Incremental stability of piecewise affine systems

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Context

Since the 90s $\rightarrow$ important theoretical and methodological developments in control theory

- Emergence of robust control methods
- Appearance of efficient solvers $\rightarrow$ optimization problems

Systematically tackle a large number of engineering specifications for linear systems

Tight specifications $\rightarrow$ non negligible nonlinear effects

Engineering expertise (heuristics) $\rightarrow$ no *a priori* guarantees

Need to develop efficient methods for nonlinear **performance analysis**

Context

Extension of robust control to nonlinear systems

- Most of the literature concerns stability
  - $\hookrightarrow$ Not able to guarantee some qualitative specifications
- Proposal of incremental stability
- For linear systems: stability $=$ incremental stability

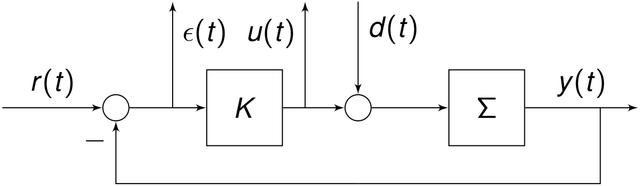
Complexity of necessary and sufficient conditions for nonlinear systems

$\hookrightarrow$ Development of efficient sufficient conditions $\rightarrow$ conservatism

Reduce conservatism $\rightarrow$ piecewise affine representations

- Describe a wide range of nonlinear system dynamics
- Similar to linear systems $\rightarrow$ extension of efficient techniques

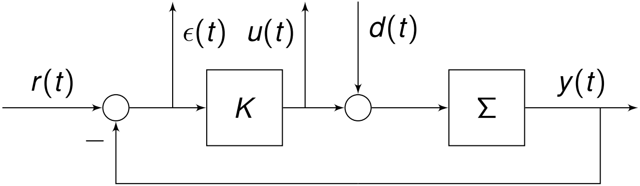
Typical control problem



Engineering specifications

- Stability
- Tracking
- Disturbance rejection
- Robustness

Typical control problem



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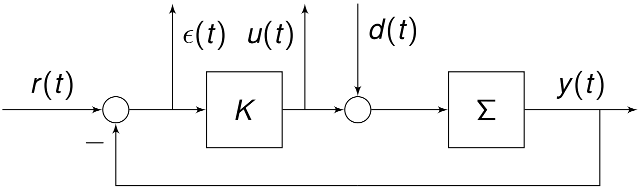

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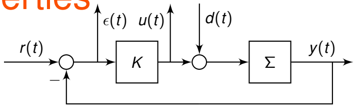
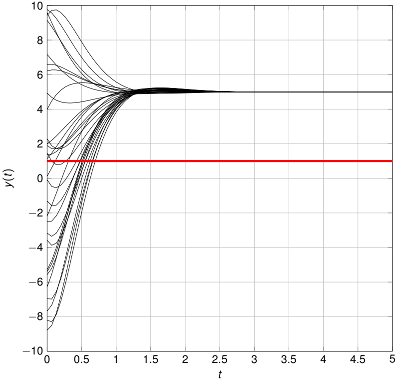
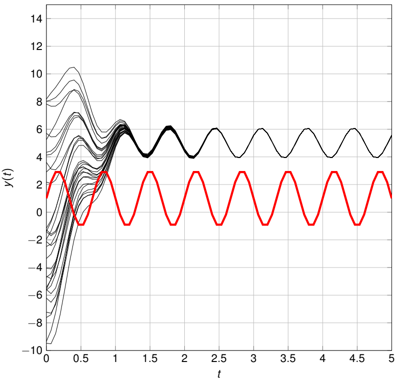
Qualitative and Quantitative properties

LTI: stability implies qualitative properties





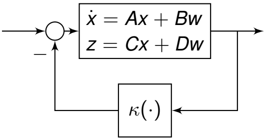
Constant response to constant input

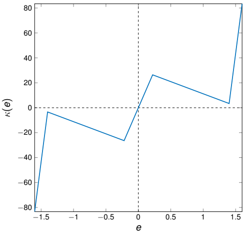


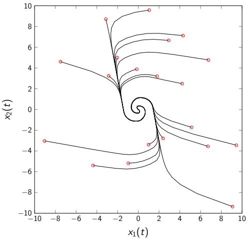
T–periodic response to T–periodic input

Independence of initial conditions

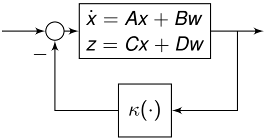
NL: Does stability imply qualitative properties?

