# Towards more general stability analysis of systems with delay-dependent coefficients

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# Outline

1. Introduction
2. Stability Analysis
   - Assumptions
   - Conditions for Imaginary Roots
   - Number of Unstable Roots
   - A Root Migration Direction Criterion
   - Application
3. Future work

# Problem definition

- In this work we study a special type of delay systems, namely systems with the delay parameter also appearing in the coefficients.
- We consider systems whose linearized dynamics has a characteristic equation of the following form

$$P(\lambda, \tau) + Q(\lambda, \tau)e^{-\tau\lambda} = 0$$

where $\tau$ is restricted to Interval $\mathcal{I}$, $I = [\tau^l, \tau^u]$

- The objective is to find all the sub-intervals of $\mathcal{I}$ that guarantees the asymptotical stability of the system.

# Motivating Examples

The stella dynamo model:

$$\dot{B}_\phi(t) = c_1 e^{-c_2 T_0} A(t - T_0) - c_2 B_\phi(t)$$

$$\dot{A}(t) = c_3 e^{-c_2 T_1} B_\phi(t - T_1) - c_2 A(t)$$

where $B_\phi$ is the strength of toroidal field, and $A$ is the strength of poloidal field, and $c_1$, $c_2$, $c_3$, $T_0$, $T_1$ are positive constants. The characteristic equation of the above system can be easily obtained as

$$\lambda^2 + 2c_2\lambda + c_2^2 - c_1 c_3 e^{-c_2\tau} e^{-\tau\lambda} = 0,$$

where $\tau = T_0 + T_1$.

# Motivating Examples continued

A model of hematopoietic stem cell:

$$\dot{S}(t) = -\delta S(t) + e^{-\delta\tau}\beta(S(t-\tau))N(t-\tau)$$
$$\dot{N}(t) = -\delta N(t) - \beta(S(t))N(t) + 2e^{-\delta\tau}\beta(S(t-\tau))N(t-\tau)$$

The model is nonlinear, and possess two equilibria. The linearized equation in the neighborhood of the nonzero equilibrium has the following characteristic equation

$$\lambda + A(\tau) - B(\tau)e^{-\lambda\tau} = 0$$

where $A$, $B$ are nonlinear functions of $\tau$. Therefore delay-dependent coefficients may arise from the linearized dynamics of a nonlinear system.

# The $\tau$-decomposition Approach

The idea of the $\tau-$decomposition approach is to regard the time delay as a variable while the system coefficients are fixed.

The general idea:

- Identify the critical delay values corresponding to the imaginary roots
- The delay domain $\mathcal{I}$ are divided into subintervals.
- On the boundary of each subinterval, determine the migration direction of the imaginary roots.

## Assumption I

**Assumption I.** For all $\tau \in \mathcal{I}$, $P_\tau$ satisfies

$$\mathrm{ord}(P_\tau) = n,$$

where $\mathrm{ord}(\cdot)$ is the order of the polynomial, and

$$\lim_{\omega \to \infty} \left| \frac{Q_\tau(j\omega)}{P_\tau(j\omega)} \right| < 1.$$

## Assumption II

No $(\omega, \tau) \in \mathbb{R}_+ \times \mathcal{I}$ satisfies simultaneously

$$\begin{aligned} P(j\omega, \tau) &= 0, \\ Q(j\omega, \tau) &= 0 \end{aligned}$$

Define $F(\omega,\tau) = P(j\omega,\tau)P(-j\omega,\tau) - Q(j\omega,\tau)Q(-j\omega,\tau)$

## Assumption III

For any $(\omega^*,\tau^*) \in \mathbb{R}_+ \times \mathcal{I}$ that satisfies $D(j\omega^*,\tau^*) = 0$ then

$$\frac{\partial}{\partial\omega}F(\omega,\tau)\bigg|_{\substack{\tau=\tau^* \\ \omega=\omega^*}} \neq 0.$$

Furthermore, $D(j\omega,\tau) = 0$ admits a finite number of solutions for $(\omega,\tau) \in \mathbb{R} \times \mathcal{I}$

## Assumption IV

There are only a finite number of $(\omega,\tau)$ in $\mathbb{R}_+ \times \mathcal{I}$ that simultaneously satisfy $F(\omega,\tau) = 0$ and

$$\frac{\partial}{\partial\omega}F(\omega,\tau) = 0.$$

Let $\tau^{(1)}$,...,$\tau^{(L)}$ be all the delay values contained in $\mathcal{I}$ that satisfy

$$F(\omega,\tau)=0, \partial_\omega F(\omega,\tau)=0$$

for some real $\omega$.

