{j,1} = f(\mu^{j,1}) \quad j = 1, \ldots, n$$

$$Y_{\{i\}}^{j,2} = f(\mu_{\{i\}}^{j,2}) \quad j = 1, \ldots, n\,, \; i = 1, \ldots, p.$$

To estimate all $Y^{j,1}$ and $Y_{\{i\}}^{j,2}$, we have to evaluate $(1+p)n$ times the output, where n is the sample size.

Estimator of first order indices

Monte Carlo estimator: [Monod *et al.*; 06], [Janon *et al.*, 14]

$$\hat{\mathrm{S}}_{\{i\},n} = \frac{\frac{1}{n}\sum_{j=1}^{n} Y^{j,1} Y^{j,2}_{\{i\}} - \left(\frac{1}{n}\sum_{j=1}^{n} \frac{Y^{j,1}+Y^{j,2}_{\{i\}}}{2}\right)^2}{\frac{1}{n}\sum_{j=1}^{n} \frac{(Y^{j,1})^2+(Y^{j,2}_{\{i\}})^2}{2} - \left(\frac{1}{n}\sum_{j=1}^{n} \frac{Y^{j,1}+Y^{j,2}_{\{i\}}}{2}\right)^2}.$$

Global Sensitivity Analysis, a many-query context:
We have to evaluate $(1+p)n$ *times the output*
$\Longrightarrow$ *a model reduction is needed.*

This reduction is done by using a metamodel, that is instead of considering the linear hyperbolic equation, we discretize it

Theorem

Assume that $\mathbb{E}(Y^4) < \infty$. Let $\alpha \in (0,1)$ (typically $\alpha = 0.05$ or 0.10). Then an asymptotic confidence interval of level $1-\alpha$ for $\mathrm{S}_{\{i\}}$ is given by

$$\left[\hat{\mathrm{S}}_{\{i\},n} - z_{1-\frac{\alpha}{2}}\frac{\hat{\sigma}_{\{i\}}}{\sqrt{n}}, \hat{\mathrm{S}}_{\{i\},n} + z_{1-\frac{\alpha}{2}}\frac{\hat{\sigma}_{\{i\}}}{\sqrt{n}}\right],$$

with $z_{1-\frac{\alpha}{2}}$ is the $1-\frac{\alpha}{2}$ quantile of the $\mathcal{N}(0,1)$ distribution and where $\hat{\sigma}$ is any consistant estimator of $\sigma_{\{i\}}$ defined by

$$\sigma^2_{\{i\}} = \frac{\mathrm{Var}\left((Y-\mathbb{E}Y)(Y_{\{i\}}-\mathbb{E}Y) - \frac{\mathrm{S}_u}{2}\left((Y-\mathbb{E}Y)^2 + (Y_{\{i\}}-\mathbb{E}Y)^2\right)\right)}{(\mathrm{Var}Y)^2}.$$

Analogous result for S_i^{total}.

First-order Sobol' indices

$T^\star = 75$s, numerical results of the uncertain parameters $\mu = \left(h_s, B, p_0, p_L, S_b, C, z_{up}, \xi_1^0, \xi_2^0\right)$

Parameter	95% confidence interval
h_s	$[0.090335; 0.11281]$
B	$[0.12559; 0.14833]$
p_0	$[-0.014771; 0.0078236]$
p_L	$[0.27415; 0.29626]$
S_b	$[0.067575; 0.090251]$
C	$[0.32127; 0.34233]$
z_{up}	$[-0.01477; 0.0078226]$
ξ_1^0	$[0.017666; 0.040285]$
ξ_2^0	$[-0.0079254; 0.014667]$

Thus, the parameters h_s, B, p_L, S_b, C and ξ_1^0 are influent on the *to-controlled-output* Y. C is the most influent one.
The other parameter are not influent (with a confidence of 95%): p_0, z_{up}, ξ_2^0

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Total Sobol' indices

What about the combined influence on Y?

Parameter	95% confidence interval for
h_s	$[0.098833; 0.11604]$
B	$[0.13818; 0.15573]$
p_0	$[-0.0077521; 0.0077808]$
p_L	$[0.28348; 0.30269]$
S_b	$[0.074616; 0.091309]$
C	$[0.33443; 0.35435]$
z_{up}	$[-0.007758; 0.0077727]$
ξ_1^0	$[0.025303; 0.041345]$
ξ_2^0	$[-0.0018308; 0.013773]$

For each parameter, as the difference between the total and the first-order indices is not significant

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It implies that the interactions are negligible.
Thus the to-be-controlled output is additive in the parameters:

$$f_{discrete}(\mu) \approx \sum_{\mu_i \in \{h_s, B, p_0, p_L, S_b, C, z_{up}, \xi_1^0, \xi_2^0\}} g_i$$

where g_i are appropriate univariate functions. And even:

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Conclusion, perspectives

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- Global sensitivity analysis (GSA) requires many evaluations of the output of interest, thus the need of model reduction
- GSA allows to determine which parameters are the most influent on a given *to-be-controlled output*
- B, S_b, C are the most influent parameters, with an additive effect.

Perspectives

- Analyze the sensitivity using the decay rate as the output?
- Considering the tuning parameters k_0 and k_L as random variables, to find the "best" insensitizing controller?
- GSA for nonlinear output? For nonlinear model?

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