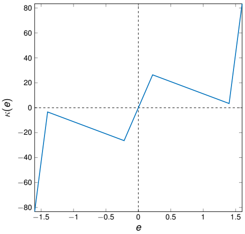


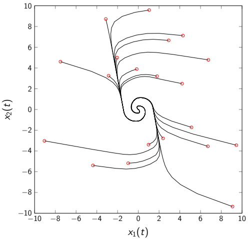




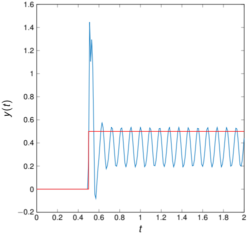
NL: Does stability imply qualitative properties?







Oscillating response to constant input



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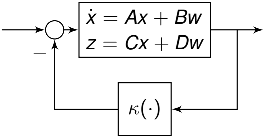
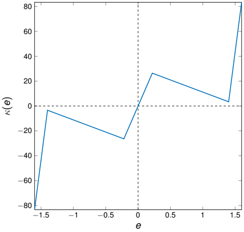
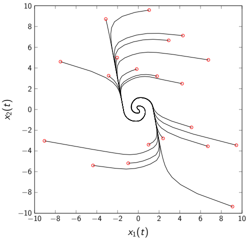
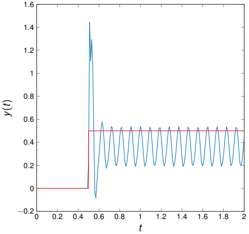






Oscillating response to constant input

Need of a stronger notion of stability



Towards nonlinear H_∞ control

LTI systems		NL systems
H_∞	$\longrightarrow$	?

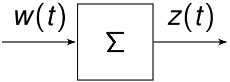
Towards nonlinear H_∞ control

LTI systems	$\longrightarrow$	NL systems
H_∞		?

$\mathcal{L}_2$-gain proposed as a natural candidate $\rightarrow$ energetic ratio between input and output

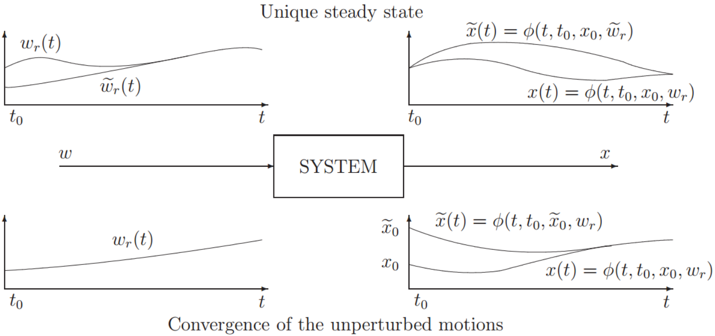
$\mathcal{L}_2$-gain

$$\exists \gamma \geq 0 \ / \ \forall w \in \mathcal{L}_2 : \qquad \int_0^{\infty} \|z(t)\|^2 \, dt \leq \gamma^2 \int_0^{\infty} \|w(t)\|^2 \, dt$$



Qualitative properties

- Constant input $\rightarrow$ constant output
- T-periodic input $\rightarrow$ T-periodic output
- Unique steady state / Convergence of the unperturbed motions



	LTI	NL
$\downarrow$ Specs $\setminus$ Norm $\rightarrow$	H_∞	$\mathcal{L}_2$-gain
Constant input $\longrightarrow$ constant output	YES	NO
T periodic input $\longrightarrow$ T periodic output	YES	NO
Unique steady state	YES	NO
Convergence of the unperturbed motions	YES	NO

$\mathcal{L}_2$-gain stability is not enough

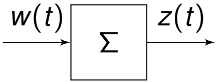
	LTI	NL	NL
↓ Specs \ Norm →	H_∞	$\mathcal{L}_2$-gain	Incremental $\mathcal{L}_2$-gain
Constant input ⟶ constant output	YES	NO	YES
T periodic input ⟶ T periodic output	YES	NO	YES
Unique steady state	YES	NO	YES
Convergence of the unperturbed motions	YES	NO	YES

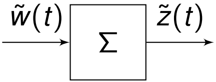
$\mathcal{L}_2$-gain stability is not enough ⟶ Incremental $\mathcal{L}_2$-gain

Incremental $\mathcal{L}_2$-gain

$\exists \eta \geq 0 \;/\; \forall w, \tilde{w} \in \mathcal{L}_2$:

$$\int_0^\infty \|z(t) - \tilde{z}(t)\|^2 \, dt \leq \eta^2 \int_0^\infty \|w(t) - \tilde{w}(t)\|^2 \, dt$$





