Stability Analysis of Neutral Type Time-Delay Positive Systems

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Outline

I. Fundamentals of Finite-Dimensional Positive Systems (FDPSs)

- Definition of Positivity
- Condition for Positivity

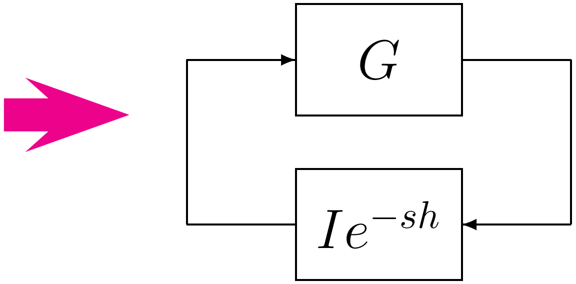
Outline

II. Representation of LTI Neutral Type Time-Delay Systems (TDSs)

- Delay-Differential Equation (DDE)
- Definition of Solution
  - Continuous Concatenated Solution (CCS)
- Time-Delay Feedback System (TDFS)
- Conversion from DDE to TDFS (Hagiwara and Kobayashi, IJC2011)

DDE

$$\dot{q}(t) = Jq(t) + K\dot{q}(t-h) + Lq(t-h)$$



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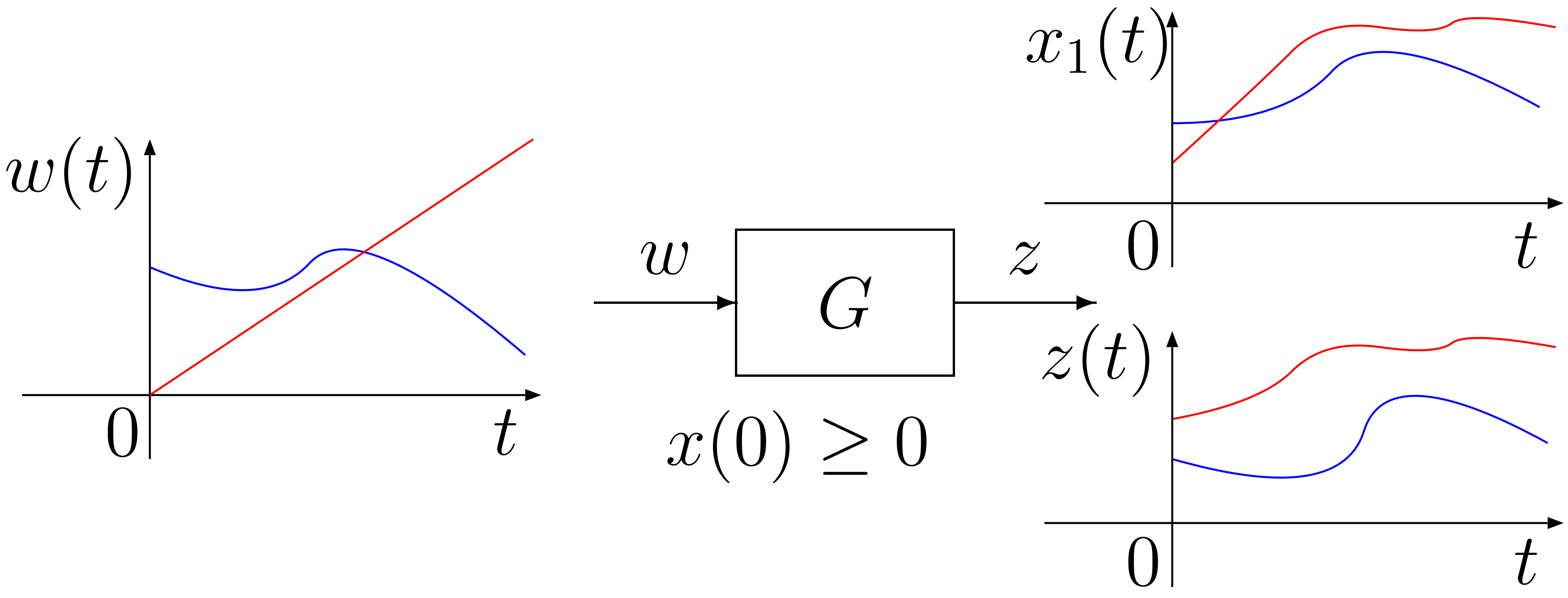
III. Stability Analysis of Neutral Type Time-Delay Positive Systems (TDPSs)

- Definition and Condition for Positivity of TDS
- Necessary and Sufficient Condition for Stability
- Connection to Preceding Results for Delay-Free Interconnected Positive Systems
- Strange Phenomenon
  - stable delay-free PS can be unstable by introducing arbitrarily small delay
- Conclusion

Positive System?

Definition (FDLTI Positive System)

An FDLTI system is said to be positive if its state and output are both nonnegative for any nonnegative initial state and nonnegative input.



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Theorem (Farina, 2000)

An FDLTI system is positive iff

- $A \geq 0$, $B \geq 0$, $C \geq 0$, $D \geq 0$: discrete-time
- $A \in \mathbb{M}$, $B \geq 0$, $C \geq 0$, $D \geq 0$: continuous-time

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$\mathbb{M} = \{A : A_{i,j} \geq 0 \; \forall (i,j), i \neq j\}$ Metzler Matrix (off-diagonal elements are nonnegative)

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Wide Range of Application Areas

- population dynamics
- compartment system
- systems in economics, biology, etc.
  (states are essentially nonnegative)

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Simple Linear Systems are Positive

- integrator, first-order lag, their serial/parallel connections

$$\frac{1}{s}, \quad \frac{K}{1+Ts}, \quad \frac{K}{s(1+Ts)}, \quad \frac{K}{(1+T_1 s)(1+T_2 s)}, \cdots$$

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Hot Topic in Control Community

- POSTA 2016 at Rome - over 30 papers!
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Our Preceding Results

- Analysis using LMIs
  EPA, *Systems & Contr. Letters,* 2014
- Analysis of Retarded Type Time-Delay PSs
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Neutral Type Time-Delay PSs

Representation of LTI Neutral Type TDSs

Delay-Differential Equation (DDE)

$$\dot{q}(t) = Jq(t) + K\dot{q}(t-h) + Lq(t-h),$$
$$q(t) \in \mathbb{R}^n,\ \ J, K, L \in \mathbb{R}^{n\times n}$$

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  - $\phi(t)\ \ (t \in [-h, 0))$: continuously differentiable

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Solution?

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Continuous Concatenated Solution
(Hagiwara and Kobayashi, IJC2011)

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$$\dot{q}(t) = Jq(t) + K\dot{q}(t-h) + Lq(t-h),$$
$$q(0) = \xi,\ q(t) = \phi(t)\ (-h \le t < 0)$$

Definition: Continuous Concatenated Solution (CCS)

Suppose $\phi(t)$ is bounded, continuously differentiable on $[-h, 0)$, and has $\lim_{t\to 0-0}\phi(t)$. Then, $q(t)$ $(t \ge -h)$ is a CCS of DDE if

(i) it is continuous for $t \ge 0$ and

(ii) it is differentiable and satisfies DDE for $t \ge 0$ except possibly for $t = kh$ $(k \in \mathbb{N})$.

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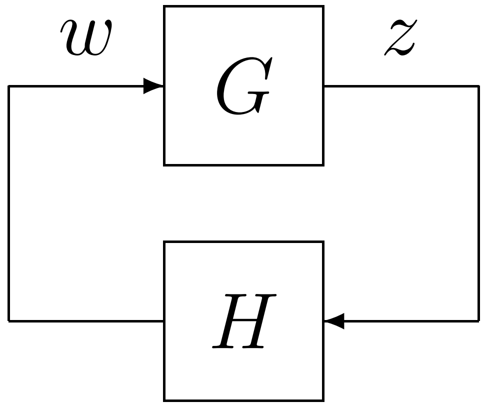
CCS exists and is unique

(Hagiwara and Kobayashi, IJC2011)

Representation of LTI Neutral Type TDSs

Time-Delay Feedback System (TDFS)



- G: FDLTI

$$\begin{cases} \dot{x}(t) = Ax(t) + Bw(t), \\ z(t) = Cx(t) + Dw(t). \end{cases}$$

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$$w(t) = z(t-h) \ (H(s) = Ie^{-sh})$$

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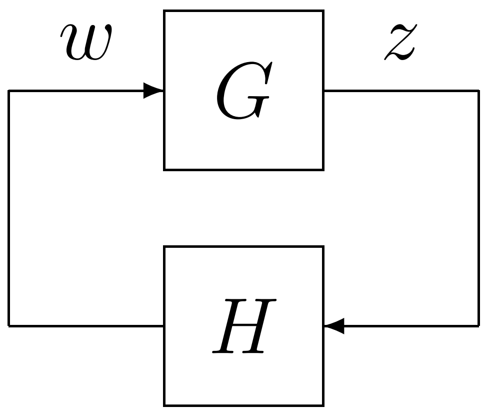

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natural representation in control community

- control-oriented analysis technique applicable

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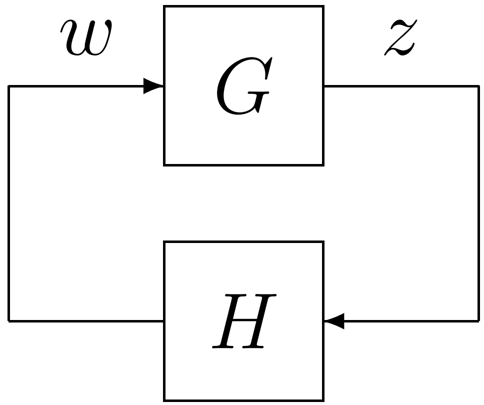

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Conversion from DDE to TDFS??

