Global optimization approach to H_∞ synthesis problem under structural constraint

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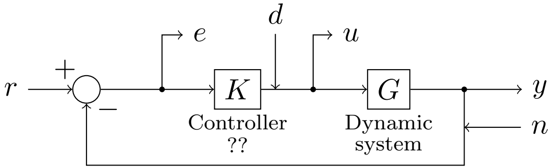
Plan

General problem

We consider a LTI plant $G(s)$. We suppose that $G(s)$ is stabilizable and detectable.



r: reference, e: tracking error, d: control disturbance, u: control, y: measurement output, n: sensor noise.
We want to ensure:

- Stability of the system.
- Performance/Robustness constraints.

by tuning a structured controller $K(k, s)$ that depends on free parameters k.

H_∞ approach to Robstness/Performance constraints

Robustness and performance constraints express the desired behavior of the closed loop system:

- small tracking error.
- small sensitivity to noise and disturbance.
- frequency constraints on actuators.

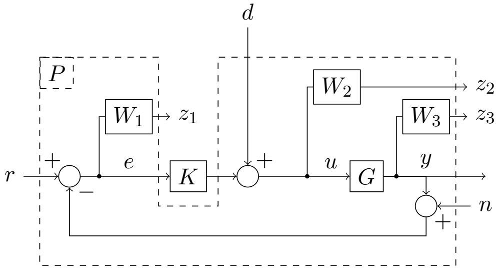
These constraints are formulated as frequency constraints on transfer functions.

$$||T_{w\to e}(k,s)W_e(s)||_\infty \leq 1$$
$$\iff \forall \omega, \sqrt{\lambda(T_{w\to e}(i\omega)T_{w\to e}(i\omega)^*)} \leq |W_e(i\omega)^{-1}|$$

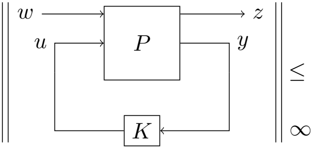
We ensure that the answer of the system to inputs w is bounded in term of gain by the modulus of a constraint function.

Classical approach

The classical approach is to built an augmented system $P(s)$



and to solve under the stability constraint:



$\leq 1 \iff ||T_{w \to z}(k,s)||_\infty \leq 1$

Our H_∞ approach

The classical approach may not compute a satisfying controller:

$$||T_{w\to z}(k,s)||_\infty = || \begin{pmatrix} T_{w\to z_1}(k,s) \\ \vdots \\ T_{w\to z_p}(k,s) \end{pmatrix} ||_\infty$$

$$\geq \max(||T_{w\to z_1}(k,s)||_\infty, ..., ||T_{w\to z_p}(k,s)||_\infty)$$

We aim to solve the problem:

$$\begin{cases} \min\limits_{k} \max(||T_{w\to z_1}(k,s)||_\infty, ..., ||T_{w\to z_p}(k,s)||_\infty), \\ s.t. \quad K(k,s) \text{ stabilizes the closed-loop system} \end{cases}$$

Minmax formulation

Constraints on weighted outputs z_i are expressed as inequalities on analytic functions:

$$||T_{w\to z_j}(k,s)||_\infty = \sup_{\omega\geq 0}\sqrt{\lambda_{max}(T_{w\to z_j}(k,i\omega)\overline{T_{w\to z_j}(k,i\omega)}^T)}$$

$$= \sup_{\omega\geq 0}\sqrt{\sum_{i=1}^{n} Re(T_{w_i\to z_j}(k,i\omega))^2 + Im(T_{w_i\to z_j}(k,i\omega))^2}$$

The problem is a minmax problem:

$$\begin{cases} \min_{k}\sup_{\omega\geq 0} f(k,i\omega), \\ s.t. \quad K(k,s) \text{ stabilizes the closed-loop system} \end{cases}$$

Where $f(k,i\omega) = \max(||T_{w\to z_1}(k,i\omega)||_\infty, \ldots, ||T_{w\to z_p}(k,i\omega)||_\infty)$

Plan

Interval Analysis

- $\boldsymbol{\omega}$ is an interval of $\mathbb{R}^+$: $\boldsymbol{\omega} = [\underline{\omega}, \overline{\omega}]$.
- $\boldsymbol{k}$ is a vector of interval of controller free parameters k.