Computation of $\mathcal{L}_2$-gain through dissipativity

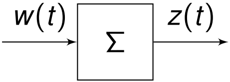
Dissipative systems

A system Σ is said to be dissipative with respect to the supply rate $s(w, z)$ if there exists a nonnegative storage function S such that

$$S(x(t_0)) + \int_{t_0}^{t_1} s(w(t), z(t))\, dt \geq S(x(t_1)), \quad \forall t_1 \geq t_0 \geq 0$$

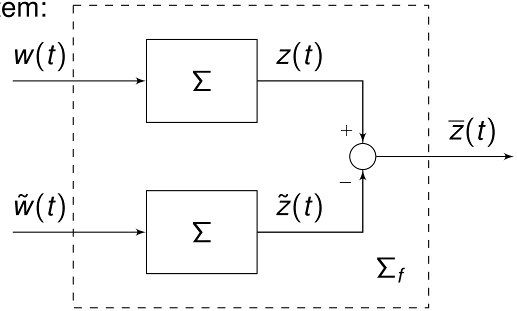
For $\mathcal{L}_2$-gain stability:

$$s(w, z) = \gamma^2 \left\| w(t) \right\|^2 - \left\| z(t) \right\|^2$$



Computation of the incremental $\mathcal{L}_2$-gain

Augmented system:



For incremental $\mathcal{L}_2$-gain: $s(w, \tilde{w}, \overline{z}) = \eta^2 \left\| w - \tilde{w} \right\|^2 - \left\| \overline{z} \right\|^2$

$$S(x_0, \tilde{x}_0) + \eta^2 \int_0^t \left\| w(\tau) - \tilde{w}(\tau) \right\|^2 d\tau - \int_0^t \left\| \overline{z}(\tau) \right\|^2 d\tau \geq S(x(t), \tilde{x}(t))$$

Finding the storage function

$\mathcal{L}_2$-gain: Find $S : \mathbb{R}^n \to \mathbb{R}_+$ such that:

$$\sup_{w \in \mathcal{L}_2} \left\{ \frac{\partial S(x)}{\partial x} \cdot f(x,w) - \gamma^2 \|w\|^2 + \|z\|^2 \right\} \leq 0$$

Incremental $\mathcal{L}_2$-gain: Find $S : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ such that:

$$\sup_{w,\tilde{w} \in \mathcal{L}_2} \left\{ \frac{\partial S(x,\tilde{x})}{\partial x} \cdot f(x,w) + \frac{\partial S(x,\tilde{x})}{\partial \tilde{x}} \cdot f(\tilde{x},\tilde{w}) - \eta^2 \|w - \tilde{w}\|^2 + \|\overline{z}\|^2 \right\} \leq 0$$

Not easy to solve in the general (nonlinear) case!

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Not easy to solve in the general (nonlinear) case!

Relaxation $\to$ Sufficient conditions $\to$ Upper bound $\to$ Conservatism

$\hookrightarrow$ Piecewise Affine (PWA) representation

PWA representation

PWA regional representation

$$\begin{cases} \dot{x}(t) = A_i x(t) + a_i + B_i w(t) \\ z(t) = C_i x(t) + c_i + D w(t) \\ x(0) = x_0 \end{cases} \quad \text{for } x(t) \in X_i$$

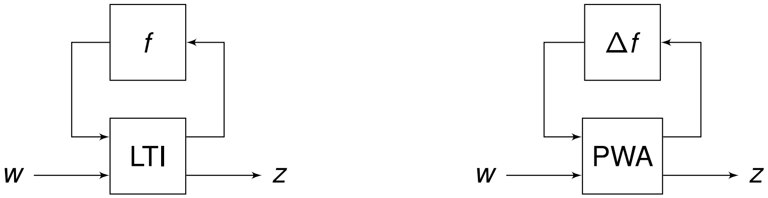
Allows us to:

- describe systems with saturations, relays, dead zones, etc.
- embed more generic nonlinear systems $\rightarrow$ differential inclusions
- assess performance with less conservatism

$\mathcal{S}$-procedure $\rightarrow$ Piecewise quadratic storage function[1]

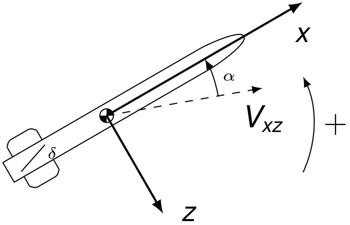
[1] M. Johansson and A. Rantzer, IEEE Trans. Autom. Control, 1998.

PWA approximations



$\hookrightarrow$ Finer description of nonlinear perturbations $+$ Piecewise quadratic storage function $\Rightarrow$ Less conservatism!

