Synthèse de correcteurs par retour d'état observé robustes

pour les systèmes à temps discret rationnels en les incertitudes

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Séminaire MOSAR, 16 mars 2016

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Extends results from two papers presented at 19th IFAC World Congress (Cape Town).

Results submitted to Automatica: hal.archives-ouvertes.fr/hal-01225068v1

■ Literature full of robust state-feedback design results, few for robust observer design

■ Output filter $\neq$ Observer   (no open loop stability assumption)

■ Observers of states in given state-space + assuming MIMO systems

    i.e. not restricted to SISO systems in canonical form (integrators in series)

$$\dot{x} = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ & & & 1 \\ 0 & \cdots & & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} (f(x,\theta) + g(x,\theta)u)$$

■ Discrete-time linear system with uncertainties

$$x_{k+1} = A_r(\theta)x_k + B_r(\theta)u_k, \qquad y_k = Cx_k$$

■ Luenberger-like observer $\hat{x}_{k+1} = A_o\hat{x}_k + B_ou_k + L(C\hat{x}_k - y_k)$

■ Observed-state feedback $u_k = K\hat{x}_k$

● Our goals:

▲ Build a separation-like heuristic with first, K design, then, A_o, B_o, L design

▲ Use up-to-date SV-LMI tools

▲ For systems rational in the uncertainties θ

■ Closed-loop dynamics (state x and error $e = x - \hat{x}$) driven by the state matrix

$$\begin{pmatrix} x_{k+1} \\ e_{k+1} \end{pmatrix} = \begin{bmatrix} A_r(\theta) + B_r(\theta)K & -B_r(\theta)K \\ \Delta_A(\theta) + \Delta_B(\theta)K & A_o + LC - \Delta_B(\theta)K \end{bmatrix} \begin{pmatrix} x_k \\ e_k \end{pmatrix}$$

where $\Delta_A(\theta) = A_r(\theta) - A_o$ and $\Delta_B(\theta) = B_r(\theta) - B_o$.

● Separation obtained when $\Delta_A(\theta) = 0$ and $\Delta_B(\theta) = 0$

$$\begin{pmatrix} x_{k+1} \\ e_{k+1} \end{pmatrix} = \begin{bmatrix} A_r(\theta) + B_r(\theta)K & -B_r(\theta)K \\ 0 & A_o + LC \end{bmatrix} \begin{pmatrix} x_k \\ e_k \end{pmatrix}$$

▲ Impossible when θ are uncertainties

(Notion of observed-state not quite defined for uncertain systems)

■ Closed-loop dynamics (state x and error $e = x - \hat{x}$) driven by the state matrix

$$\begin{pmatrix} x_{k+1} \\ e_{k+1} \end{pmatrix} = \begin{bmatrix} A_r(\theta) + B_r(\theta)K & -B_r(\theta)K \\ \Delta_A(\theta) + \Delta_B(\theta)K & A_o + LC - \Delta_B(\theta)K \end{bmatrix} \begin{pmatrix} x_k \\ e_k \end{pmatrix}$$

where $\Delta_A(\theta) = A_r(\theta) - A_o$ and $\Delta_B(\theta) = B_r(\theta) - B_o$.

▲ Choices from the literature: $A_o = A_r(\theta_{nom})$, but why?

▲ Possible choice $\min_{A_o} \max_{\theta} \|A_r(\theta) - A_o\|$, but what properties?

● Our choice: optimize the input/output performances of

$$e_{k+1} = (A_o + LC - \Delta_B(\theta)K)e_k + (\Delta_A(\theta) + \Delta_B(\theta)K)x_k, \ \ \epsilon_k = Ke_k$$

where x_k is treated as the input and ϵ_k is the output.

❶ Descriptor multi-affine modeling of rational systems

❷ LMI results for robust design and robust analysis

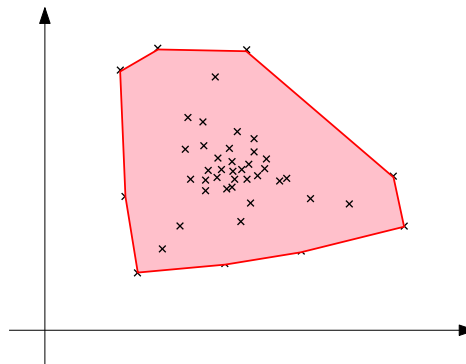❸ Observed-state feedback design heuristic