Interval analysis provides an inclusion function $[f]$ of f:

$$f(\boldsymbol{k}, i\boldsymbol{\omega}) = \{f(k, i\omega), k \in \boldsymbol{k}, \omega \in \boldsymbol{\omega}\} \subseteq [f](\boldsymbol{k}, i\boldsymbol{\omega})$$

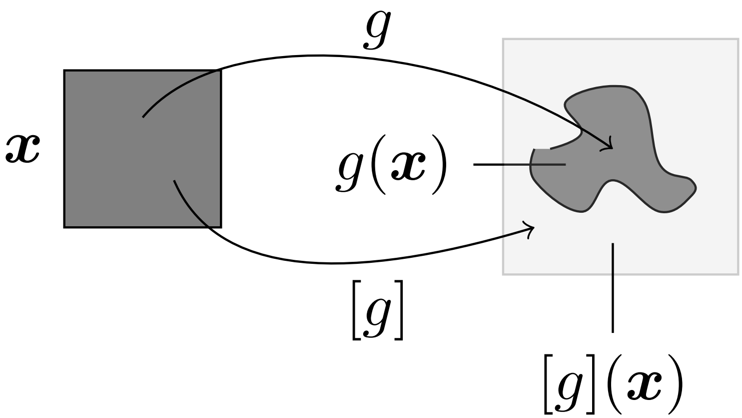
Stability of the system can be verified with Routh-Hurwitz criterion, expressed as the satisfaction of a system:

$$R_1(k) \leq 0 \wedge ... \wedge R_t(k) \leq 0$$

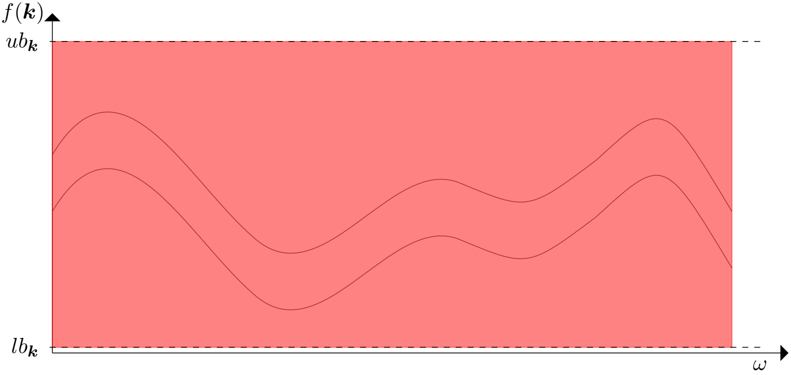
We also have inclusion function $[R_i]$ of R_i.
$\Longrightarrow$ We can certify that $\forall k \in \boldsymbol{k}$, $K(k, s)$ stabilizes or not the system.

Inclusion function

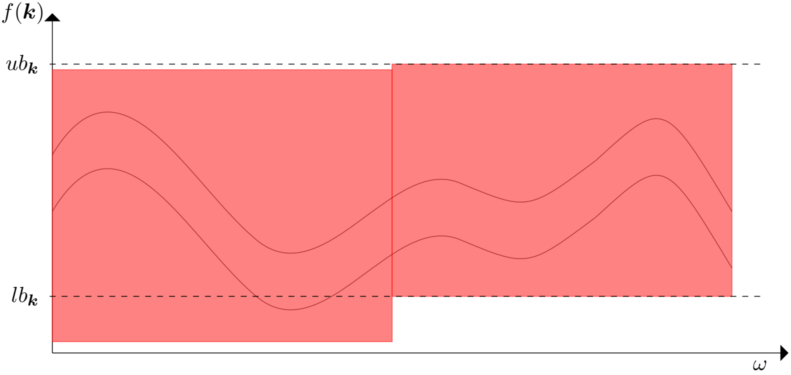
An inclusion function provides an over approximation



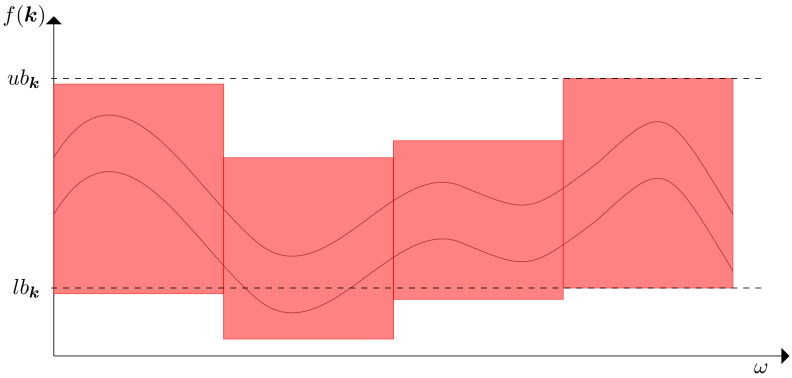
Computing $\sup_{\omega \geq 0} f(\boldsymbol{k}, i\omega)$