Example: Computing the $\mathcal{L}_2$-gain of NL missile



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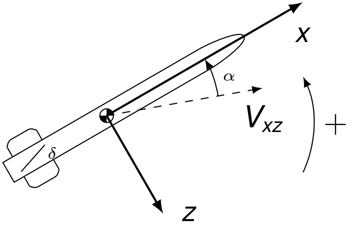

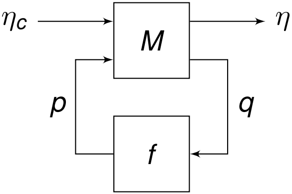

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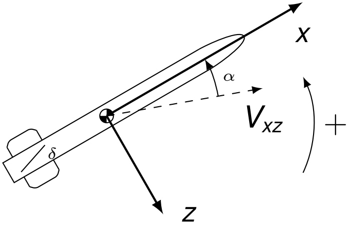

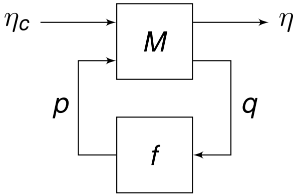

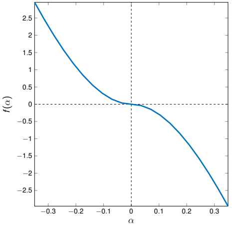

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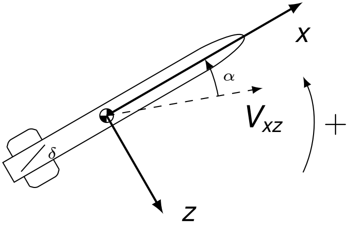

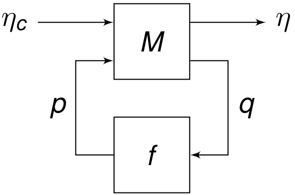

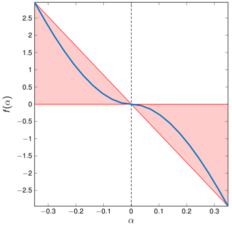

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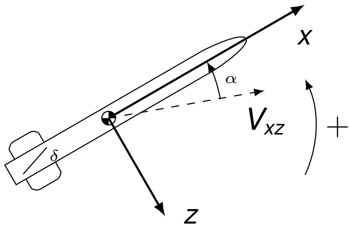

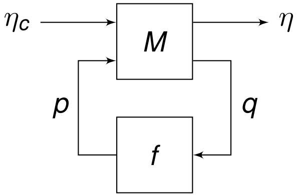

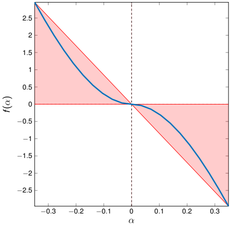

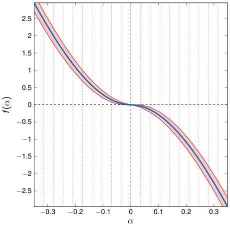

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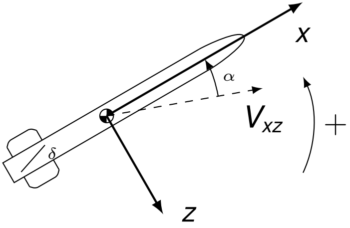

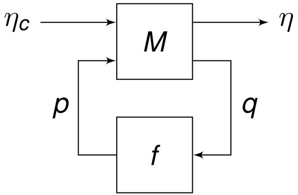

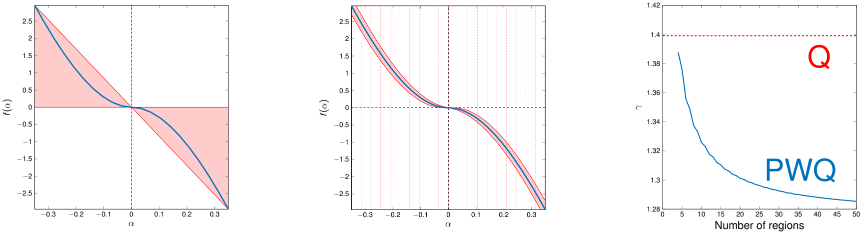

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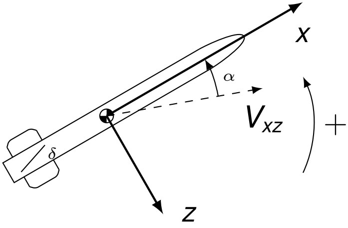
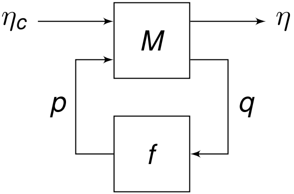
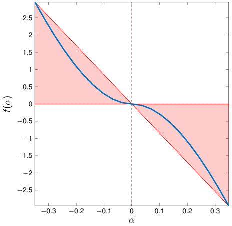
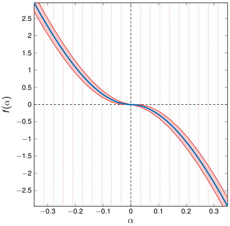
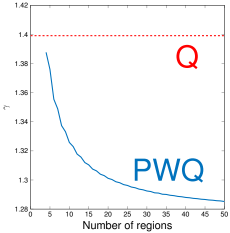