**Proposition**

*The number of real roots of $F(\omega,\tau)=0$ in $\omega$ are the same for all $\tau \in \mathcal{I}_o^{(i)}=(\tau^{(i)},\tau^{(i+1)})$, and they are all simple. These real simple roots are continuously differentiable functions of $\tau$, and may be expressed as $\pm\omega_k^{(i)}(\tau)$, $k=1,2,\ldots,m^{(i)}$, where $m^{(i)}\le n$, and $\omega_k^{(i)}(\tau)>0$ for all $\tau \in \mathcal{I}_o^{(i)}$.*

- The signature of $\partial_\omega F(\omega_k^{(i)}(\tau),\tau)$ is constant in each subinterval $\mathcal{I}^{(i)}$.

Recall the characteristic equation is $P + Qe^{-\lambda\tau} = 0$
Now for each subinterval $\mathcal{I}^{(i)}$ and some critical frequency function $\omega_k^{(i)}$ we define the corresponding phase function:

$$\theta_k^{(i)}(\omega_k^{(i)}(\tau), \tau) = \angle P(j\omega_k^{(i)}\tau, \tau) - \angle Q(j\omega_k^{(i)}\tau, \tau) + \omega\tau + \pi$$

Then it is easy to see that the sufficient and necessary condition for $j\omega^*$ to be an imaginary root when $\tau = \tau^* \in \mathcal{I}^{(i)}$ is that for some integer $k$ and $l$

$$\omega_k^{(i)}(\tau^*) = \omega^*, \theta_k^{(i)}(\tau^*) = 2l\pi$$

For each critical delay $\tau^*$, and imaginary root $\lambda = j\omega^*$
We define

$$\mathrm{Inc}(\tau^*, \omega^*) = \frac{\mathrm{sgn}\left(\Re(\lambda(\tau^{*+}))\right) - \mathrm{sgn}(\Re(\lambda(\tau^{*-})))}{2}.$$

for all $\tau^* \neq \tau^l$. If $\tau_1 = \tau^l$ we define instead

$$\mathrm{Inc}(\tau_1, \omega^*) = \mathrm{sgn}\left(\Re(\lambda(\tau^{*+}))\right)$$

Further for each critical delay $\tau_l$, define

$$\mathrm{Inc}(\tau_l) = 2 \sum_{h=1}^{H_l} \mathrm{Inc}(\tau^*, \omega_{lh})$$

where $\omega_{lh}$, $h = 1, ..., H_l$ are all the non-negative frequencies of the imaginary roots corresponding to $\tau_l$.

Now we can count the number of unstable roots for delay $\tau$ as

$$N^u(\tau) = N^u(\tau^l) + \sum_{l=1}^{L(\tau)} \mathrm{Inc}(\tau_l)$$

where $L(\tau)$ is the largest number such that $\tau_{L(\tau)} < \tau$
It is easy to see that if $\Re\left(\lambda'_{lh}(\tau)\right)_{\tau=\tau_l} \neq 0$, then the following holds:

$$\mathrm{Inc}(\tau_l, \omega_{lh}) = \begin{cases} \mathrm{sgn}\left(\Re\left(\lambda'_{lh}(\tau)\right)_{\tau=\tau_l}\right), & \text{if } \tau_l > \tau^l, \\ \max\left\{0, \mathrm{sgn}\left(\Re\left(\lambda'_{lh}(\tau)\right)_{\tau=\tau_l}\right)\right\}, & \text{if } \tau_l = \tau^l, \end{cases}$$

## Lemma

$\partial_\omega F(\omega_k^{(i)}(\tau),\tau) \neq 0$ *implies* $\partial_\lambda D(j\omega_k^{(i)}(\tau),\tau) \neq 0$

It follows from the this lemma that if $\tau^*$ is a critical delay and $j\omega^*$ is a corresponding imaginary root, then for $\tau$ in a neigbourhood of $\tau^*$, $\lambda$ can be viewed as a function of $\tau$ denoted as $\lambda(\tau)$.

