$\mathcal{H}_2$ optimal model approximation by structured time-delay reduced order models

I. Pontes Duff, Charles Poussot-Vassal, C. Seren

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27th September, GT MOSAR and GT SAR meeting

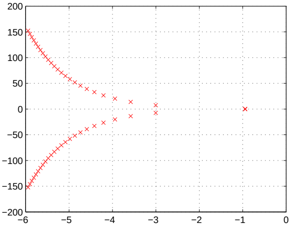

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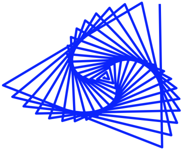
Table of Contents



Outlines

Introduction

Preliminaries on $\mathcal{H}_2$ model approximation

$\mathcal{H}_2$ optimal model reduction with I/O delay structure

$\mathcal{H}_2$ optimality conditions for reduced state-delay systems

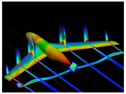
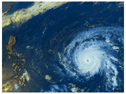
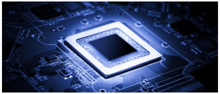
Conclusions and perspectives

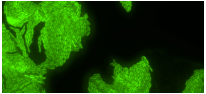
Large-scale models

Some motivating examples in the simulation & control domains...

Digitalized and computer-based modeling and studies are crucial steps for any system / concept or physical phenomena understanding



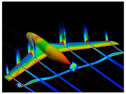
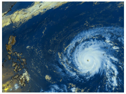
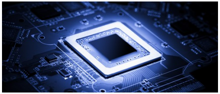


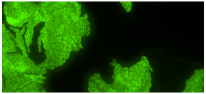

Large-scale models

Some motivating examples in the simulation & control domains...

Digitalized and computer-based modeling and studies are crucial steps for any system / concept or physical phenomena understanding






Problem: involved numerical dynamical models are too complex

Due to finite machine precision, computation burden and memory management,

- difficulties with system simulation, analysis, optimization, controller design
- results are not accurate, time consumption
- inappropriate actual numerical tools

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Large-scale models

... *e.g.* in aeronautics

Physical system

rigid behaviour

fluid mechanics, structure, etc.

Differential Algebraic Equations (DAEs) or Rational Functions

$$E\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

$$H(s) = H_1(s) + H_d e^{-\tau s} + \dots$$

Partial Differential Equations (PDEs)

$$-\overrightarrow{\nabla} p + \rho \overrightarrow{g} = \rho \overrightarrow{a}$$

$$\frac{\partial \rho}{\partial t} + \overrightarrow{\nabla}.(\rho \overrightarrow{v}) = 0$$

- Highly accurate and/or flexible aircraft
- Spacecraft, launcher, satellites,
- Fluid dynamics physics (Navier and Stokes)

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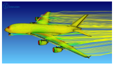
Discretisation

finite elements, finite volume, etc.



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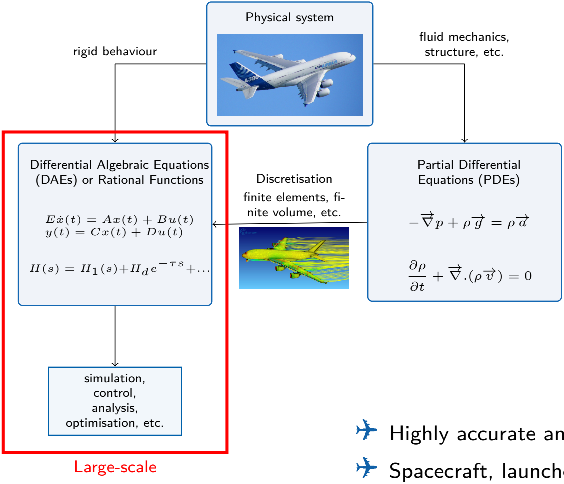

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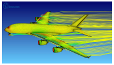

Partial Differential Equations (PDEs)

$$-\vec{\nabla} p + \rho \vec{g} = \rho \vec{a}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}.(\rho \vec{v}) = 0$$

~~simulation, control, analysis, optimisation, etc.~~

Large-scale

objective: alleviate numerical burden

- Highly accurate and/or flexible aircraft
- Spacecraft, launcher, satellites,
- Fluid dynamics physics (Navier and Stokes)

Introduction

Topics addressed in this presentation :

- $\mathcal{H}_2$-model approximation with I/O delay structure.
- $\mathcal{H}_2$ optimality conditions for reduced state-delay systems.

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Introduction

Topics addressed in this presentation :

- $\mathcal{H}_2$-model approximation with I/O delay structure.
- $\mathcal{H}_2$ optimality conditions for reduced state-delay systems.

Goal: find a nth order rational model approximation in the form

$$\hat{\mathbf{H}}(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B}$$

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Introduction

Topics addressed in this presentation :

- $\mathcal{H}_2$-model approximation with I/O delay structure.
- $\mathcal{H}_2$ optimality conditions for reduced state-delay systems.

Delay Goal: find a nth order rational I/O delay structured model approximation in the form

$$\hat{\mathbf{H}}_d(s) = \hat{\Delta}_o(s)\hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B}\hat{\Delta}_i(s)$$

where $\hat{\Delta}_i(s) = \mathbf{diag}(e^{-s\hat{\tau}_1}, \ldots, e^{-s\hat{\tau}_{n_u}})$ and $\hat{\Delta}_o(s) = \mathbf{diag}(e^{-s\hat{\gamma}_1}, \ldots, e^{-s\hat{\gamma}_{n_y}})$

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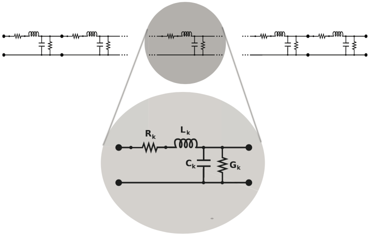
Benchmarks

Example 1: Ladder network[1]



$$\mathbf{G}_{Ladder} := \begin{cases} E\dot{\mathbf{x}}(t) & = & A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) & = & C\mathbf{x}(t) \end{cases} \quad (1)$$

- Finite dimensional model of order $N = 100$.
- It has an intrinsic input-delay behavior.