❹ Example

■ $\bar{p}$ independent uncertain vectors $\theta_p \in \mathbb{R}^{m_p}$ indexed by $p = 1 \cdots \bar{p}$

$$\theta \in \Theta = \{(\theta_1, \ldots, \theta_{\bar{p}}) \in \Theta_1 \times \ldots \times \Theta_{\bar{p}}\}.$$

■ Each θ_p in a polytope with $\bar{v}_p$ vertices $\mathcal{V}_p = \left\{ \theta_p^{[1]}, \ldots, \theta_p^{[\bar{v}_p]} \right\}$

$$\Theta_p = Co(\mathcal{V}_p) = \left\{ \theta_p = \sum_{v=1}^{\bar{v}_p} \xi_{p,v} \theta_p^{[v]} \; : \; \xi_{p,v} \geq 0, \; \sum_{v=1}^{\bar{v}_p} \xi_{p,v} = 1 \right\}.$$

● Example: scalar uncertainty in an interval: $\theta_p \in [\, \theta_p^{[1]}, \, \theta_p^{[2]} \,]$.

● Example: 2D vector in convex hull of points issued from identification process

■ Multi-affine matrices: affine in each θ_p

● Example for two scalar uncertainties $\theta_1 \in [\theta_1^{[1]}, \theta_1^{[2]}], \theta_2 \in [\theta_2^{[1]}, \theta_2^{[2]}]$

$$1 + \theta_1 + \theta_1\theta_2 = \begin{array}{l} \xi_{1,1}\xi_{2,1}(1 + \theta_1^{[1]} + \theta_1^{[1]}\theta_2^{[1]}) \\ +\xi_{1,1}\xi_{2,2}(1 + \theta_1^{[1]} + \theta_1^{[1]}\theta_2^{[2]}) \\ +\xi_{1,2}\xi_{2,1}(1 + \theta_1^{[2]} + \theta_1^{[2]}\theta_2^{[1]}) \\ +\xi_{1,2}\xi_{2,2}(1 + \theta_1^{[2]} + \theta_1^{[2]}\theta_2^{[2]}). \end{array}$$

▲ Not the same as the convex hull of all possible vertices

● Example: $\begin{bmatrix} \theta_1 & \theta_1\theta_2 & \theta_2 \end{bmatrix}$ with $\theta_1 \in [1, 2]$ and $\theta_2 \in [1, 2]$.

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2 & 4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{5}{2} & \frac{3}{2} \end{bmatrix} \neq \begin{bmatrix} \frac{3}{2} & \frac{9}{4} & \frac{3}{2} \end{bmatrix}$$

■ Any matrix rational in θ admits a descriptor multi-affine representation (DMAR)

$$R(\theta) = M_1(\theta)M_2^{-1}(\theta)M_3(\theta)$$

where $M_1(\theta)$, $M_2(\theta)$, $M_3(\theta)$ are multi-affine in θ.

● Alternative to linear-fractional representations

● Usually of smaller size, and easier to build

● Example:

$$\begin{bmatrix} \frac{\theta_1}{1+\theta_2} & \theta_1{}^2\theta_2 \\ \frac{1}{\theta_1} & 0 \end{bmatrix} = \begin{bmatrix} \theta_1 & \theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1+\theta_2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \theta_1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \theta_1\theta_2 \\ 0 & 1 \end{bmatrix}.$$

■ Discrete-time linear system, with performance I/O, rational in the uncertainties

$$x_{k+1} = A_r(\theta)x_k + B_{rw}(\theta)w_k$$

$$z_k = C_{rz}(\theta)x_k + D_{rzw}(\theta)w_k$$

● The DMAR

$$\begin{bmatrix} A_r(\theta) & B_{rw}(\theta) \\ C_{rz}(\theta) & D_{rzw}(\theta) \end{bmatrix} = \begin{bmatrix} E_x(\theta) \\ E_z(\theta) \end{bmatrix} E_\pi^{-1}(\theta) \begin{bmatrix} A(\theta) & B_w(\theta) \end{bmatrix}$$

■ gives the following descriptor multi-affine representation of the system

$$\begin{bmatrix} I & 0 & E_x(\theta) & 0 & 0 \\ 0 & I & E_z(\theta) & 0 & 0 \\ 0 & 0 & E_\pi(\theta) & A(\theta) & B_w(\theta) \end{bmatrix} \begin{pmatrix} x_{k+1} \\ z_k \\ \pi_k \\ x_k \\ w_k \end{pmatrix} = E(\theta)\eta_k = 0$$

▲ π_k: exogenous vector $= E_\pi^{-1}(\theta)(A(\theta)x_k + B_w(\theta)w_k)$

■ If there exists $P^{[v]} = P^{[v]T} \succ 0$, S and μ^2 such that for all vertices $\theta^{[v]} \in \mathcal{V}$

$$\text{diag} \begin{bmatrix} P^{[v]} & I & 0 & -P^{[v]} & -\mu^2 I \end{bmatrix} \prec (SE(\theta^{[v]})) + (SE(\theta^{[v]}))^T$$

then the system is robustly stable (i.e. $\forall \theta \in \Theta$) with robust H_∞ performance μ.