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→

TDFS??

$$A, B, C, D$$

$$x(0), w(t)\ (t \in [0, h))$$

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DDE		TDFS??
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$q(0) = \xi,\ q(t) = \phi(t)\ (t \in [-h, 0))$		$x(0), w(t)\ (t \in [0, h))$

Theorem (Hagiwara and Kobayashi, IJC2011)

For given $J, K, L, \xi, \phi(t)\ (-h \le t < 0)$, define

$$\left[\begin{array}{c:c} A & B \\ \hdashline C & D \end{array}\right] = \left[\begin{array}{c:c} J & I \\ \hdashline L + KJ & K \end{array}\right],$$

$$x(0) = \xi, \quad w(t) = K\dot\phi(t-h) + L\phi(t-h)\ (0 \le t < h)$$

Then, the unique CCS $q(t)$ of DDE coincides with $x(t)$ resulting from the TDFS over $t \ge 0$.

Representation of LTI Neutral Type TDSs

DDE → **TDFS??**

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$$q(0)=\xi,\ q(t)=\phi(t)\ (t\in[-h,0))$$

$$A, B, C, D$$
$$x(0), w(t)\ (t\in[0,h))$$

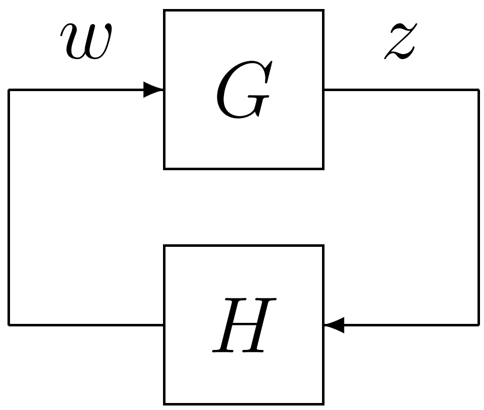
Summary

- conversion from DDE to TDFS always possible

$$\left[\begin{array}{c:c} A & B \\ \hdashline C & D \end{array}\right] = \left[\begin{array}{c:c} J & I \\ \hdashline L+KJ & K \end{array}\right]$$

- neutral type DDE $\Leftrightarrow K\neq 0 \Leftrightarrow D\neq 0$
- it suffices to focus on TDFS of $D\neq 0$

Neutral Type Time-Delay Positive Systems (TDPSs)



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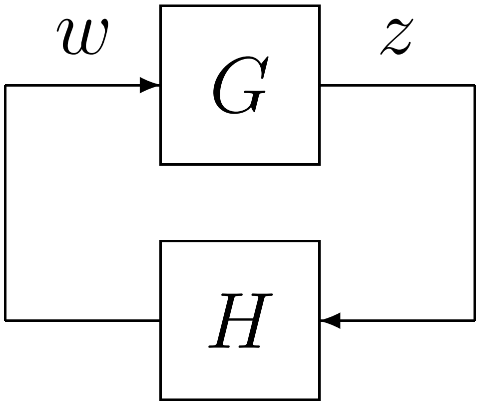

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positivity?

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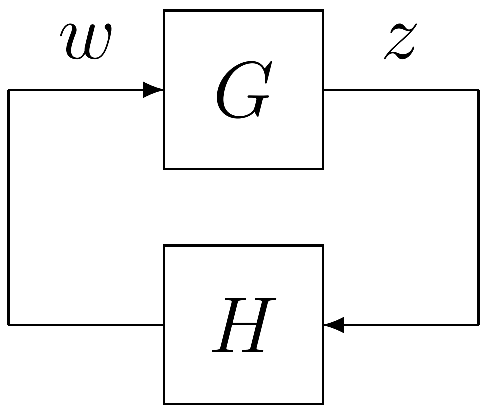

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Notation

- $\mathcal{K}_h^m := \left\{ f \in \mathcal{C}_{[0,h)}^m : \lim_{t\to h-0} f(\theta) \text{ exists} \right\}$
- $\mathcal{K}_{h+}^m := \{ f \in \mathcal{K}_h^m : f(\theta) \geq 0 \}$
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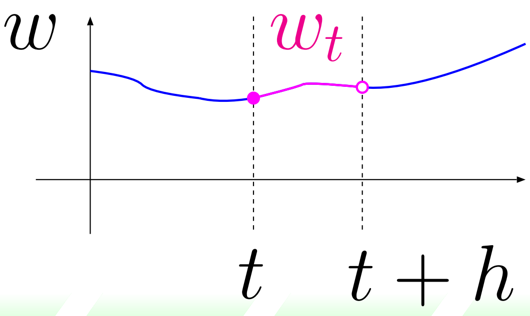

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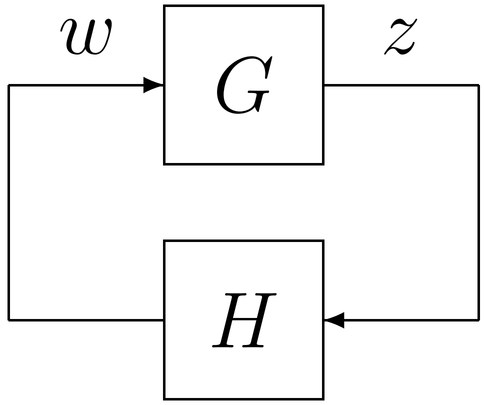

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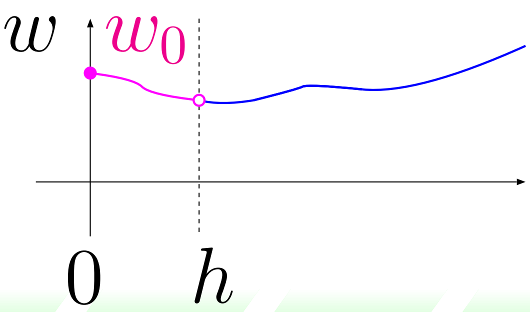
$w_0 \in \mathcal{K}_h^m$ ➡ sufficient for CCSs



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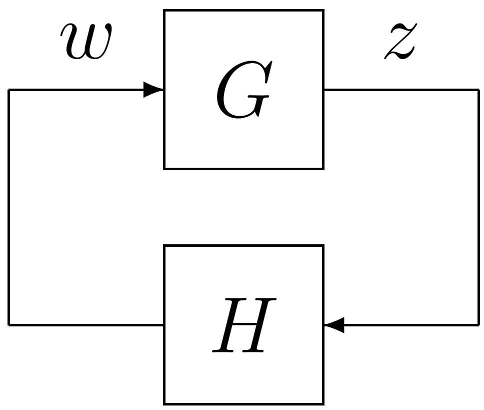

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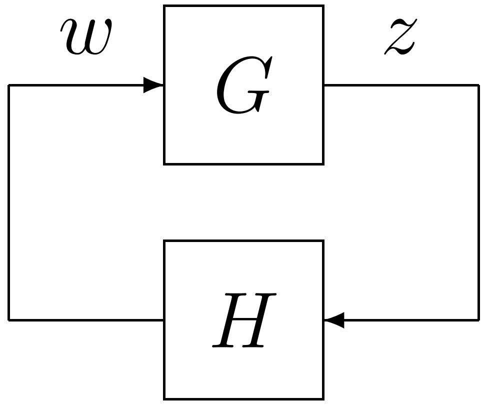

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- extension of the definition for FDLTIPSs

Neutral Type Time-Delay Positive Systems (TDPSs)

Theorem

$G \star H$ is positive if and only if G is positive, i.e., $A \in \mathbb{M}^{n\times n}$, $B \in \mathbb{R}_+^{n\times m}$, $C \in \mathbb{R}_+^{m\times n}$, $D \in \mathbb{R}_+^{m\times m}$.