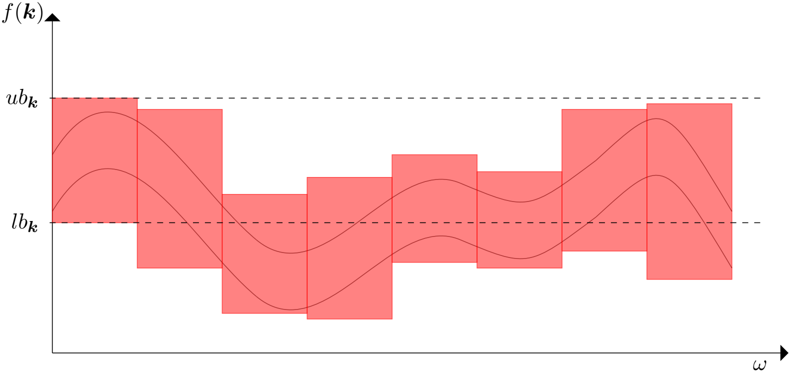
Computing $\sup_{\omega \geq 0} f(\boldsymbol{k}, i\omega)$



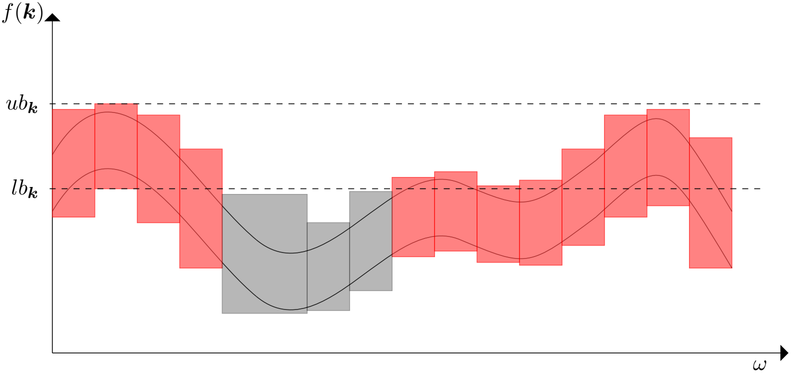
Computing $\sup_{\omega \geq 0} f(\boldsymbol{k}, i\omega)$



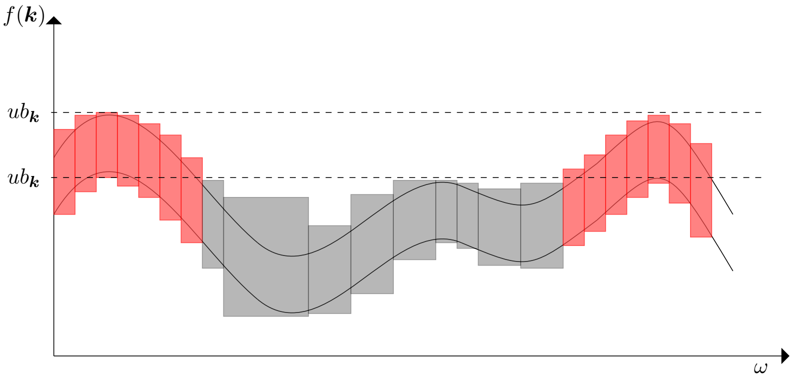
Computing $\sup_{\omega \geq 0} f(\boldsymbol{k}, i\omega)$



Computing $\sup_{\omega \geq 0} f(\boldsymbol{k}, i\omega)$



Computing $\sup_{\omega \geq 0} f(\boldsymbol{k}, i\omega)$



Main B&B algorithm

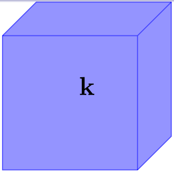
Interval B&B algorithm:

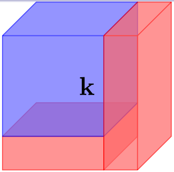
- Initial set of controller parameters $\mathbb{K}$.
- Finite frequency domain $[\omega]$.

While $\mathcal{L} \neq \emptyset$:

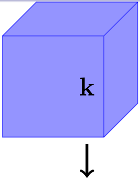
1. Choose a box $\boldsymbol{k}$ from $\mathcal{L}$.
2. Contract $\boldsymbol{k}$ w.r.t $R(\boldsymbol{k}) \leq 0$.
3. Compute $[lb_{\boldsymbol{k}}, ub_{\boldsymbol{k}}]$ an enclosure of $\sup_{\omega} [f](\boldsymbol{k}, \omega)$.
4. Try to find a good feasible solution in $\boldsymbol{k}$.
5. Update best current solution.