● Proof - step 1 - By convexity the condition holds for all $\theta \in \Theta$:

$$\text{diag} \begin{bmatrix} P(\theta) & I & 0 & -P(\theta) & -\mu^2 I \end{bmatrix} \prec (SE(\theta)) + (SE(\theta))^T$$

with multi-affine Lyapunov matrix $P(\theta) \succ 0$.

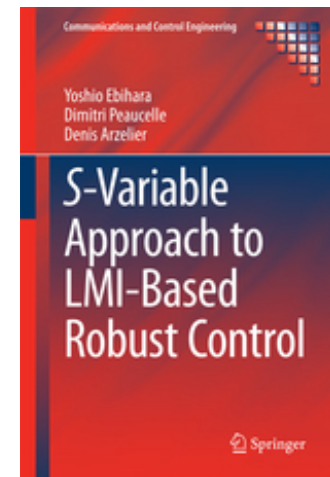● Proof - step 2 - Since $E(\theta)\eta_k = 0$ one gets

$$\eta_k^T \text{diag} \begin{bmatrix} P(\theta) & I & 0 & -P(\theta) & -\mu^2 I \end{bmatrix} \eta_k$$
$$= x_{k+1}^T P(\theta) x_{k+1} + z_k^T z_k - x_k^T P(\theta) x_k - \mu^2 w_k^T w_k < 0$$

■ If there exists $P^{[v]} = P^{[v]T} \succ 0$, S and μ^2 such that for all vertices $\theta^{[v]} \in \mathcal{V}$

$$\mathrm{diag} \begin{bmatrix} P^{[v]} & I & 0 & -P^{[v]} & -\mu^2 I \end{bmatrix} \prec (SE(\theta^{[v]})) + (SE(\theta^{[v]}))^T$$

then the system is robustly stable (i.e. $\forall \theta \in \Theta$) with robust H_∞ performance μ.

● S-variable result

● Extended in present work to multi-affine representations

● Exist tools to reduce numerical burden (sometimes lossless)

▲ Example: no S if plant is multi-affine in θ & common $P = P(\theta)$

● Extensions to mixed constant/time-varying uncertainties

Communications and Control Engineering

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Denis Arzelier

S-Variable
Approach to
LMI-Based
Robust Control

Springer

■ SV-LMI for robust state-feedback design

If there exist $P_d^{[v]} \succ 0$, S_{dx}, S_{dy}, $S_{d\pi}$ such that LMIs $\mathcal{L}_{sf}(\theta^{[v]})$ hold for all $\theta^{[v]} \in \mathcal{V}$ then $K = S_{dy}{}^T (S_{dx}{}^T)^{-1}$ is a robustly stabilizing state-feedback gain s.t.

$$
\begin{aligned}
x_{k+1} &= A_r(\theta)x_k + B_r(\theta)u_k + B_{rw}(\theta)w_k \\
z_k &= C_{rz}(\theta)x_k + D_{rzu}(\theta)u_k + D_{rzw}(\theta)w_k
\end{aligned}
\quad , \quad u_k = Kx_k
$$

has an H_∞ performance smaller than μ_d whatever $\theta \in \Theta$.

- Linearizing change of variables on S-variables
- Proof uses equivalence with dual system $x_{d,k+1} = A_r^T(\theta)x_{d,k} + \ldots$
- Result is new because for rational systems
- Easy extensions for regional pole location, H_2 performance, etc.

■ **SV-LMI for analysis of state trajectories under fixed state-feedback** $K = K$

If there exist $P^{[v]} \succ 0$, Q and S such that LMIs $\mathcal{L}_{sf,a}(\theta^{[v]})$ hold for all $\theta^{[v]} \in \mathcal{V}$, then

$$x_{k+1} = A_r(\theta)x_k + B_r(\theta)u_k \; , \;\; u_k = Kx_k + \epsilon_k$$

is robustly stable and x_k is bounded for bounded control errors ϵ_k:

$$\|Wx\|_2 \leq \|\epsilon\|_2 \text{ where } W = Q^{1/2}.$$

● Allow to estimate the state trajectories in case of corrupted state-feedback

(inevitable when feedback is with observed-state)