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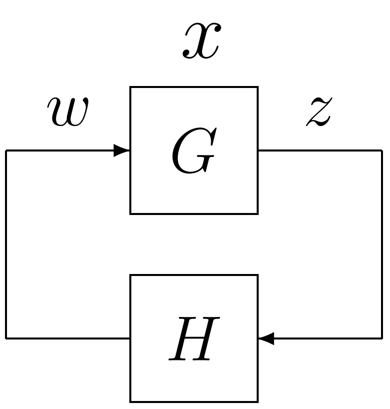
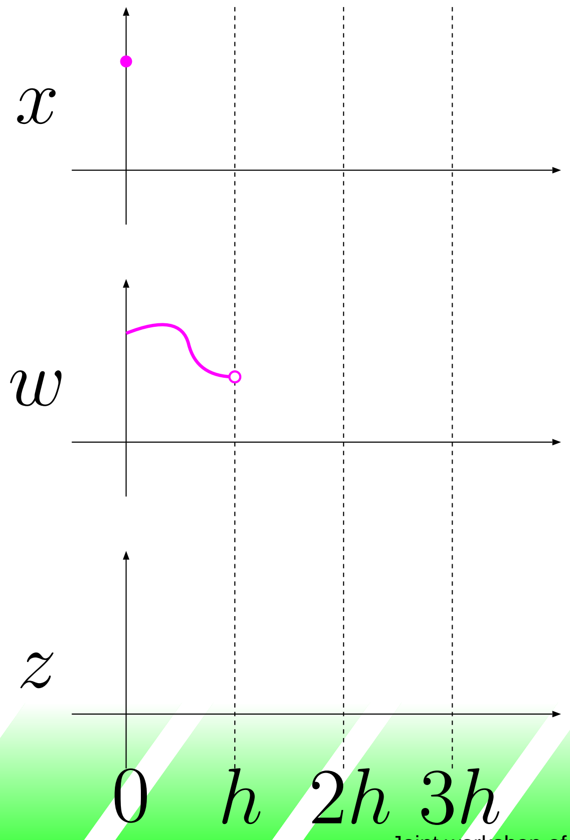
Proof (Sufficiency)

$\boxed{t = 0}$

$x(0) \in \mathbb{R}_+^n, \quad w_0 \in \mathcal{K}_{h+}^m$



$$H(s) = Ie^{-sh}$$



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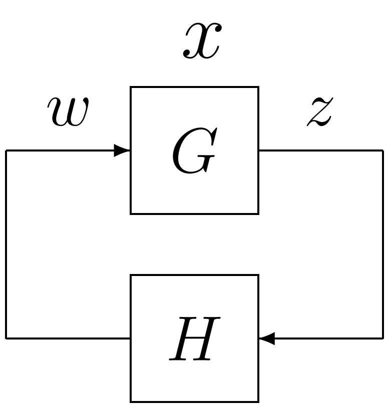

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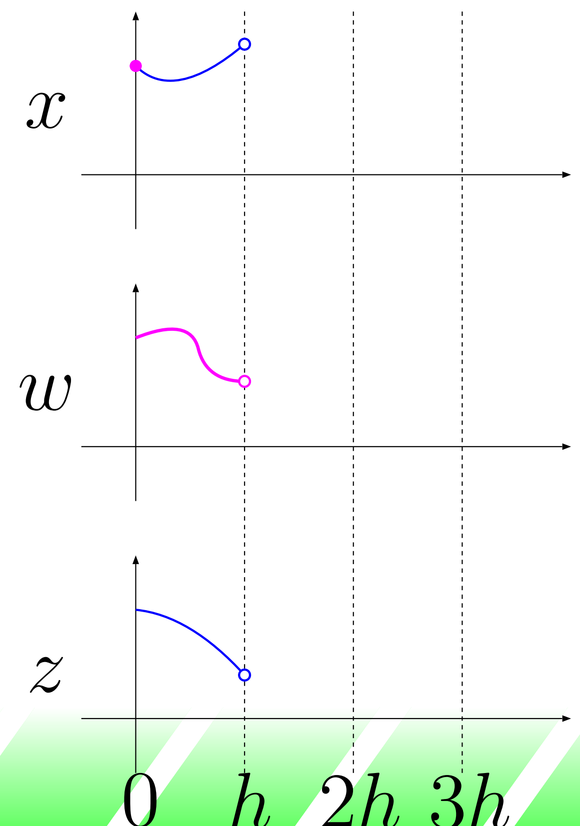

$t \in [0, h)$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bw(\tau)d\tau \geq 0$$

$$z(t) = Cx(t) + Dw(t) \geq 0$$

Remark

- $\lim_{t\to h-0} x(t)$ exists
  - $w_0 \in \mathcal{K}_{h+}^m$ ➡ $\lim_{t\to h-0} w(t)$ exists
- CCS ➡ $x(h) = \lim_{t\to h-0} x(t)$
- $z_0 \in \mathcal{K}_{h+}^m$

Neutral Type Time-Delay Positive Systems (TDPSs)

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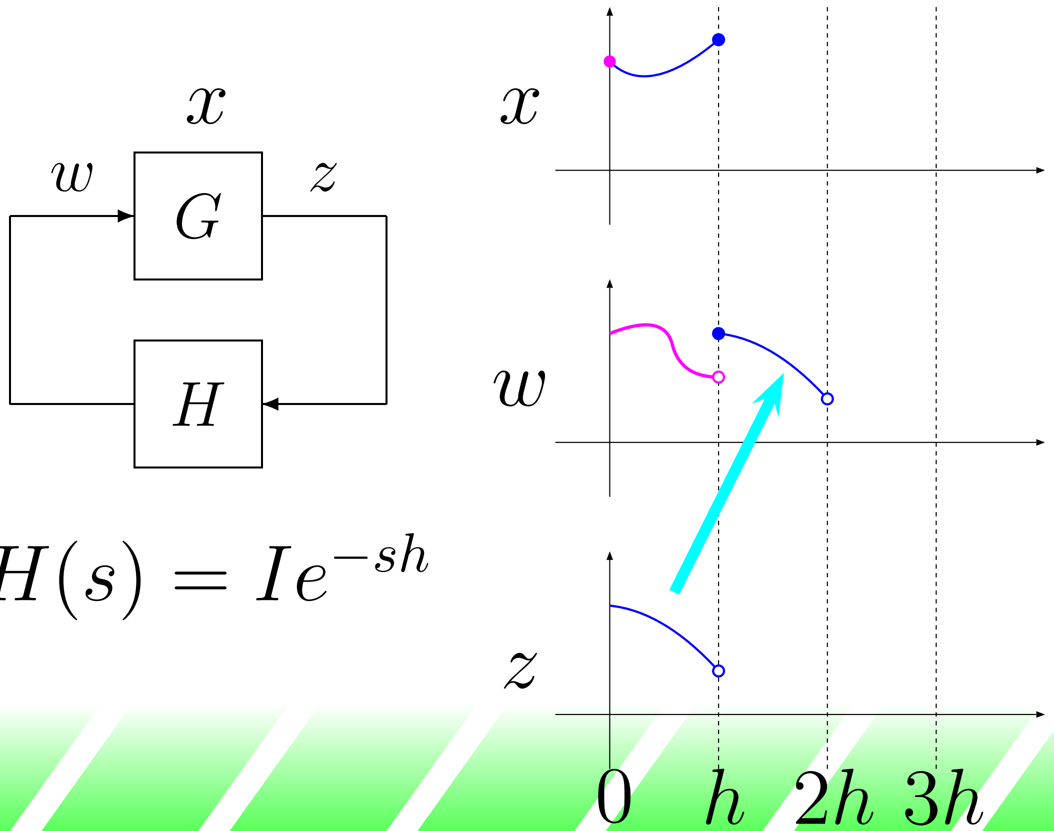

$$H(s) = Ie^{-sh}$$

$h \in [h, 2h)$

$$x(h) = \lim_{t\to h-0} x(t),$$
$$w(t) = z(t-h)$$

Since $z_0 \in \mathcal{K}_{h+}^m$,
$w(t)$ $(t \in [h, 2h))$ is

- continuous,
- nonnegative,
- $\lim_{t\to h-0} w(t)$ exists

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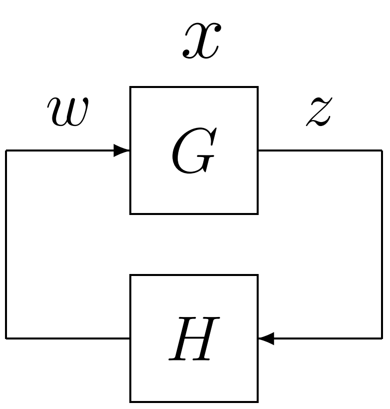

