Bessel inequality for robust stability analysis of time-delay system

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Stability of Time delay system

Let consider the following time delay system :

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-h), \ \forall t \geq 0 \\ x(t) = \phi(t), \ \forall t \in [-h, 0] \end{cases} \quad (1)$$

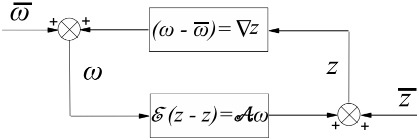
⋆ h is the delay possibly unknown and uncertain.
⋆ Goal : Give conditions on h for finding the largest interval $[h_{\min} \ \ h_{\max}]$ such that for all h in this interval the delay system is stable.

Previous work

Numerous tools for testing the stability of linear time delay systems have been successfully exploited :

- Direct approach using pole location [Sipahi2011].
  - ⊕ It can lead to an analytical solution...
  - ⊖ ...But it's only for **constant delay**,
  - ⊖ and robustness issues are still an open question.
- A Lyapunov-Krasovskii /Lyapunov- Razumikhin approach [Gu03, Fridman02, He07, Sun2010 ...].
  - A general L.K. functional exists but difficult to handle [Kharitonov].
    $\Longrightarrow$ see the work of [Gu03] for a discretized scheme of the general L.K. functional or polynomial approximation [Peet06].
  - Choice of more simple and then more conservative L.K. functional.
- Input - Output Approach
  - Small gain theorem [Zhang98,Gu03 ...],
  - IQC approach [Safonov02, Kao07],
  - Quadratic separation approach.
  - ⊕ It works either for constant or time varying delay systems,
  - ⊕ Robustness issue is straightforward,
  - ⊖ ...But some conservatism to handle.

Stability analysis using quadratic separation



Stability analysis of an interconnection between a linear transformation and an uncertain relation ∇ belonging to a given set $\mathbb{V}$.

- Whatever bounded perturbations $(\bar{z}, \bar{w})$, internal signals have to be bounded.
- Stability of the interconnection $\Leftrightarrow$Well-posedness pb[Safonov87].
- Separation of the graph of the implicit transformation and the inverse graph of the uncertain transformation.

$\Rightarrow$ key idea [Iwasaki98] for classical linear transformation, the well posedness is assessed losslessly by a quadratic separator (quadratic function of z and w).

$\Rightarrow$ extension to the implicit linear transformation proposed by [Peaucelle07,Ariba09].

Stability analysis using Quadratic Separator

Theorem ([Peaucelle07])

The uncertain feedback system of Figure 1 is well-posed and stable if and only if there exists a Hermitian matrix $\Theta = \Theta^*$ satisfying both conditions

$$\begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp *} \Theta \begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp} > 0 \tag{2}$$

$$\begin{bmatrix} 1 \\ \nabla \end{bmatrix}^{*} \Theta \begin{bmatrix} 1 \\ \nabla \end{bmatrix} \leq 0 \quad , \quad \forall \nabla \in \mathbb{W} \ . \tag{3}$$

Goal :Develop an interconnected system to use this theorem, i.e. artificially construct augmented systems to develop less conservative results.

Procedure

1. Define an appropriate modeling of time delay system by constructing the linear transformation defined by the matrices $\mathcal{E}, \mathcal{A}$, and the relation ∇, composed with chosen operators.
2. Define an appropriate separator a matrix Θ satisfying the constraint :

$$\begin{bmatrix} 1 \\ \nabla \end{bmatrix}^* \Theta \begin{bmatrix} 1 \\ \nabla \end{bmatrix} \leq 0 \quad , \quad \forall \nabla \in \mathbb{W} \ . \tag{4}$$

   The infinite numbers of constraints are then verified by construction.
3. Solve the inequality :

$$\begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp *} \Theta \begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp} > 0, \tag{5}$$

   which proves the stability of the interconnection and the time delay system.

Concerning the robust analysis for delay system, the general idea :

1. Choose an uncertain relation composed by several uncertainties depending on the delay operator e^{-hs}.
   $\rightarrow$ Often based on a rational or polynomial approximation.
2. Embed the uncertainties into a suitable norm bounded and well-known uncertainties.
   $\rightarrow$It allows to find a separator Θ, possibly conservative.
3. Application of the stability criterion.

The difficulties come from :
$\rightarrow$ The choice of the uncertainties to reduce the conservatism.
$\rightarrow$ The choice of the best embedding.

How to use the delay state ?