Stop criterion: $width(\boldsymbol{k}) < \epsilon$

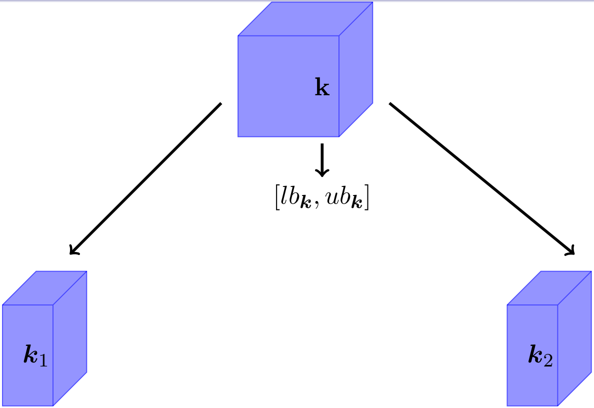

k

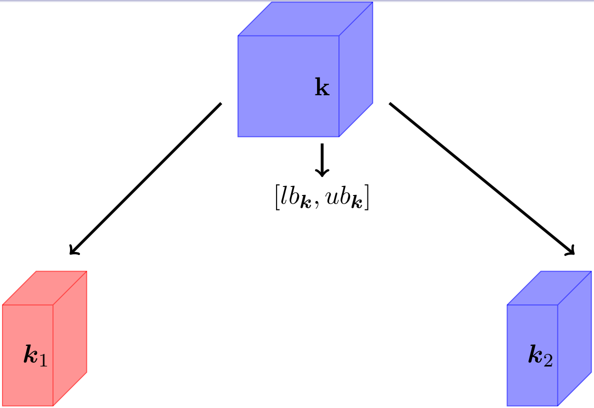


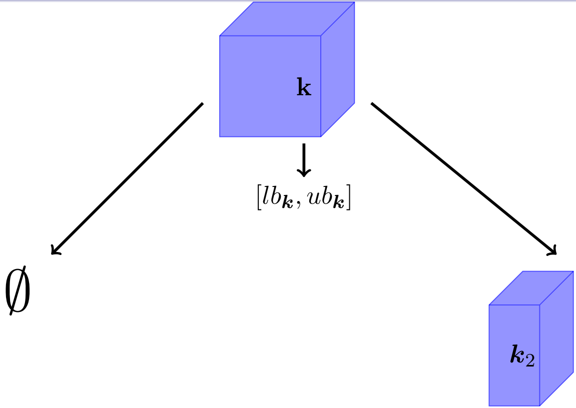
Contraction: remove unstable parameters

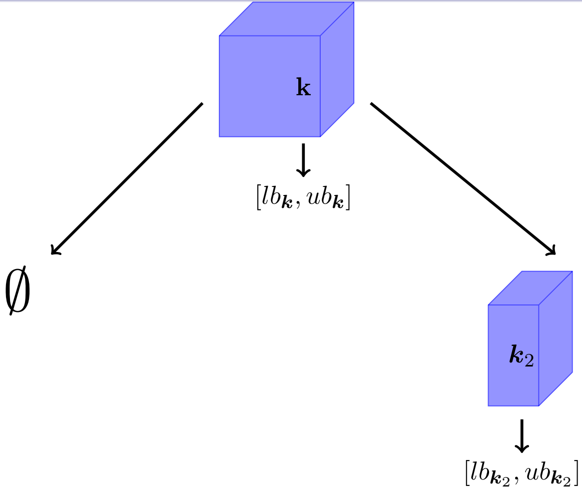


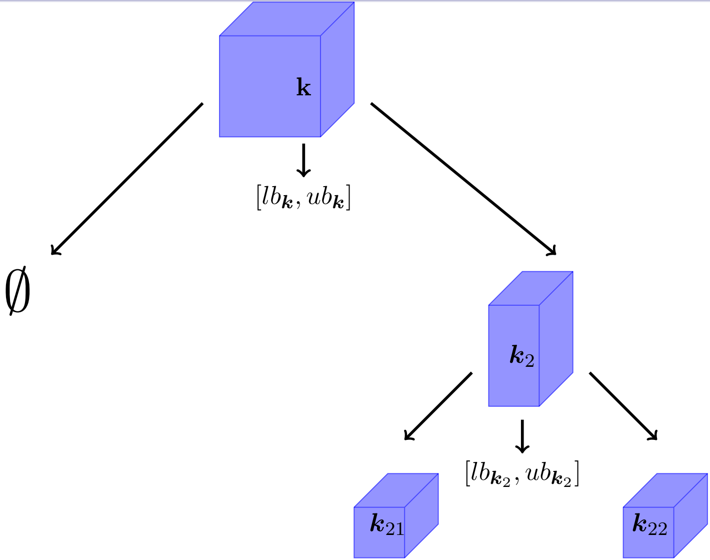
$[lb_{\boldsymbol{k}}, ub_{\boldsymbol{k}}]$