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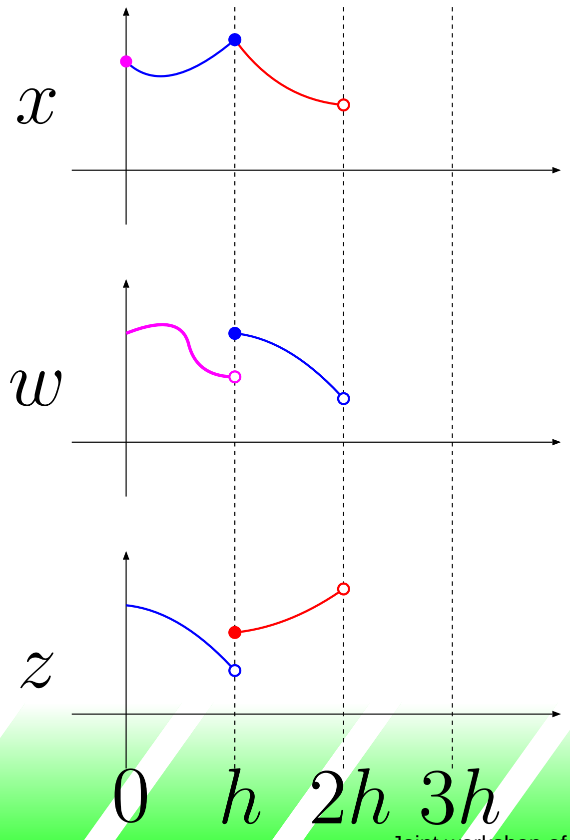


$h \in [h, 2h)$

repeating the same arguments,

- $x(t) \geq 0, \ z(t) \geq 0$
- $\lim_{t\to 2h-0} x(t)$ exists

Neutral Type Time-Delay Positive Systems (TDPSs)

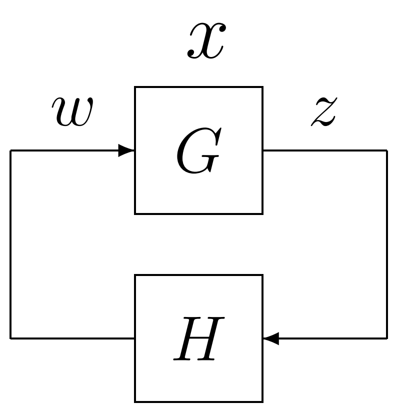
Theorem

$G \star H$ is positive if and only if G is positive, i.e.,
$A \in \mathbb{M}^{n\times n}$, $B \in \mathbb{R}_+^{n\times m}$, $C \in \mathbb{R}_+^{m\times n}$, $D \in \mathbb{R}_+^{m\times m}$.

Proof (Sufficiency)

$h \in [2h, 3h)$

$x(2h) = \lim_{t\to 2h-0} x(t)$,
$w(t) = z(t-h)$



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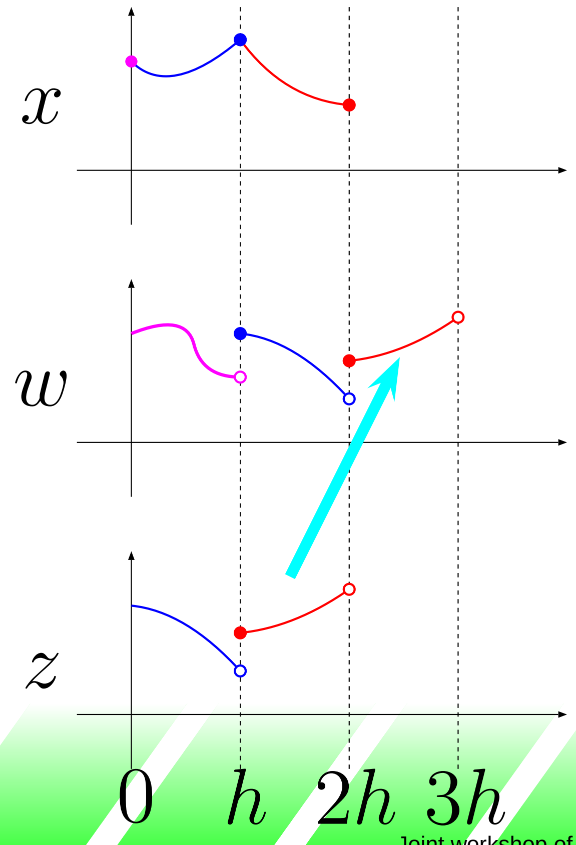

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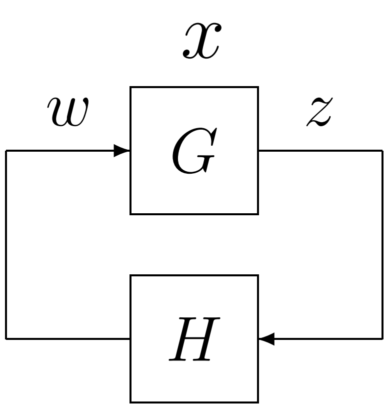

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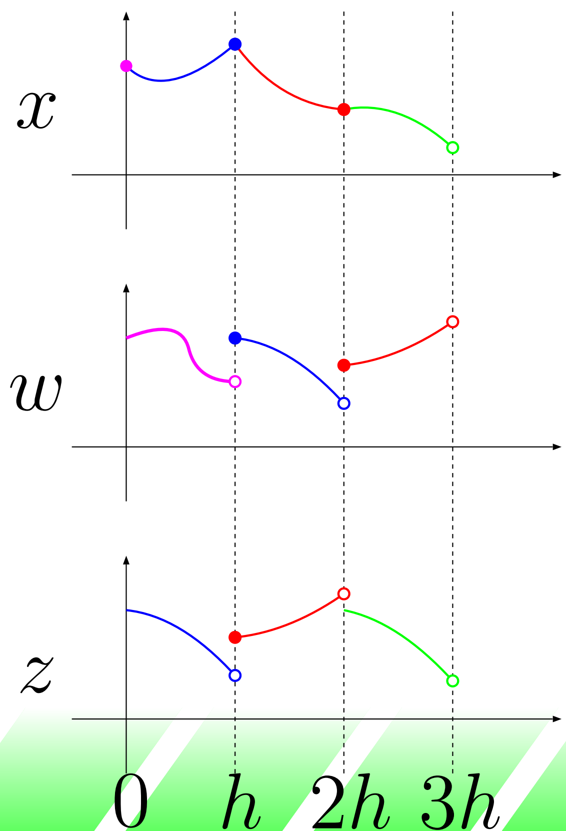

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$\vdots$

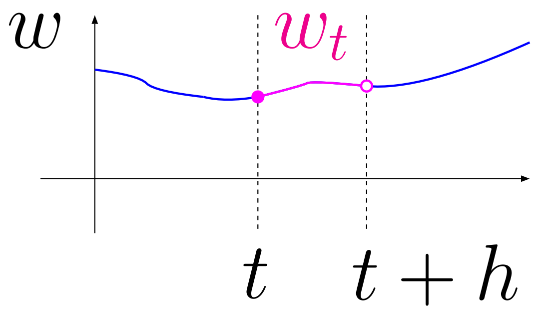
$$x(t) \geq 0,\ \ z(t) \geq 0\ (t \geq 0)$$

$\Rightarrow$ $x(t) \geq 0,\ \ w(t) \geq 0\ (t \geq 0)$

Stability Analysis of Neutral Type TDPSs

Notation

- $w_t := w_t(\theta) = w(t+\theta)\ (0 \leq \theta < h)$
- $x_t := x(t) \in \mathbb{R}^n$
- $\|x_t\|$: 1-norm of $x_t \in \mathbb{R}^n$, i.e., $\|x\| := \sum_{i=1}^{n} |x_i|$
- $\|w_t\| = \int_0^h \|w_t(\theta)\| d\theta$ ($L_1[0,h)$ norm)