Finer computation of the upper bound to the $\mathcal{L}_2$-gain

Incremental $\mathcal{L}_2$-gain of PWA systems

- Works of Romanchuk[2] $\rightarrow$ Upper bound to the incremental $\mathcal{L}_2$-gain of PWA systems by means of a global quadratic function

$$S(x, \tilde{x}) = (x - \tilde{x})^T P (x - \tilde{x})$$

- Our proposal $\rightarrow$ Continuous piecewise quadratic storage functions

$$S(x, \tilde{x}) = \bar{x}^T P_{ij} \bar{x}, \text{for } \bar{x} \in X_{ij}$$

with $\bar{x} = \begin{bmatrix} x \\ \tilde{x} \\ 1 \end{bmatrix}$ and $X_{ij} = \{(x, \tilde{x}) \mid x \in X_i, \tilde{x} \in X_j\}$

[2] B. G. Romanchuk and M. C. Smith, Automatica, 1999.

PWA augmented system

$$\overline{y} = \Sigma_f(\overline{u}) \begin{cases} \dot{\overline{x}}(t) = \overline{A}_{ij}\overline{x}(t) + \overline{B}_{ij}\overline{u}(t) \\ \overline{y}(t) = \overline{C}_{ij}\overline{x}(t) + \overline{D}\overline{u}(t) \\ \overline{x}(0) = \overline{x}_0 \end{cases} \quad \text{for } \overline{x}(t) \in X_{ij}$$

where

$$\overline{x} = \begin{bmatrix} x \\ \tilde{x} \\ 1 \end{bmatrix} \qquad \overline{u} = \begin{bmatrix} u \\ \tilde{u} \end{bmatrix}$$

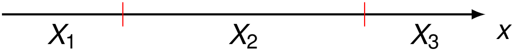
$$\overline{A}_{ij} = \begin{bmatrix} A_i & 0 & a_i \\ 0 & A_j & a_j \\ 0 & 0 & 0 \end{bmatrix} \qquad \overline{B}_{ij} = \begin{bmatrix} B_i & 0 \\ 0 & B_j \\ 0 & 0 \end{bmatrix}$$

$$\overline{C}_{ij} = \begin{bmatrix} C_i & -C_j & c_i - c_j \end{bmatrix} \qquad \overline{D} = \begin{bmatrix} D & -D \end{bmatrix}$$

and with $X_{ij} = \{\overline{x} \mid x \in X_i \text{ and } \tilde{x} \in X_j\} = \{\overline{x} \mid \overline{G}_{ij}\overline{x} \succeq 0\}$

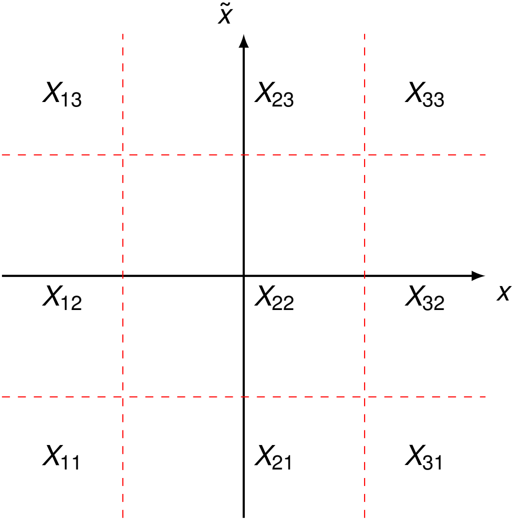
$$\overline{G}_{ij} = \begin{bmatrix} G_i & 0 & g_i \\ 0 & G_j & g_j \end{bmatrix} \qquad X_{ij} \cap X_{kl} \subseteq \{\overline{x} \in \overline{X} \mid \overline{E}_{ijkl}\overline{x} = 0\}$$

Augmented regions



PWA system:
N regions

Augmented regions



PWA system:
N regions

Augmented PWA System:
N^2 regions

Structure of the storage function

Lemma

Let the state x be reachable in finite time from the origin. Then, if S is a storage function for the augmented system Σ_f, $S(x, x) = 0$.