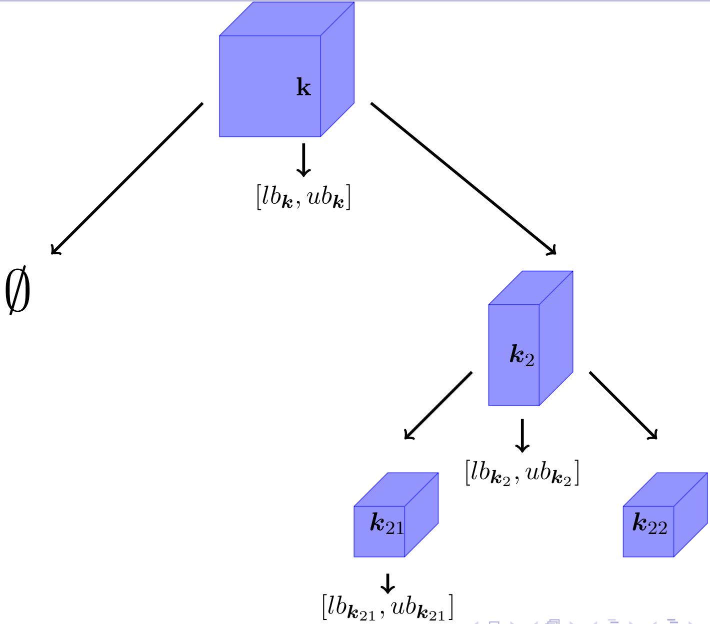

$\mathbf{k}$
$[lb_{\boldsymbol{k}}, ub_{\boldsymbol{k}}]$
$\boldsymbol{k}_1$
$\boldsymbol{k}_2$


k
[lb_k, ub_k]
k_1
k_2

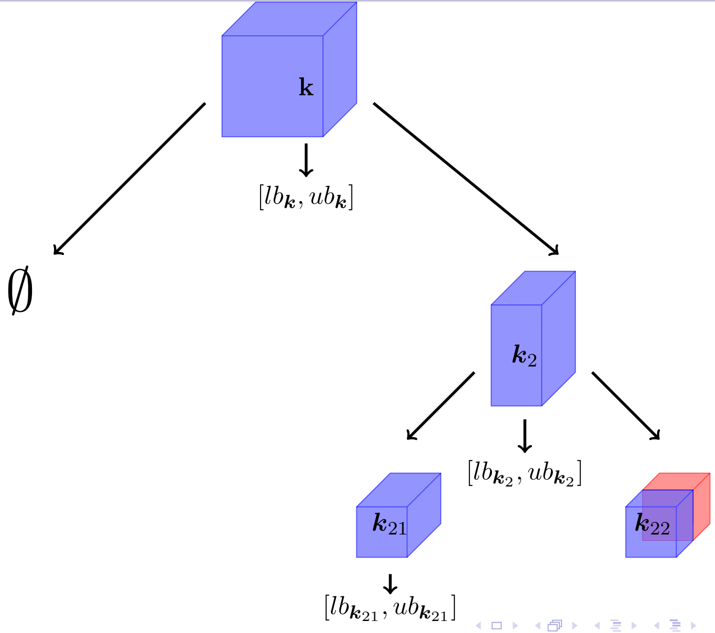

k
$[lb_{\boldsymbol{k}}, ub_{\boldsymbol{k}}]$
$\emptyset$
$\boldsymbol{k}_2$

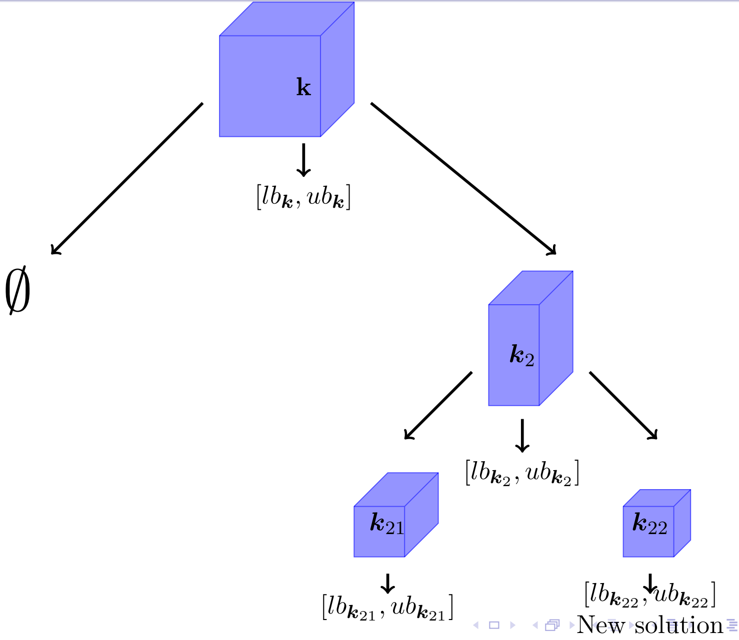

k
$[lb_{\boldsymbol{k}}, ub_{\boldsymbol{k}}]$
$\emptyset$
$\boldsymbol{k}_2$
$[lb_{\boldsymbol{k}_2}, ub_{\boldsymbol{k}_2}]$