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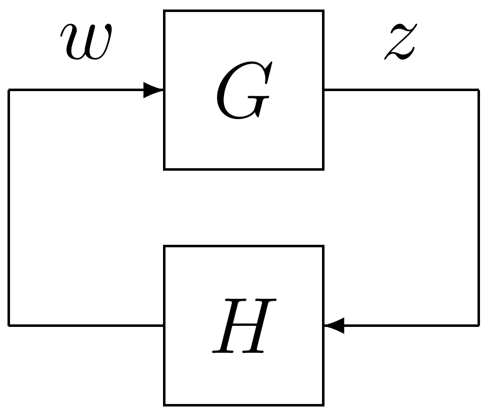
- it suffices to consider $x(0) \in \mathbb{R}_+^n$ and $w_0 \in \mathcal{K}_{h+}^m$

Stability Analysis of Neutral Type TDPSs



$G \star H$

- G: FDLTIPS

$$\begin{cases} \dot{x}(t) = Ax(t) + Bw(t), \\ z(t) = Cx(t) + Dw(t). \end{cases}$$

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$$w(t) = z(t-h)$$

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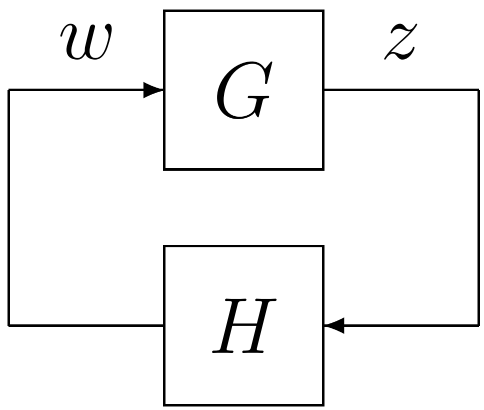
Stability Analysis of Neutral Type TDPSs



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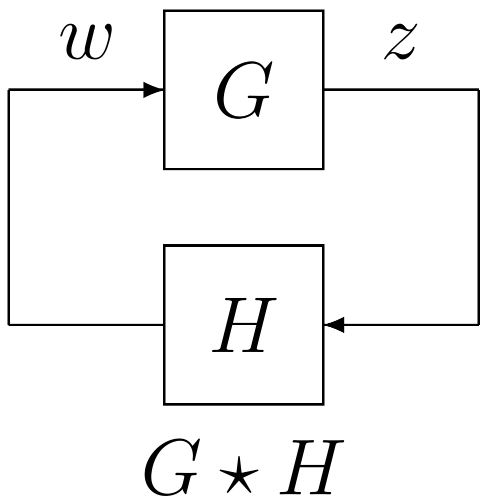

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- stability is independent of the delay length h

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brief sketch of the proof

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  ⬇

- Lyapunov functional $V(x_t, w_t) := r_x^T x_t + r_w^T \int_0^h w_t(\theta) d\theta$ proves the stability

Stability Analysis of Neutral Type TDPSs

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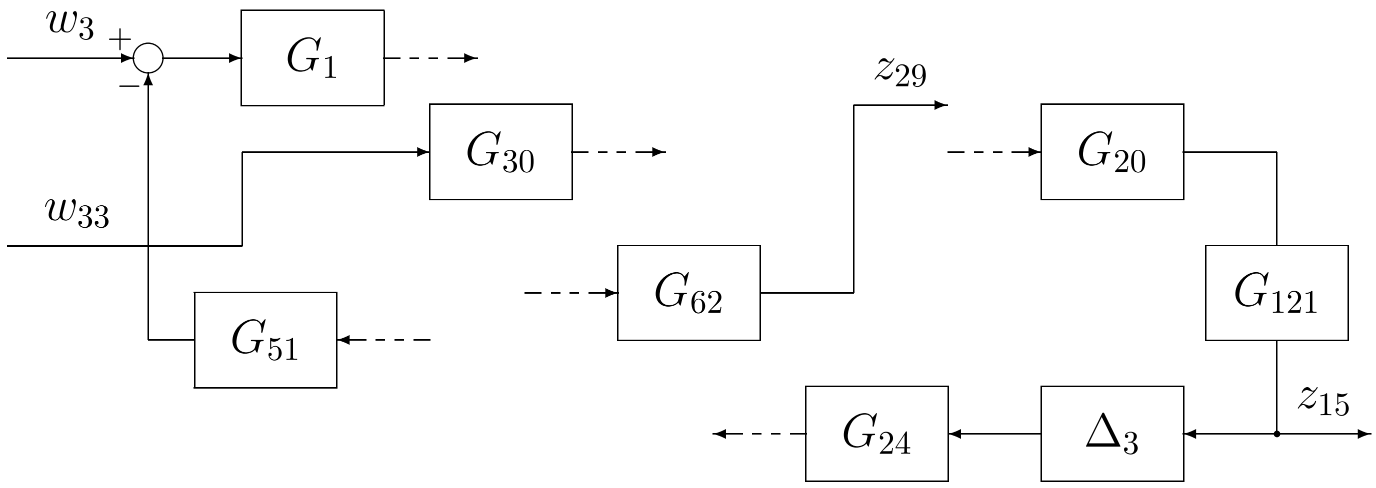
includes this well-known result
as a special case

Connection to Delay-Free Case Results (EPA, TAC2017)

Interconnected Positive Systems



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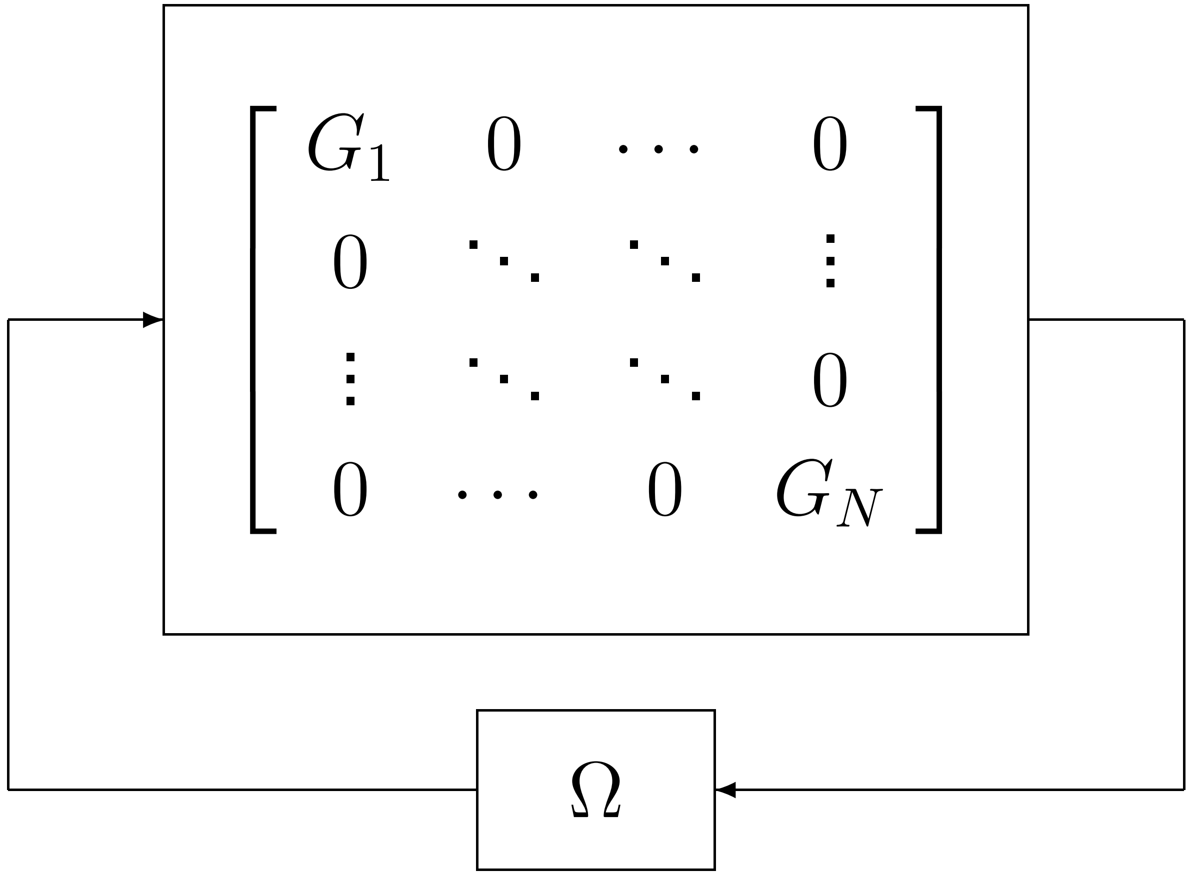