🟥 **SV-LMI for robust observer design under fixed $K = K$**

and expected state trajectories $W = W$

If there exist $P_\infty^{[v]} \succ 0$, $P_p^{[v]} \succeq K^T K$, S_x, S_a, S_b, S_l, $S_{2\pi}$, $S_{p\pi}$ such that LMIs $\mathcal{L}_{ob}(\theta^{[v]})$ hold for all $\theta^{[v]} \in \mathcal{V}$, then $A_o = S_x^{-1} S_a$, $B_o = S_x^{-1} S_b$, $L = S_x^{-1} S_l$ define an observer that guarantees:

$$\|\epsilon\|_2 \leq \gamma_2 \|Wx\|_2 \;,\;\; \|\epsilon\|_p \leq \gamma_p \|Wx\|_2$$

where $\epsilon_k = K e_k$. The properties hold whatever bounded x and whatever $\theta \in \Theta$.

🔵 Norm-to-norm perf: asymptotic coupling of observation error on system dynamics
🔵 Norm-to-peak perf: avoid waterbed effects of transient peaks

🔵 Small gain theorem: if $\gamma_2 < 1$ observed-state feedback robustly stabilizes

■ SV-LMI for observed-state feedback analysis under fixed $K = K$, $A_o = A_o$ etc.

If there exist $P_c^{[v]} \succ 0$, S_c such that LMIs $\mathcal{L}_{ob,a}(\theta^{[v]})$ hold for all $\theta^{[v]} \in \mathcal{V}$, then

$$x_{k+1} = A_r(\theta)x_k + B_r(\theta)u_k + B_{rw}(\theta)w_k$$

$$z_k = C_{rz}(\theta)x_k + D_{rzu}(\theta)u_k + D_{rzw}(\theta)w_k$$

$$\hat{x}_{k+1} = A_o\hat{x}_k + B_o u_k + L(C\hat{x}_k - y_k), \quad u_k = K\hat{x}_k$$

has an H_∞ performance smaller than μ_c whatever $\theta \in \Theta$.

● Exists also SV-LMI conditions $\mathcal{L}_{ob,da}$ for the dual system: μ_{dc} upper-bound

● No apriori relation between upper-bounds μ_d (ideal state-feedback), μ_c and μ_{dc}

■ 4 steps

● 1- Design stabilizing state-feedback K (for example using LMI $\mathcal{L}_{sf}$)

● 2- Get estimate of state trajectories represented by W (using LMIs $\mathcal{L}_{sf,a}$)

$$(\max \lambda_{\min}(W^T W) \text{ leads to tight estimates})$$

● 3- Design observer (using LMIs $\mathcal{L}_{ob}$)

$(\min \beta_2 {\gamma_2}^2 + \beta_p {\gamma_p}^2$ to adjust tradeoff between norm and peak performances)

● 4- Analyze observed-state closed-loop (using LMIs $\mathcal{L}_{ob,a}$ or $\mathcal{L}_{ob,da}$)

▲ No guarantee that next step would be feasible

▲ No guarantee to find a robustly stabilizing control when exists

▲ All step are purely LMI with clear control theory justification

▲ Each step based on new LMI conditions

■ Academic example for illustration

$$x_{k+1} = \begin{bmatrix} -\theta_1{}^2/\theta_2 & -\theta_1 \\ 1 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ \theta_2 \end{bmatrix} u_k + \begin{bmatrix} \theta_1 \\ 0 \end{bmatrix} w_k$$

$$z_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + \theta_2 u_k, \qquad y_k = \begin{bmatrix} 0 & 1 \end{bmatrix} x_k$$

● DMAR

$$E_x = \begin{bmatrix} \theta_1 & 0 \\ 0 & 1 \end{bmatrix}, \; E_z = \begin{bmatrix} 0 & 1 \end{bmatrix}, \; E_\pi = \begin{bmatrix} \theta_2 & 0 \\ 0 & 1 \end{bmatrix},$$

$$A = \begin{bmatrix} -\theta_1 & -\theta_2 \\ 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ \theta_2 \end{bmatrix}, \; B_w = \begin{bmatrix} \theta_2 \\ 0 \end{bmatrix}.$$

● $\theta_1 \in [\,1-\delta_1,\, 1+\delta_1\,], \theta_2 \in [\,1-\delta_2,\, 1+\delta_2\,]$ at limit of stability when $\theta_1 = \theta_2 = 1$