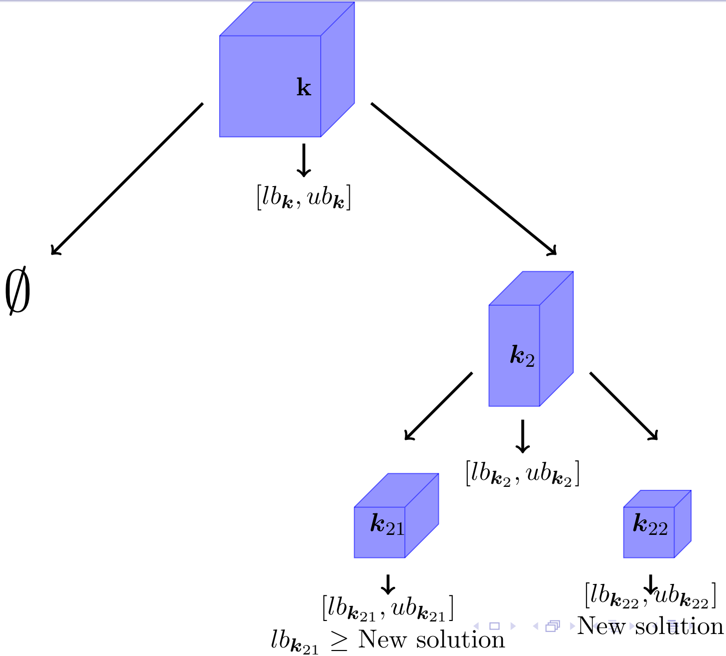

$\mathbf{k}$
$[lb_{\boldsymbol{k}}, ub_{\boldsymbol{k}}]$
$\emptyset$
$\boldsymbol{k}_2$
$[lb_{\boldsymbol{k}_2}, ub_{\boldsymbol{k}_2}]$
$\boldsymbol{k}_{21}$
$\boldsymbol{k}_{22}$



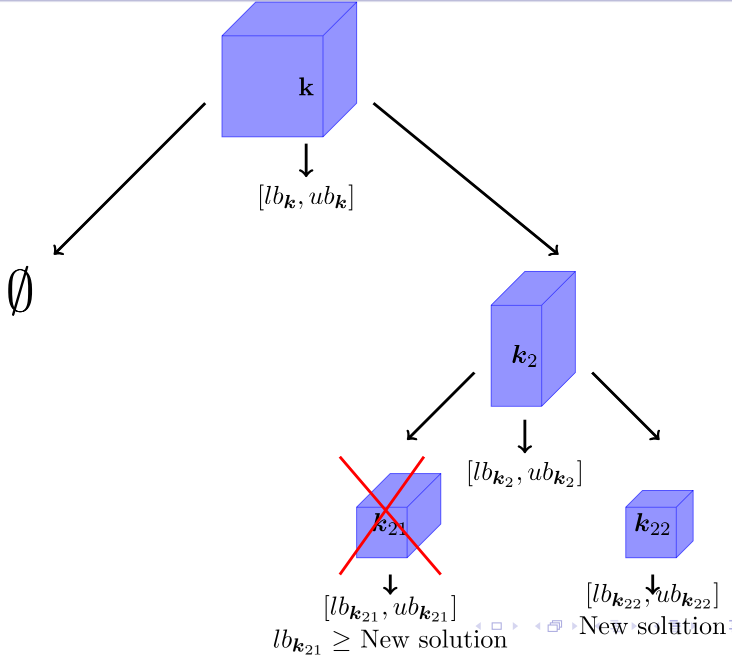
k
$[lb_{\boldsymbol{k}}, ub_{\boldsymbol{k}}]$
$\emptyset$
$\boldsymbol{k}_2$
$[lb_{\boldsymbol{k}_2}, ub_{\boldsymbol{k}_2}]$
$\boldsymbol{k}_{21}$
$\boldsymbol{k}_{22}$
$[lb_{\boldsymbol{k}_{21}}, ub_{\boldsymbol{k}_{21}}]$


k
$[lb_{\boldsymbol{k}}, ub_{\boldsymbol{k}}]$
$\emptyset$
$\boldsymbol{k}_2$
$[lb_{\boldsymbol{k}_2}, ub_{\boldsymbol{k}_2}]$
$\boldsymbol{k}_{21}$
$\boldsymbol{k}_{22}$
$[lb_{\boldsymbol{k}_{21}}, ub_{\boldsymbol{k}_{21}}]$


k
$[lb_{\boldsymbol{k}}, ub_{\boldsymbol{k}}]$
$\emptyset$
$\boldsymbol{k}_2$
$[lb_{\boldsymbol{k}_2}, ub_{\boldsymbol{k}_2}]$
$\boldsymbol{k}_{21}$
$\boldsymbol{k}_{22}$
$[lb_{\boldsymbol{k}_{21}}, ub_{\boldsymbol{k}_{21}}]$
$[lb_{\boldsymbol{k}_{22}}, ub_{\boldsymbol{k}_{22}}]$
New solution