## Theorem

*Let* $(\omega^*,\tau^*)$, $\omega^* \in \mathbb{R}$, $\tau^* \in \mathcal{I}$, *satisfies* $D(j\omega^*,\tau^*) = 0$. *Let* $i,k$ *be the integers such that* $\tau^* \in \mathcal{I}^{(i)}$,$\omega_k^{(i)}(\tau^*) = \omega^*$

$$sgn\left(\Re\left(\frac{d\lambda}{d\tau}\right)_{\tau=\tau^*}\right) = sgn\left(\frac{\partial}{\partial\omega}F(\omega,\tau)\right) \times sgn\left(\frac{d\theta_k^{(i)}(\tau)}{d\tau}\right)_{\substack{\tau=\tau^*\\ \omega=\omega^*}}$$

We summarize our stability analysis procedure.

- Solve $F(\omega,\tau)=0$ and $\partial_\omega F(\omega,\tau)=0$ to obtain $\tau^{(1)}$,...,$\tau^{(L)}$ and thus $\mathcal{I}^{(i)}$, $i=1,...,L$.
- In each sub-interval $\mathcal{I}^{(i)}$, solve $\theta_k^{(i)}(\tau)=2l\pi$, for some integer $l$ and thus obtain all the critical delay values $\tau_1, \tau_2, ..., \tau_H$.
- At each critical delay value $\tau_l$, apply our root crossing criterion to obtain $\mathrm{Inc}(\tau_l)$.
- For any $\tau\in\mathcal{I}$, count the number of unstable roots using

$$N^u(\tau)=N^u(\tau^l)+\sum_{l=1}^{L(\tau)}\mathrm{Inc}(\tau_l)$$

Consider the stability of the stellar dynamos model with the delay interval $\mathcal{I} = [0, 2]$. The characteristic equation can be written as

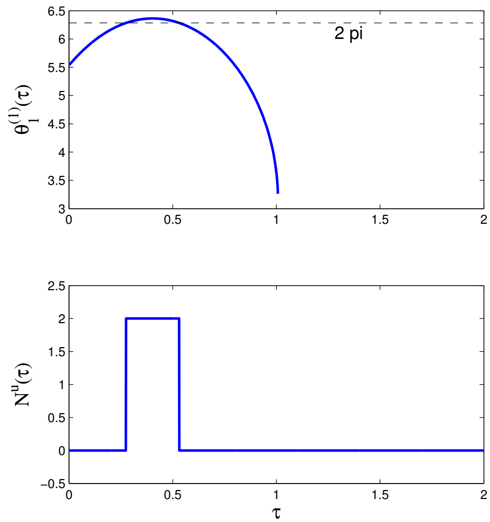
$$\lambda^2 + 2c_2\lambda + c_2^2 - c_1c_3e^{-c_2\tau}e^{-\lambda\tau} = 0. \qquad (0)$$

We have:

$$F(\omega, \tau) = \omega^4 + 2c_2^2\omega^2 + c_2^4 - c_1^2c_3^2e^{-2c_2\tau},$$

$$\frac{\partial}{\partial\omega}F(\omega, \tau) = 4\omega(\omega^2 + c_2^2).$$

Solving these equations, we have $\tau^{(1)} \approx 1.006.$. $\mathcal{I}^{(1)} = [0, \tau^{(1)}]$, $\mathcal{I}^{(2)} = [\tau^{(1)}, \tau^{(2)}]$, $\tau^{(2)} = 2$.




$\omega_1^{(1)}$ is the only positive frequency curve defined in $\mathcal{I}^{(1)}$.
the phase angle curve $\theta_1^{(1)}$ intersects the horizontal line $2\pi$ at $\tau_1 \approx 0.2748$ and $\tau_2 \approx 0.5314$. Therefore, $H_1 = 1$, $\omega_{11} = \omega_1^{(1)}(\tau_1)$, and $H_2 = 1$, $\omega_{21} = \omega_1^{(1)}(\tau_2)$.

## Future research directions

- Our results indicate a strong correlation between roots crossing the imaginary axis and the phase curve crossing $2l\pi$. We want to find the essential reason behind this interesting correlation.
- Extend our root migration direction criterion to the case where $\Re\left(\frac{d}{d\tau}\lambda(\tau^*)\right) = 0$. High-order analysis is necessary.
- Extend the analysis approach for systems of more general forms.