● Results when $\mu_d = 10$ at first step

(δ_1, δ_2)	(β_2, β_p)	(γ_2, γ_p)	μ_c	μ_{dc}
$(0, 0)$	$(1, 1)$	$(10^{-4}, 10^{-4})$	2.8152	2.8152
$(0.1, 0)$	$(1, 1)$	$(1.0747, 1.0410)$	2.6672	2.6672
$(0, 0.1)$	$(1, 1)$	$(0.3947, 0.3524)$	2.4546	2.4546
$(0.1, 0.1)$	$(10, 1)$	$(1.2736, 1.1765)$	6.5901	6.5901
$(0.1, 0.1)$	$(1, 1)$	$(1.2962, 1.0985)$	6.1252	6.1252
$(0.1, 0.1)$	$(1, 10)$	$(1.3339, 1.0809)$	5.3226	5.3226
$(0.2, 0.1)$	$(1, 1)$	$(1.3006, 1.2181)$	11.2285	11.2285
$(0.1, 0.2)$	$(1, 1)$	$(1.3242, 1.1553)$	6.8505	6.8505
$(0.2, 0.2)$	$(1, 1)$	$(3.5392, 3.0228)$	∞	14.3142

⬤ Results when μ_d is minimized at first step and $(\beta_2, \beta_p) = (1,1)$.

(δ_1, δ_2)	μ_d	$\tilde{\mu}_d$	(γ_2, γ_p)	μ_c	μ_{dc}
$(0,0)$	1	1	$(10^{-4}, 10^{-4})$	3.3731	3.3731
$(0.1, 0)$	1.5303	1.5303	$(0.2349, 0.2349)$	2.8013	2.8013
$(0, 0.1)$	1.2497	1.2502	$(0.3061, 0.2777)$	2.5644	2.5644
$(0.1, 0.1)$	2.0284	2.0284	$(0.5611, 0.5181)$	4.3398	4.3398
$(0.2, 0.1)$	3.8506	3.8506	$(1.1902, 1.0970)$	10.1302	10.1302
$(0.1, 0.2)$	3.1161	3.1161	$(1.3334, 1.1567)$	7.7934	7.7934
$(0.2, 0.2)$	8.7776	8.7776	$(3.0445, 2.5166)$	∞	∞

🔺 $\tilde{\mu}_d$ computed on SV-LMI with reduced size

● Size of LMI conditions (applying non-conservative size reduction procedures)

(δ_1, δ_2)	$\mathcal{L}_{sf}$	$\tilde{\mathcal{L}}_{sf}$	$\mathcal{L}_{sf,a}$	$\mathcal{L}_{ob}$	$\mathcal{L}_{ob,a}$	$\mathcal{L}_{ob,da}$
$(0,0)$	$\{5,0\}$	$\{5,0\}$	$\{3,0\}$	$\{6,0\}$	$\{5,0\}$	$\{5,0\}$
$(1,0)$	$\{6,1\}$	$\{6,1\}$	$\{5,2\}$	$\{7,1\}$	$\{7,2\}$	$\{7,2\}$
$(0,1)$	$\{7,2\}$	$\{6,1\}$	$\{5,2\}$	$\{8,2\}$	$\{7,2\}$	$\{10,5\}$
$(1,1)$	$\{7,2\}$	$\{6,1\}$	$\{6,3\}$	$\{8,2\}$	$\{8,3\}$	$\{11,6\}$

▲ $\{N_r, N_c\}$ with N_r rows in each LMI (to be multiplied by nb of vertices)

▲ $\{N_r, N_c\}$ with N_c columns of S-variables

■ System model

$$
x_{k+1} = \begin{bmatrix} -{\theta_1}^2/\theta_2 & -\theta_1 \\ 1 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ \theta_2 \end{bmatrix} u_k + \begin{bmatrix} \theta_1 \\ 0 \end{bmatrix} w_k
$$

● Observer model (design for $\delta_1 = \delta_2 = 0.2$)

$$
A_o = \begin{bmatrix} -0.7832 & -1.1081 \\ 1.1373 & 0.9274 \end{bmatrix}, \quad B_o = \begin{bmatrix} 0.1425 \\ 0.2195 \end{bmatrix}
$$

■ Revisited Luenberger observer design in case of uncertain systems

● Separation principle replaced by a mixed norm/peak performance measure

● LMI design of state-feedback and observer gains one after the other

● Heuristic with no guarantee of success

▲ Surprisingly, K design for fixed observer is more complex (prospective work)

▲ Extensions for continuous-time systems:

  raises issues about S-variable conditions for design (tuning parameters)

■ Promising descriptor multi-affine representation combined to SV-LMIs