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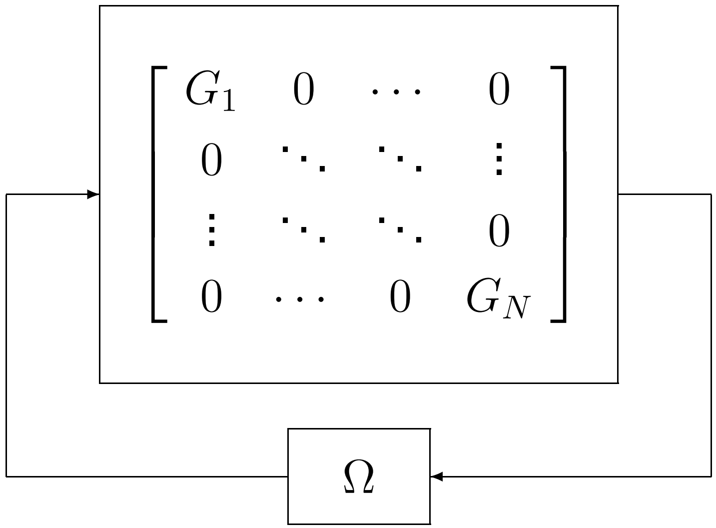


- subsystem G_i: positive and stable

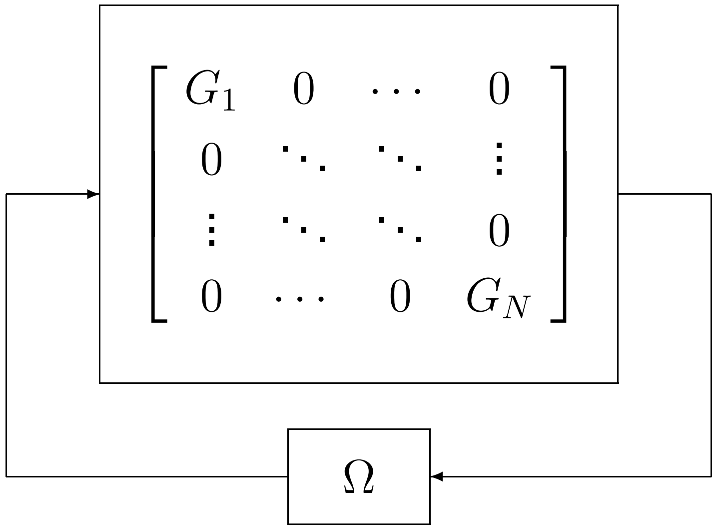
$$G_i : \begin{cases} \dot{x}_i = A_i x_i + B_i w_i, \\ z_i = C_i x_i + D_i w_i, \end{cases}$$

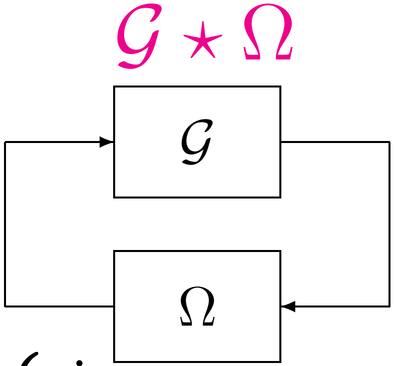
$$A_i \in \{\mathbb{M}^{n_i} \cap \mathbb{H}^{n_i}\},\ B_i \in \mathbb{R}_+^{n_i \times n_{w_i}},\ C_i \in \mathbb{R}_+^{n_{z_i} \times n_i},\ D_i \in \mathbb{R}_+^{n_{z_i} \times n_{w_i}}.$$

- interconnection matrix Ω: nonnegative

Analysis of Interconnected Positive Systems

Interconnected Positive Systems





$$\mathcal{G}: \begin{cases} \dot{\widehat{x}} = \mathcal{A}\widehat{x} + \mathcal{B}\widehat{w}, \\ \widehat{z} = \mathcal{C}\widehat{x} + \mathcal{D}\widehat{w} \end{cases}$$

$$\Omega \in \mathbb{R}_+^{n_{\widehat{w}} \times n_{\widehat{z}}}$$

Analysis of Interconnected Positive Systems

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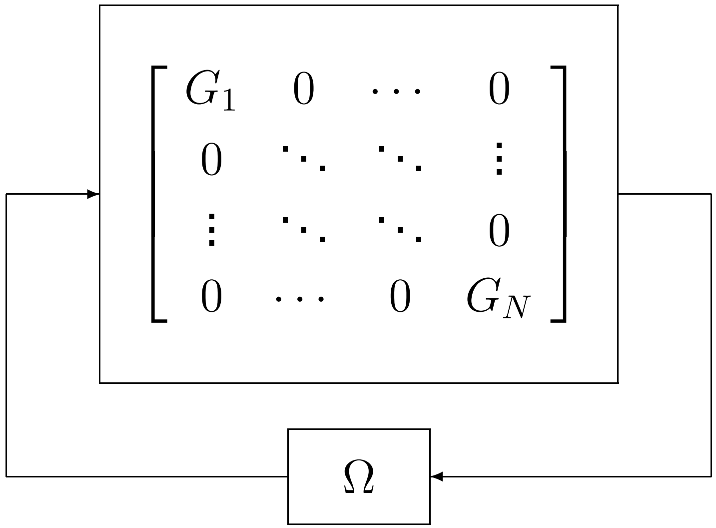
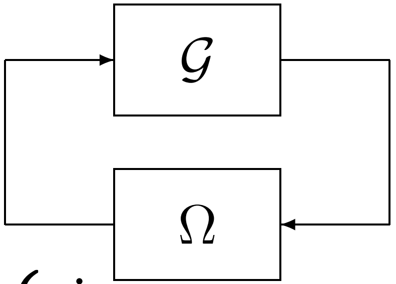


$\mathcal{G} \star \Omega$



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Definition The interconnection is admissible if the matrix $\Omega\mathcal{D} - I$ is Hurwitz stable.

Analysis of Interconnected Positive Systems

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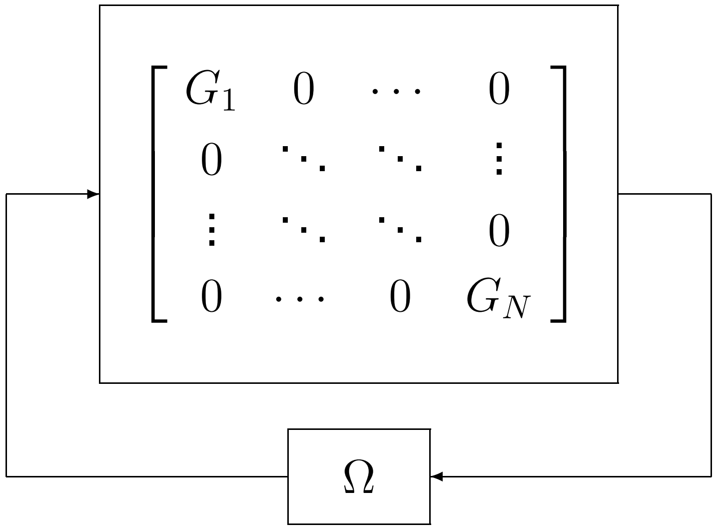

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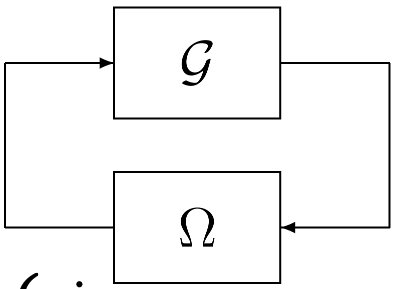

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- sufficient condition for well-posedness

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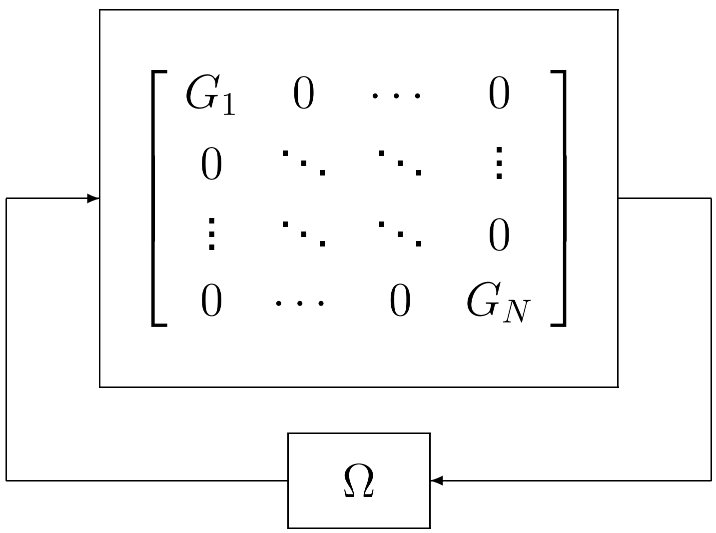

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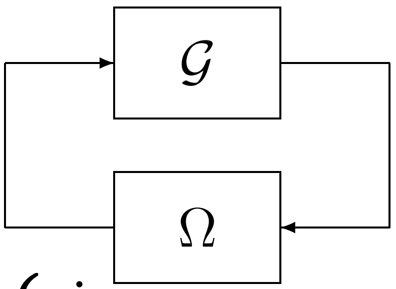

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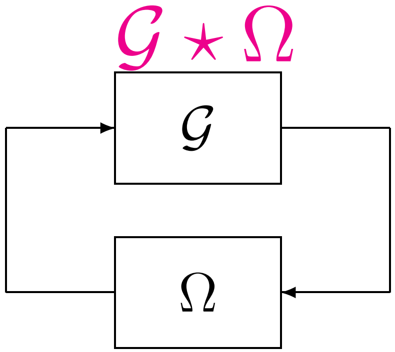
Definition The interconnection is admissible if the matrix $\Omega\mathcal{D} - I$ is Hurwitz stable.