k
$[lb_{\boldsymbol{k}}, ub_{\boldsymbol{k}}]$
$\emptyset$
$\boldsymbol{k}_2$
$[lb_{\boldsymbol{k}_2}, ub_{\boldsymbol{k}_2}]$
$\boldsymbol{k}_{21}$
$\boldsymbol{k}_{22}$
$[lb_{\boldsymbol{k}_{21}}, ub_{\boldsymbol{k}_{21}}]$
$lb_{\boldsymbol{k}_{21}} \geq$ New solution
$[lb_{\boldsymbol{k}_{22}}, ub_{\boldsymbol{k}_{22}}]$
New solution


k
$[lb_{\boldsymbol{k}}, ub_{\boldsymbol{k}}]$
$\emptyset$
$\boldsymbol{k}_2$
$[lb_{\boldsymbol{k}_2}, ub_{\boldsymbol{k}_2}]$
$\boldsymbol{k}_{21}$
$[lb_{\boldsymbol{k}_{21}}, ub_{\boldsymbol{k}_{21}}]$
$lb_{\boldsymbol{k}_{21}} \geq$ New solution
$\boldsymbol{k}_{22}$
$[lb_{\boldsymbol{k}_{22}}, ub_{\boldsymbol{k}_{22}}]$
New solution

Plan

1. Problem
2. Global Optimization resolution
3. Example
4. Conclusion and next steps

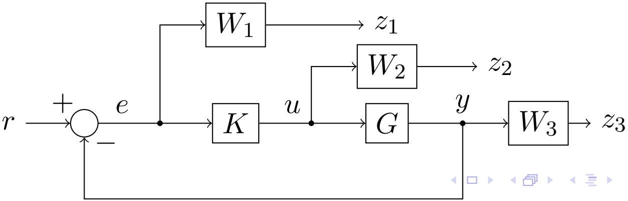
Problem under study

We consider a second-order system controlled by a PID, with contraints on error and control signals:

$$G(s) = \frac{1}{s^2 + 1.4s + 1}, \quad K(k,s) = k_p + \frac{k_i}{s} + \frac{k_d s}{1+s}$$

$$W_1(s) = \frac{10s + 100}{1000s + 1}, \qquad W_2(s) = \frac{10s + 1}{s + 10} \qquad W_3(s) = \frac{100s + 1}{s + 10}$$

Initial set of coefficient: $\mathbb{K} = \begin{pmatrix} [-10, 10] \\ [-10, 10] \\ [-10, 10] \end{pmatrix}$, $[\omega] = [10^{-3}, 10^3]$



Results

Method	Cpu (s)	$\|\|F(P,K)\|\|_\infty$	$\max_i(\|\|T_{w\to z_i}\|\|_\infty)$
H_∞ full (Matlab)	2	1.0258	1.01611
H_∞ structured (Matlab)	99, 400 rs	1.0411	1.04108
GO structured	83	1.0811	0.99782

- H_∞ full controller:

$$\frac{0.8356s^4 + 17.88s^3 + 107.8s^2 + 133.7s + 83.56}{s^5 + 22.73s^4 + 175.1s^3 + 564.3s^2 + 858.4s + 0.8578}$$

- H_∞ structured:

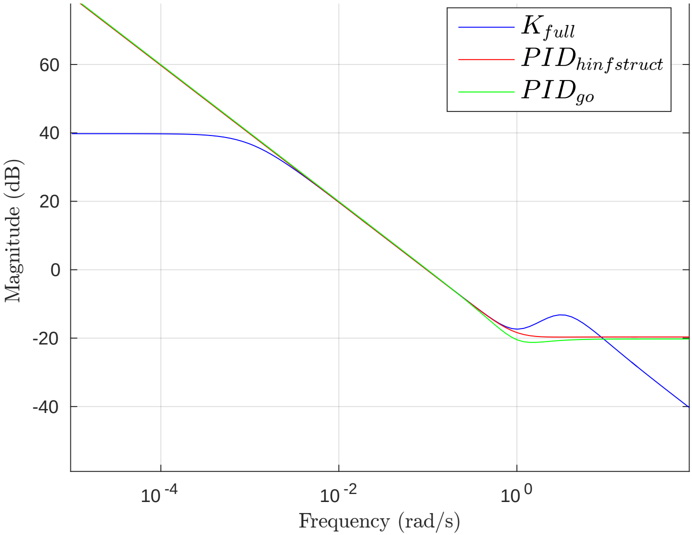
$$0.0736 + \frac{0.0969}{s} + \frac{0.0305s}{1+s}$$

- GO structured:

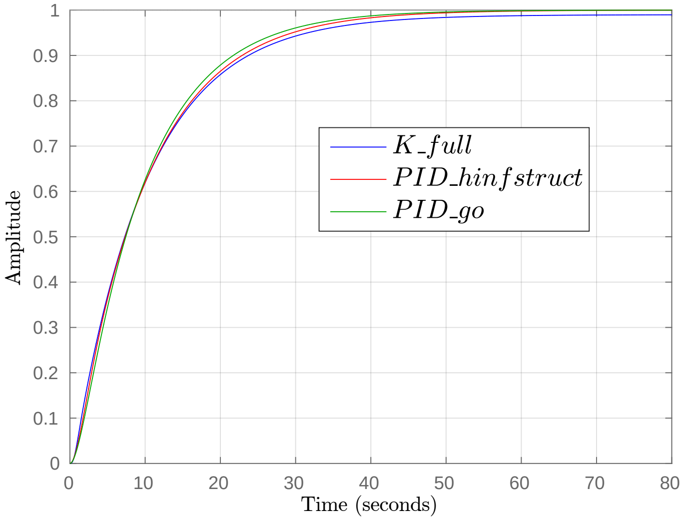
$$0.0348 + \frac{0.0993}{s} + \frac{0.0625}{1+s}$$

Moreover, we guarantee: $\max_i(||T_{w\to z_i}||_\infty) \in [0.905531, 0.997827]$

Bode diagram of controllers

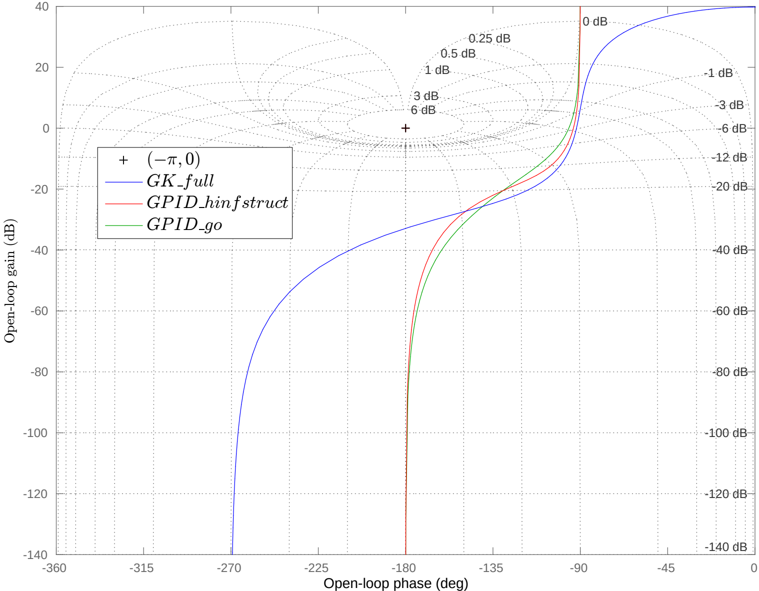


Simulation

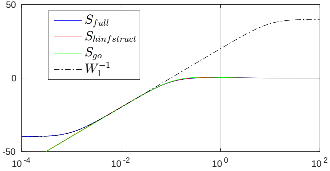


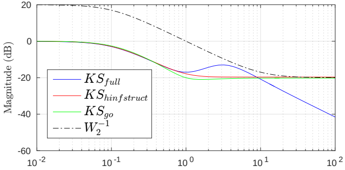
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Amplitude
0
10
20
30
40
50
60
70
80
Time (seconds)
K_full
PID_hinfstruct
PID_go

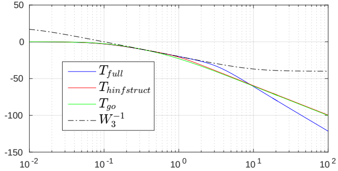
Nichols chart, open loop study



Simulation


50
0
-50
S_{full}
$S_{hinfstruct}$
S_{go}
W_1^{-1}
10^{-4}
10^{-2}
10^{0}
10^{2}


Magnitude (dB)
20
0
-20
-40
-60
KS_{full}
$KS_{hinfstruct}$
KS_{go}
W_2^{-1}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}


50
0
-50
-100
-150
T_{full}
$T_{hinfstruct}$
T_{go}
W_3^{-1}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}

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Conclusion

- Our approach is not sensitive to coupled constraints.
- We can compute a guaranteed enclosure of $\max_i(||T_{w\to z_i}||_\infty)$, and therefore prove the existence or not of a solution to $\max_i(||T_{w\to z_i}||_\infty) \leq 1$.
- We are able to compute solutions that classical methods cannot possibly find.
- Our method is suited for small order controller: the complexity of our algorithm grows exponentially with the number of free parameters.

Outlooks and future works

- Take parametric uncertainties into account: $G(p, s) \to G(\boldsymbol{p}, s)$.
- Time-domain constraints.
- Criteria on stability margins.
- H_∞ constraints/frequency-domain constraint on one input/one output channel $T_{w_i \to z_j}$.
- Weighting functions $W_i(s)$ are not limited to rational functions.