Hence:

$$S(x, \tilde{x}) = \begin{cases} (x - \tilde{x})^T P_i (x - \tilde{x}) & \text{for } \overline{x} \in X_{ii} \\ \overline{x}^T \overline{P}_{ij} \overline{x} & \text{for } \overline{x} \in X_{ij},\ i \neq j \end{cases}$$

Problem: Find P_i and $\overline{P}_{ij}$ such that S is a storage function for the augmented system

Theorem

If there exist symmetric matrices $P_i \in \mathbb{R}^{n\times n}$ *and* $\overline{P}_{ij} \in \mathbb{R}^{(2n+1)\times(2n+1)}$*;* $U_{ij}, R_{ij}, W_{ij} \in \mathbb{R}^{p_{ij}\times p_{ij}}$ *with nonnegative coefficients and zero diagonal;* $L_{ijkl} \in \mathbb{R}^{(2n+1)\times 1}$ *and* $\sigma_1, \sigma_2, \sigma_3 > 0$ *such that*

$$\begin{cases} \mathbf{P_i} - \sigma_1 I_n \succeq 0 \\ \mathbf{P_i} - \sigma_2 I_n \preceq 0 \\ \begin{bmatrix} A_i^T \mathbf{P_i} + \mathbf{P_i} A_i + C_i^T C_i + \sigma_3 I_n & \mathbf{P_i} B_i + C_i^T D \\ \bullet & D^T D - \eta^2 I_p \end{bmatrix} \preceq 0 \end{cases} \tag{TH1}$$

$$\begin{cases} \overline{\mathbf{P}}_{\mathbf{ij}} - \sigma_1 \overline{J}_n - \overline{G}_{ij}^T \mathbf{U_{ij}} \overline{G}_{ij} \succeq 0 \\ \overline{\mathbf{P}}_{\mathbf{ij}} - \sigma_2 \overline{J}_n + \overline{G}_{ij}^T \mathbf{R_{ij}} \overline{G}_{ij} \preceq 0 \\ \begin{bmatrix} \overline{A}_{ij}^T \overline{\mathbf{P}}_{\mathbf{ij}} + \overline{\mathbf{P}}_{\mathbf{ij}} \overline{A}_{ij} + \overline{C}_{ij}^T \overline{C}_{ij} + \sigma_3 \overline{J}_n + \overline{G}_{ij}^T \mathbf{W_{ij}} \overline{G}_{ij} & \overline{\mathbf{P}}_{\mathbf{ij}} \overline{B}_{ij} + \overline{C}_{ij}^T \overline{D} \\ \bullet & \overline{D}^T \overline{D} - \eta^2 \overline{I}_p \end{bmatrix} \preceq 0 \end{cases} \tag{TH2}$$

$$\overline{\mathbf{P}}_{\mathbf{ij}} = \overline{\mathbf{P}}_{\mathbf{kl}} + \mathbf{L_{ijkl}} \overline{E}_{ijkl} + \overline{E}_{ijkl}^T \mathbf{L_{ijkl}}^T \tag{TH3}$$

are satisfied, then Σ *is incrementally* $\mathcal{L}_2$*-gain stable, and has an incremental* $\mathcal{L}_2$*-gain less than or equal to* η*.*

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Sketch of proof

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Norm bounds:

$$\sigma_1 \|x - \tilde{x}\|^2 \leq S(x, \tilde{x})$$

$$P_i - \sigma_1 I_n \succeq 0$$

$$\overline{P}_{ij} - \sigma_1 \overline{J}_n - \overline{G}_{ij}^T U_{ij} \overline{G}_{ij} \succeq 0$$

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Continuity of S

$$\overline{P}_{ij} = \overline{P}_{kl} + L_{ijkl}\overline{E}_{ijkl} + \overline{E}_{ijkl}^T {L_{ijkl}}^T$$

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Continuity of S

Dissipativity of the augmented system $\Rightarrow$ η gives an upper bound to the incremental $\mathcal{L}_2$-gain

Ampere UNIVERSITÉ DE LYON ÉCOLE CENTRALE LYON INSA Lyon 1 cnrs

Incremental stability

Several different definitions extending Lyapunov stability to the incremental framework:

- Contraction analysis (W. Lohmiller and J.-J. E. Slotine, Automatica, 1998.)
- Convergence (A. Pavlov *et al.*, Sys. & Cont. Let., 2004)
- Incremental asymptotic stability (Angeli, IEEE Trans. Autom. Contol, 2002)
- ...