- sufficient condition for closed-loop positivity

$$\dot{\widehat{x}} = \mathcal{A}_{\mathrm{cl}}\widehat{x}, \quad \mathcal{A}_{\mathrm{cl}} := \mathcal{A} + \mathcal{B}(I - \Omega\mathcal{D})^{-1}\Omega\mathcal{C}.$$

Stability Analysis of $\mathcal{G} \star \Omega$



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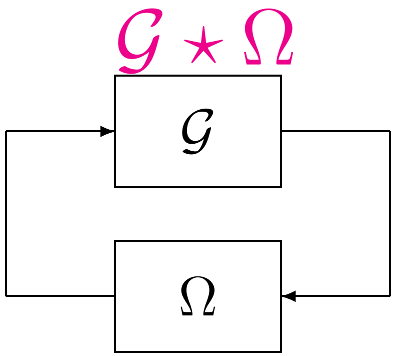

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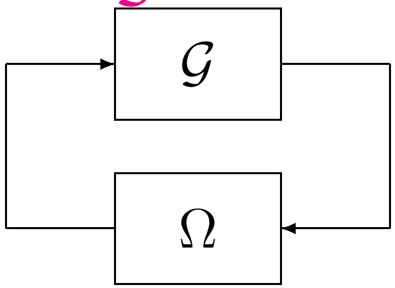
Lemma (EPA, TAC2017)

$\mathcal{G} \star \Omega$ is admissible and stable iff

$\Omega\mathcal{D} - I \in \mathbb{H}^{n_{\widehat{w}} \times n_{\widehat{w}}}$, $\mathcal{A} + \mathcal{B}(I - \Omega\mathcal{D})^{-1}\Omega\mathcal{C} \in \mathbb{H}^{n_{\widehat{x}} \times n_{\widehat{x}}}$.

Stability Analysis of $\mathcal{G} \star \Omega$

$\mathcal{G} \star \Omega$



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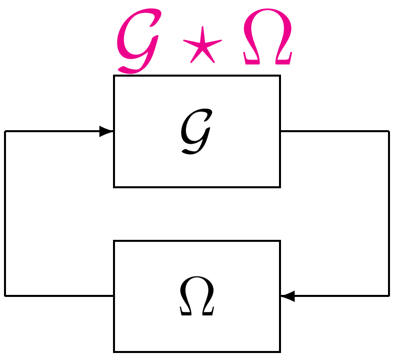
Stability Analysis of $\mathcal{G} \star \Omega$



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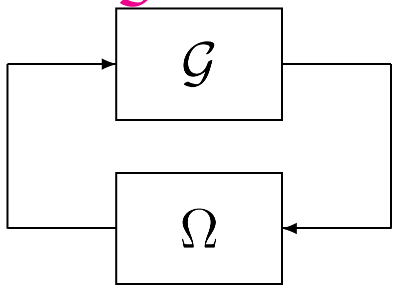
necessary and sufficient conditions for

- MIMO case (extension of EPA, CDC2011).
- SISO case.
- SISO and $G_1(0) = \cdots = G_N(0)$ case.

Stability Analysis of $\mathcal{G} \star \Omega$

$\mathcal{G} \star \Omega$



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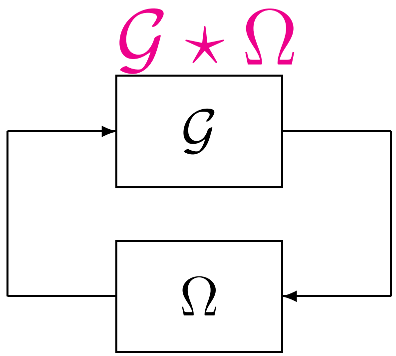
admissibility ??

- artificially introduced to ensure positivity?
- indeed relevant to stability?

Stability Analysis of $\mathcal{G} \star \Omega$



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admissibility ??

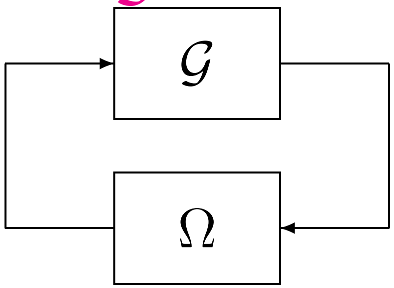
➡ indispensable!!

- ~~artificially introduced to ensure positivity?~~
- ~~indeed relevant to stability?~~

Soundness of Admissibility

Finite Dimensional Interconnected Positive System

$\mathcal{G} \star \Omega$



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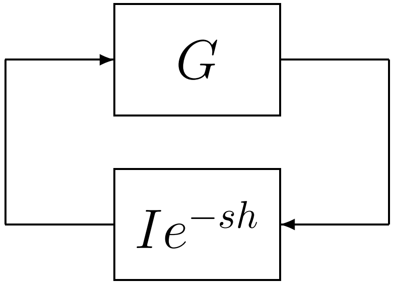
TDPS

$G \star H$



$$G : \begin{cases} \dot{x} = Ax + Bw, \\ z = Cx + Dw \end{cases}$$

$$H = Ie^{-sh}$$

stable iff

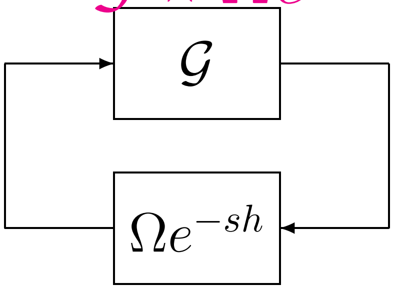
$D - I \in \mathbb{H}^{n_w \times n_w}$

$A + B(I - D)^{-1}C \in \mathbb{H}^{n \times n}$

Soundness of Admissibility

Finite Dimensional Interconnected Positive System

$\mathcal{G} \star \Omega e^{-sh}$



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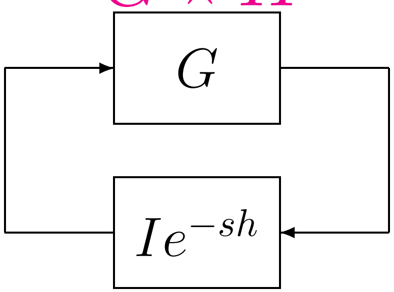
➡ perturbed to Ωe^{-sh}

TDPS

$G \star H$



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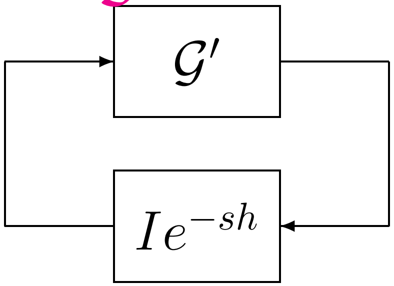
Soundness of Admissibility

Finite Dimensional Interconnected Positive System

$\mathcal{G}' \star H$



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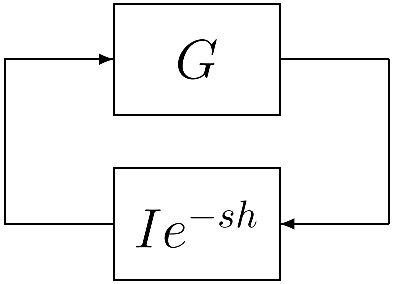
TDPS

$G \star H$



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Soundness of Admissibility

Finite Dimensional Interconnected Positive System

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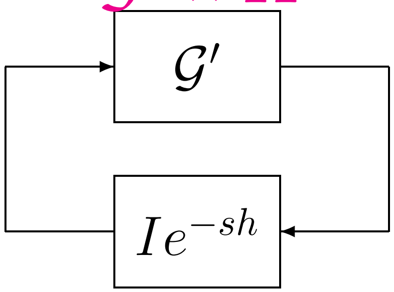