⋆ In the literature on the robust analysis of time delay system, we approximate e^{-hs} (often based on polynomial or rational approximations)

⋆ But, the delay state is defined by

$$x_t : \left\{ \begin{array}{l} [-h, 0] \to \mathbb{R}^n \\ \theta \mapsto x_t(\theta) = x(t+\theta) \end{array} \right.$$

⋆ Using Laplace transform, it should be better to consider the approximation of :

$$\mathcal{D} : \left\{ \begin{array}{l} [-h, 0] \to \mathbb{C} \\ \theta \mapsto e^{s\theta} \end{array} \right.$$

Approximation of the delay operator $\mathcal{D}$

idea : Approximate function $\mathcal{D}$ rather than e^{-sh}.

$\star$ Let H the vector space of complex valued square integrable functions on $[-h, 0]$, associated with the hermitian inner product :

$$\langle f, g \rangle = \int_{-h}^{0} f(\theta) g^*(\theta) d\theta,$$

where f and g belonging to H. Let recall the bessel inequality :

Lemma (Bessel inequality)

let $\{e_0, e_1, e_2, ..., e_n\}$ an orthogonal sequence of H, then $\forall f \in H$:

$$\langle f, f \rangle \geq \sum_{i=0}^{n} |\langle f, e_i \rangle|^2$$

$\rightarrow$ A way to approximate function $\mathcal{D}$ by orthogonal polynomials.

Choice of uncertainties (1)

idea Use of orthogonal polynomials in order to define uncertainties.

⋆ Bessel inequality will provides with a fine embedding of the resulting uncertainties.

⋆ Firstly, note the following inequality :

$$\langle \mathcal{D}, \mathcal{D} \rangle \leq h,$$

ie

$$\int_{-h}^{0} e^{s\theta} e^{s^*\theta} d\theta \leq h$$

⋆ We choose the first two Legendre polynomials :

$$e_0(\theta) = \frac{1}{\sqrt{h}}, \qquad \forall \theta \in [-h, 0], \qquad \langle e_0, e_0 \rangle = 1.$$

$$e_1(\theta) = \sqrt{\frac{3}{h}} \left(\frac{2}{h}\theta + 1 \right), \ \langle e_1, e_1 \rangle = 1, \langle e_0, e_1 \rangle = 0.$$

⋆ (e_0, e_1) is an orthogonal sequence, $\rightarrow$ Bessel inequality :

$$\langle \mathcal{D}, \mathcal{D} \rangle \geq |\langle \mathcal{D}, e_0 \rangle|^2 + |\langle \mathcal{D}, e_1 \rangle|^2.$$

Choice of uncertainties (2)

⋆ Bessel inequalities give :

$$\langle \mathcal{D}, \mathcal{D} \rangle \geq |\langle \mathcal{D}, e_0 \rangle|^2,$$

⋆ It leads to the definition of an uncertainty set $[\delta_0, \delta_1]^T$:

$$\delta_0 = \sqrt{h}\, \langle \mathcal{D}, e_0 \rangle \qquad = \int_{-h}^{0} e^{s\theta} d\theta, \tag{6}$$

$$\delta_1 = \sqrt{\frac{h}{3}} \langle \mathcal{D}, e_1 \rangle = \int_{-h}^{0} e^{s\theta} \left(\frac{2}{h}\theta + 1\right) d\theta.$$

⋆ This last inequality is very similar to the extended Wirtinger inequality employed by [Seuret12] to derive less conservative results in LKF framework.

How to use these "uncertainties" as operators ?

⋆ Let note that

$$\delta_0[x(t)] = \int_{-h}^{0} x(s)ds$$

$$\delta_0[\dot{x}(t)] = x(t) - x(t-h)$$

$$\begin{aligned}
\delta_1[x(t)] &= \int_{-h}^{0} x(t+\theta)\Big(\frac{2}{h}\theta + 1\Big)d\theta, \\
\delta_1[\dot{x}(t)] &= \int_{-h}^{0} \dot{x}(t+\theta)\Big(\frac{2}{h}\theta + 1\Big)d\theta, \\
&= x(t) + x(t-h) - \frac{2}{h}\,\delta_0[x(t)]. \qquad (7)
\end{aligned}$$

Choice of the overall uncertainties

⋆ Bessel inequality applied to the delay operator $\mathcal{D}$ give some clues to consider an uncertainty ∇

$$\nabla = \begin{bmatrix} s^{-1}1_n & & & \\ & s^{-1}1_n & & \\ & & e^{-hs}1_n & \\ & & & \delta_0 1_n \\ & & & \delta_1 1_n \end{bmatrix} \quad (8)$$

It allows also to define the relation $w(t) = \nabla z(t)$,

$$w(t) = \begin{bmatrix} x(t) \\ \int_{t-h}^{t} x(\theta)d\theta \\ x(t-h) \\ \alpha(t) \\ \delta_1[\dot{x}(t)] \end{bmatrix} \text{ and } z(t) = \begin{bmatrix} \dot{x}(t) \\ \alpha(t) \\ x(t) \\ \dot{x}(t) \end{bmatrix}, \quad (9)$$

with $\alpha(t) = x(t) - x(t-h)$.