Definition (Incremental asymptotic stability)

$\exists \beta \in \mathcal{KL}$ / $\forall x_0, \tilde{x}_0 \in X$, $\forall w \in \mathcal{L}_2^e$, $\forall t \geq 0$:

$$\|x(t) - \tilde{x}(t)\| \leq \beta(\|x_0 - \tilde{x}_0\|, t)$$

with $x(t) = \phi(t, 0, x_0, w)$ and $\tilde{x}(t) = \phi(t, 0, \tilde{x}_0, w)$.

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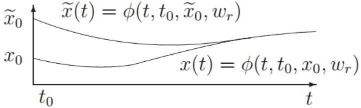
The transient response fades away



Characterization of incremental asymptotic stability

Incremental Lyapunov function

$\exists V : X \times X \to \mathbb{R}_+$, called incremental Lyapunov function, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ s.t.

$$\alpha_1\big(\|x - \tilde{x}\|\big) \leq V(x, \tilde{x}) \leq \alpha_2\big(\|x - \tilde{x}\|\big)$$

and $\forall w \in \mathcal{L}_2^e$, $\forall t \geq 0$

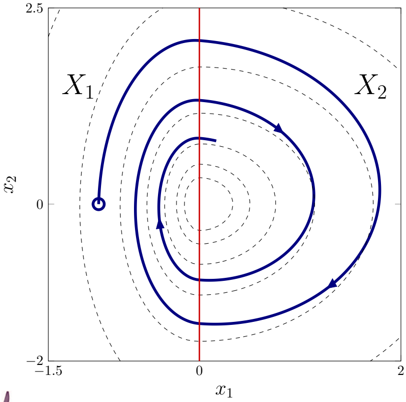
$$V(x(t), \tilde{x}(t)) - V(x_0, \tilde{x}_0) \leq -\int_0^t \rho\big(\|x(\tau) - \tilde{x}(\tau)\|\big)\, d\tau$$

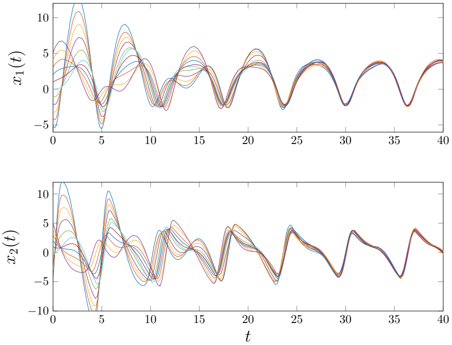
with $x(t) = \phi(t, 0, x_0, w)$, $\tilde{x}(t) = \phi(t, 0, \tilde{x}_0, w)$ and ρ a positive definite function.

S is also an incremental Lyapunov function

Example 1

$$A_1 = \begin{bmatrix} -0.1 & 1 \\ -5 & -0.1 \end{bmatrix} \quad A_2 = \begin{bmatrix} -0.1 & 1 \\ -1 & -0.1 \end{bmatrix}$$





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Quadratic storage function: $S(x, \tilde{x}) = (x - \tilde{x})^T P(x - \tilde{x})$

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Piecewise quadratic storage function:
$$S(x,\tilde{x}) = \begin{cases} (x-\tilde{x})^T P_i (x-\tilde{x}) & \text{for } \overline{x} \in X_{ii} \\ \overline{x}^T \overline{P}_{ij} \overline{x} & \text{for } \overline{x} \in X_{ij},\ i \neq j \end{cases}$$

Upper bound computed: $\eta = 5.005$

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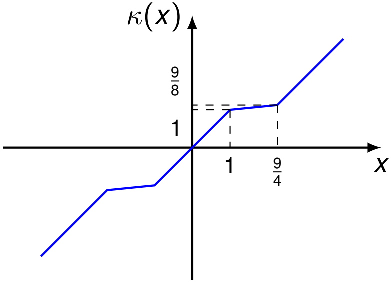
$\hookrightarrow$ less conservative!

Example 2

$$\begin{cases} \dot{x} = -\kappa(x) + u \\ y = x \end{cases}$$



$$\kappa(x) = \begin{cases} x - \frac{9}{8} & x > \frac{9}{4} \\ \frac{1}{10}x + \frac{9}{10} & 1 < x \leq \frac{9}{4} \\ x & |x| \leq 1 \\ \frac{1}{10}x - \frac{9}{10} & -\frac{9}{4} \leq x < -1 \\ x + \frac{9}{8} & x < -\frac{9}{4} \end{cases}$$

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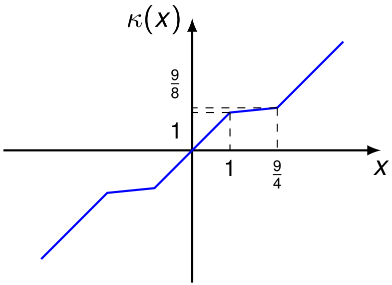