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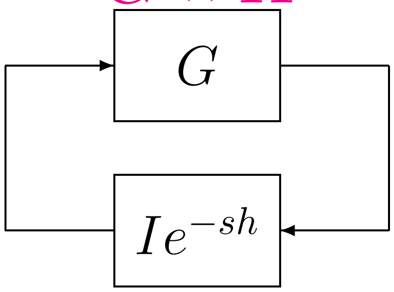
necessary and sufficient condition for stability

TDPS

$G \star H$



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Finite Dimensional Interconnected Positive System

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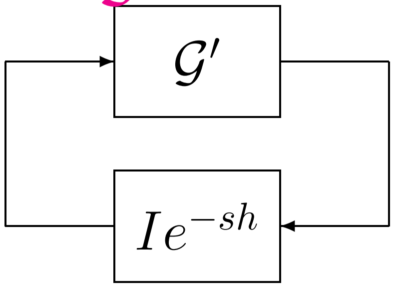

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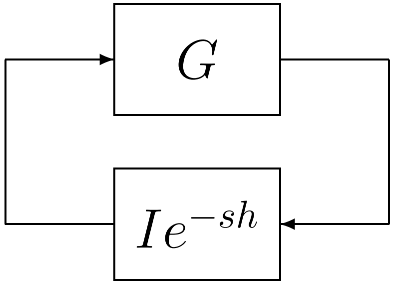

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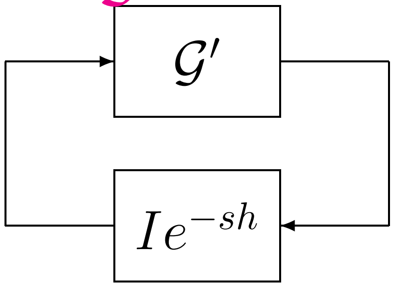
- in practice communication delay unavoidable
- FDIPS without admissibility becomes unstable even for arbitrarily small delay

Soundness of Admissibility

Finite Dimensional Interconnected Positive System

$\mathcal{G}' \star H$



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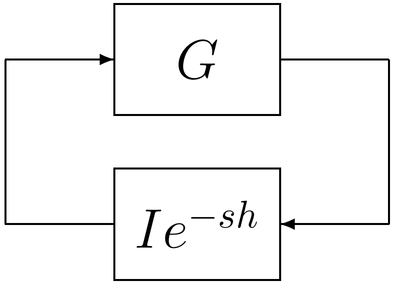

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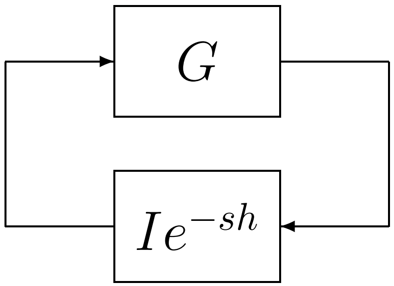
admissibility is a fundamental requirement for IPS

Strange Phenomenon on Stability of Neutral Type TDPS

TDPS

$G \star H$



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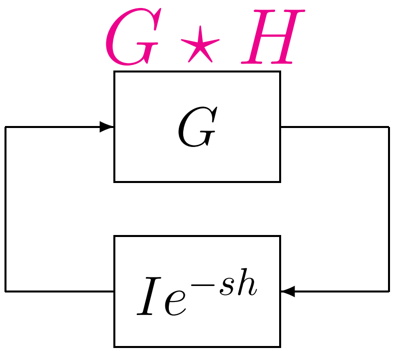
Strange Phenomenon on Stability of Neutral Type TDPS

TDPS



$$G : \begin{cases} \dot{x} = Ax + Bw, \\ z = Cx + Dw \end{cases}$$

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stable iff

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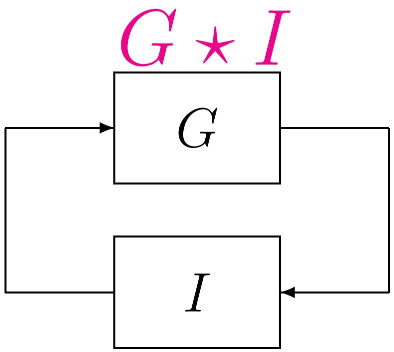
FDPS



$$G : \begin{cases} \dot{x} = Ax + Bw, \\ z = Cx + Dw \end{cases}$$

can be stable even if

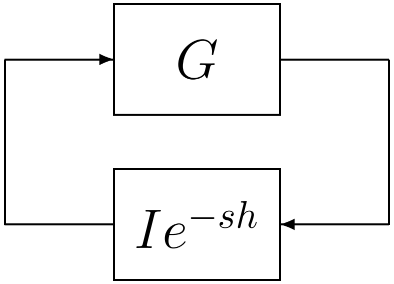
$D - I \notin \mathbb{H}^{n_w \times n_w}$

$A + B(I - D)^{-1}C \in \mathbb{H}^{n \times n}$

Strange Phenomenon on Stability of Neutral Type TDPS

TDPS

$G \star H$



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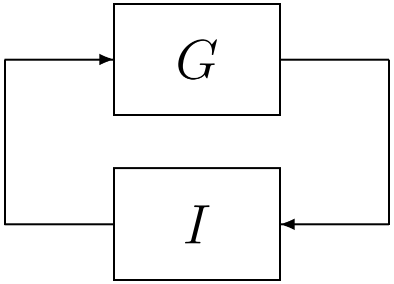
FDPS

$G \star I$



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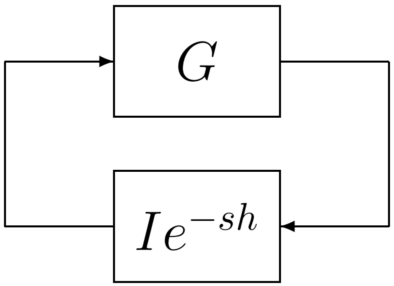
$$G \star I : \dot{x} = A_{\mathrm{cl}} x, \ \ A_{\mathrm{cl}} := A + B(I - D)^{-1}C$$

Example $A = -1$, $B = 1$, $C = 1$, $D = 2$ ➡ $A_{\mathrm{cl}} = -2 < 0$

Strange Phenomenon on Stability of Neutral Type TDPS

TDPS

$G \star H$



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FDPS

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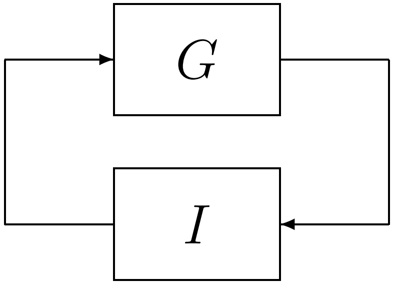

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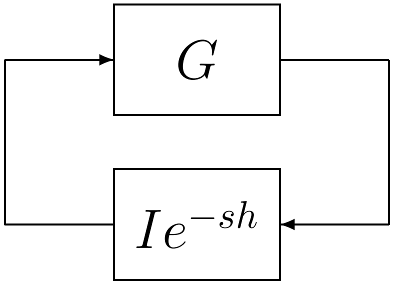
$A + B(I - D)^{-1}C \in \mathbb{H}^{n \times n}$

stable FDPS can be (suddenly) unstable under arbitrarily small delay perturbation

Strange Phenomenon on Stability of Neutral Type TDPS

TDPS

$G \star H$



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FDPS

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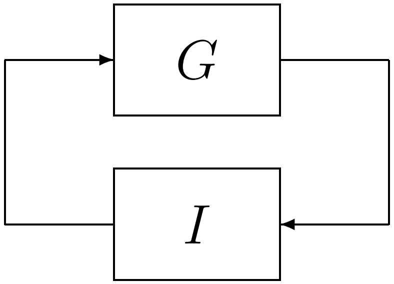

$$G : \begin{cases} \dot{x} = Ax + Bw, \\ z = Cx + Dw \end{cases}$$

can be stable even if

$D - I \notin \mathbb{H}^{n_w \times n_w}$

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- FDPS to infinite-dimensional system
- how (infinitely many) unstable poles appear?

Conclusion

Stability Analysis of Neutral Type Time-Delay Positive Systems

- DDE ➔ TDFS
- necessary and sufficient condition for the stability of neutral type TDPS in TDFS form

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Stability Analysis of Neutral Type Time-Delay Positive Systems

- DDE ➡ TDFS
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Future Work

- instability under “small delay perturbation”
- convergence rate analysis of neutral type stable TDPS