Choice of the singular linear transformation

The linear transformation is straightforwardly described by :

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathcal{E}} z(t) = \underbrace{\begin{bmatrix} A & 0 & A_d & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ A & 0 & A_d & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ -1 & 2/h & -1 & 0 & 1 \end{bmatrix}}_{\mathcal{A}} w(t). \tag{10}$$

Choice of a separator (conservative choice) (1)

⋆ As soon as the modeling is chosen, we look for a separator :

Lemma

A quadratic constraint for the operator s^{-1} is given by

$$\begin{bmatrix} 1_n \\ s^{-1}1_n \end{bmatrix}^* \begin{bmatrix} 0 & -P \\ -P & 0 \end{bmatrix} \begin{bmatrix} 1_n \\ s^{-1}1_n \end{bmatrix} \leq 0, P > 0.$$

Lemma

A quadratic constraint for the operator e^{-hs} is given by

$$\begin{bmatrix} 1_n \\ e^{-hs}1_n \end{bmatrix}^* \begin{bmatrix} -Q & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} 1_n \\ e^{-hs}1_n \end{bmatrix} \leq 0, Q > 0.$$

⋆ Well known result from [Iwasaki 98, Peaucelle 07]

Choice of a separator (conservative choice) (2)

Lemma

A quadratic constraint for the operator $[\delta_0, \delta_1]^T$ *is given by*

$$\begin{bmatrix} 1_n \\ \delta_0 1_n \\ \delta_1 1_n \end{bmatrix}^* \begin{bmatrix} -h^2 R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & 3R \end{bmatrix} \begin{bmatrix} 1_n \\ \delta_0 1_n \\ \delta_1 1_n \end{bmatrix} \leq 0, R > 0.$$

This inequality comes from Bessel inequality :

$$\delta_0 R \delta_0^* + 3\delta_1 R \delta_1^* - h^2 R \leq 0.$$

Choice of a separator (conservative choice) (3)

Gathering these three inequalities, a separator for

$$\nabla = \begin{bmatrix} s^{-1}1_n & & & & \\ & s^{-1}1_n & & & \\ & & e^{-hs}1_n & & \\ & & & \delta_0 1_n & \\ & & & \delta_1 1_n & \end{bmatrix} \tag{11}$$

could be built :

$$\Theta = \left[\begin{array}{ccc|cccc} 0 & 0 & 0 & -P & 0 & 0 & 0 \\ 0 & -Q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -h^2 R & 0 & 0 & 0 & 0 \\ \hline -P & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & R & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3R \end{array}\right], \tag{12}$$

A stability criterion for a known delay h

⋆ Using quadratic separation theorem, we get :

$$\Phi_{N=2} = \begin{bmatrix} \mathcal{E} & -\mathcal{A}(h) \end{bmatrix}^{\perp *} \Theta(h) \begin{bmatrix} \mathcal{E} & -\mathcal{A}(h) \end{bmatrix}^{\perp} > 0,$$

where $N = 2$ indicates the number of polynomials used in Bessel inequality.
⋆ $\mathcal{A}$ and Θ are depending on h.
⋆ fixing h, this criterion is an LMI in P, Q, R, three positive definite matrices.

The case of N orthogonal polynomials

$\star$ The general case uses $N+1$ Legendre polynomials $\{e_0, e_1, e_N\}$ and the inequality :

$$\sum_{k=0}^{N} |\langle f, e_k\rangle|^2 \leq \langle f, f\rangle, \tag{13}$$

and therefore :

$$\sum_{k=0}^{N}(2k+1)\,\delta_k\delta_k^* \leq h^2.$$

where

$$\delta_k = \sqrt{\frac{h}{2k+1}}\,\langle f, e_k\rangle = \int_{-h}^{0}(-1)^k \sum_{l=0}^{k} p_l^k \left(\frac{\theta+h}{h}\right)^l e^{s\theta}d\theta.$$

$\rightarrow$ stability criterion for a given delay h.

The pointwise delay case to the delay range case

⋆ The stability criterion is only valid for a given delay h.

→ Stability for a pointwise delay.