$\mathcal{L}_2$-gain

Upper bound: $\gamma = 2$

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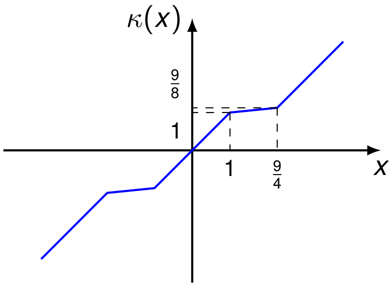

$\mathcal{L}_2$-gain

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Incremental $\mathcal{L}_2$-gain

Upper bound: $\eta = 10$

Lower bound: $\eta = 10$

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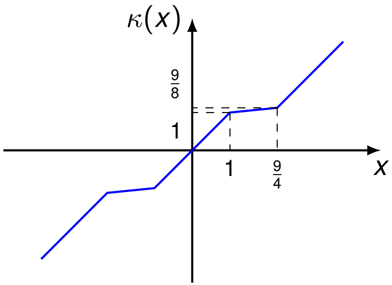

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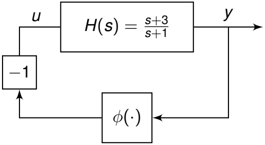
Stronger property!

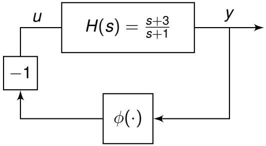
Example 3



with

$$\phi(e) = \begin{cases} h & e > \frac{h}{\kappa} \\ \kappa e & |e| \leq \frac{h}{\kappa} \\ -h & e < -\frac{h}{\kappa} \end{cases}$$

Example 3



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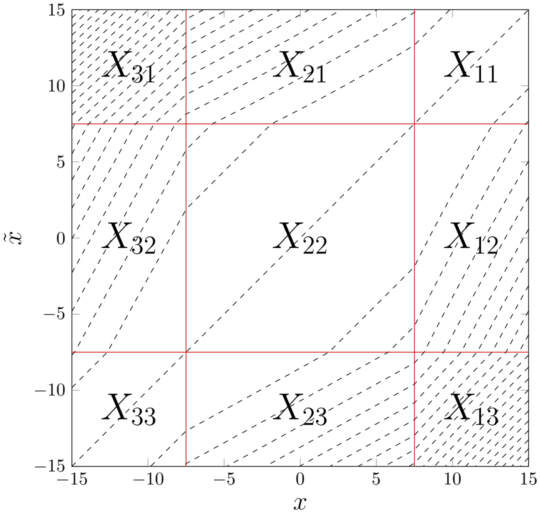

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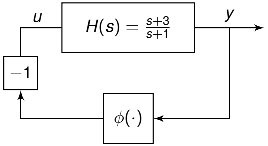

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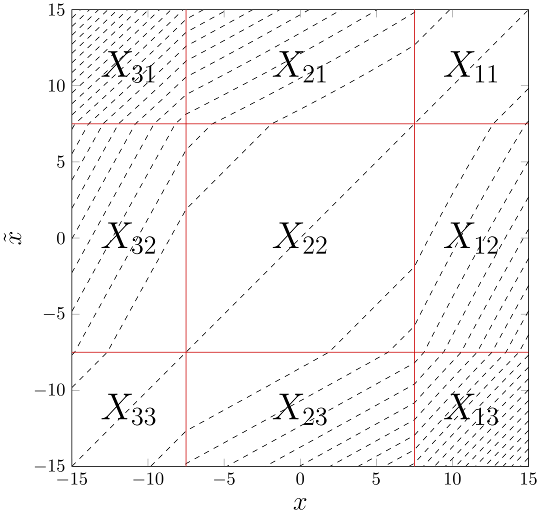
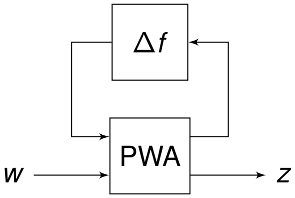


S is a continuous piecewise quadratic function of x and $\tilde{x}$

Concluding remarks

- Extension of previous results concerning $\mathcal{L}_2$-gain stability of PWA systems to the incremental framework
- Choice of piecewise quadratic storage function yields less conservative results
- Perspectives:
    - Representation of nonlinear systems as PWA with ”perturbations”
    - Efficient computation of an upper bound to the incremental $\mathcal{L}_2$-gain for general nonlinear systems



Thank you for your attention

Questions?