⋆ If the delay h is belonging to a prescribed interval $[h_{\min}, h_{\max}]$, the proposed criterion can be transformed into an LMI criterion linear with respect to the delay h (use of the vertex separator proposed by [Iwasaki 98]).
⋆ Possible alternative : use of slack variables. → stability of delay system with

interval delay (possibly excluding zero).

A classical example

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t-h). \qquad (14)$$

Theorems	h_{max}	number of variables
Gu03 ($\tilde{N}=1$)	6.053	$5.5n^2 + 2.5n$
Gu03 ($\tilde{N}=2$)	6.156	$9.5n^2 + 3.5n$
Gu03 ($\tilde{N}=3$)	6.162	$17.5n^2 + 4.5n$
Gu03 ($\tilde{N}=4$)	6.163	$20.5n^2 + 5.5n$
Th.1 ($N=0$)	4.472	$1.5n^2 + 1.5n$
Th.1 ($N=1$)	6.059	$3n^2 + 2n$
Th.1 ($N=2$)	6.166	$5.5n^2 + 2.5n$
Th.1 ($N=3$)	6.1719	$9n^2 + 3n$
Th.1 ($N=4$)	6.17250	$13.5n^2 + 3.5n$

TABLE: Pointwise method : maximal allowable delay h_{max}.

A classical example

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t-h). \tag{15}$$

Theorems	h_{max}	number of variables
Gouaisbaut06	4.472	$1.5n^2 + 1.5n$
He07b	4.472	$3n^2 + 3n$
Shao2009	4.472	$2.5n^2 + 1.5n$
Kim2011	4.97	$69n^2 + 5n$
Sun2010	5.02	$18n^2 + 18n$
Ariba10	5.120	$7n^2 + 4n$
Kao07	6.1107	$1.5n^2 + 9n + 9$

TABLE: Delay-range method : maximal allowable delay h_{max}.

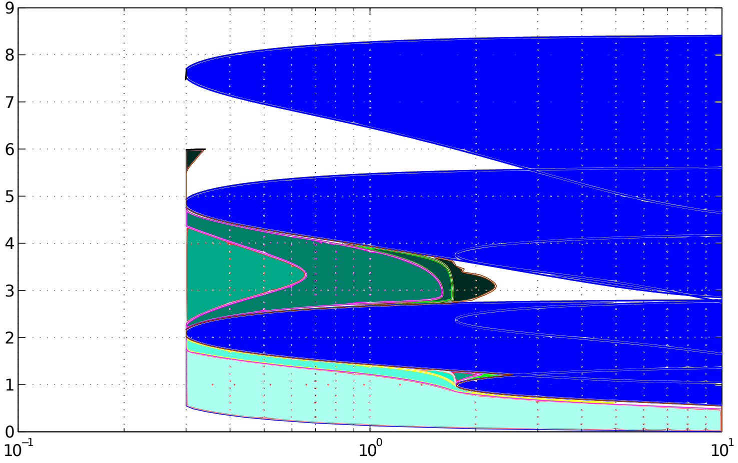
Example 2

⋆ Let consider

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(10+K) & 10 & 0 & 0 \\ 5 & -15 & 0 & -0.25 \end{bmatrix}, A_d = \begin{bmatrix} 0 \\ 0 \\ K \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

where K is a control parameter

Example 2

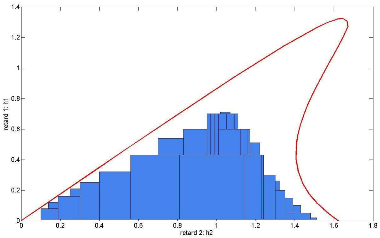


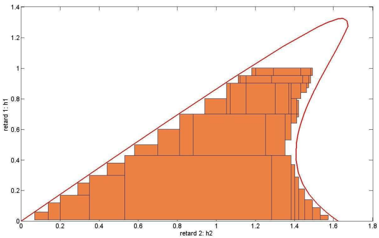
- ⋆ The stability regions are calculated via generalized Delay margins.
- ⋆ For different N et K, inner stability regions are estimated.

An example with two delays via a direct extension

$$\ddot{y}(t) + 2y(t - h_1) - 1.75y(t - h_2) = 0$$

⋆ for $h_1 = h_2 = 0$, the system is unstable.





Conclusion

- We have proposed two criteria for assessing the pointwise and delay-range stability of time delay systems.
- The approach is based on quadratic separation and Legendre orthogonal polynomials and Bessel inequality.
- It provides a sequence of LMIs conditions which are less and less conservative, at least on examples.
- Future work will be devoted to the proof of the conservatism reduction and to the extension of this work to the time varying delay.