[1] Gugercin, S., Polyuga R., G., Beattie C., van der Schaft, A., *"Structure-preserving tangential interpolation for model reduction of port-Hamiltonian systems"*, Automatica,, 48(9):1963–1974.

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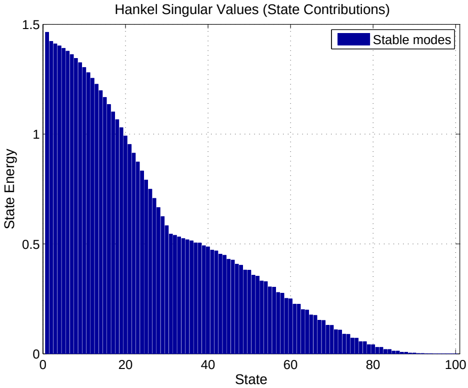

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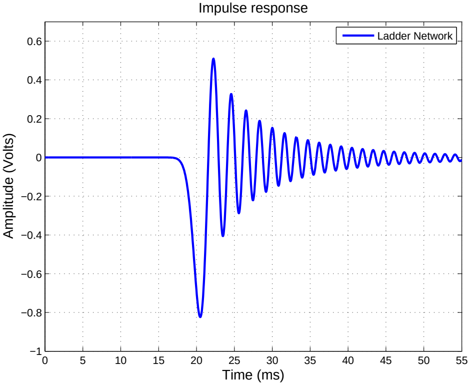

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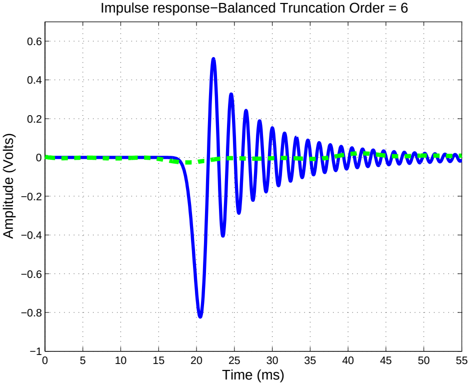

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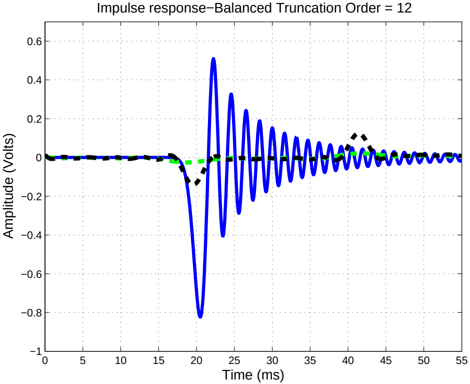

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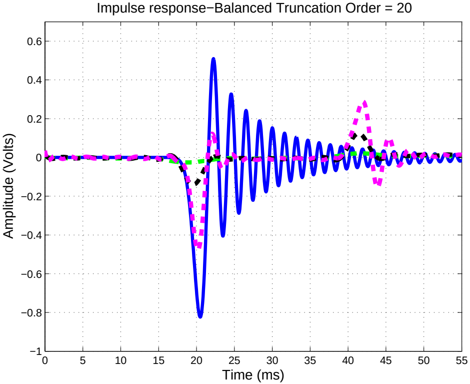

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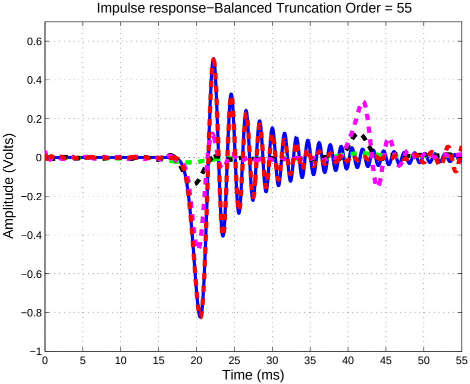

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Benchmarks

St-Venant[2]

PDE St-Venant equations... toward linearisation

$$\begin{aligned} \frac{\partial S}{\partial t} + \frac{\partial Q}{\partial x} &= 0 \\ \frac{\partial Q}{\partial t} + \frac{\partial (Q^2/S)}{\partial x} + gS\frac{\partial H}{\partial x} &= gS(I-J), \end{aligned} \quad (1)$$

- $x \in [0; L]$ is the spatial variable, $H(x,t)$ the water depth,
- $S(x,t)$ the wetted area,
- $Q(x,t)$ the discharge...

Step 1: Apply linearisation at (H_0, Q_0), which are both x dependent.

Step 2: Apply Laplace transformation around equilibrium.

Step 3: Find solutions of $h(s,x)$, $q(s,x)$ and identify coefficient with boundary conditions

[2] Dalmas, V., Robert, G., Poussot-Vassal, C., Pontes Duff, I. and Seren, C. , *"Parameter dependent irrational and infinite dimensional modelling and approximation of an open-channel dynamics"*, in Proceedings of the 15th European Control Conference, (ECC'16), Aalborg, Denmark, July, 2016..

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Benchmarks

St-Venant

$$\mathbf{H}(s,x,Q_0) = \begin{bmatrix} \mathbf{G_e}(s,x,Q_0) & -\mathbf{G_s}(s,x,Q_0) \end{bmatrix} \begin{bmatrix} q_e(s) \\ q_s(s) \end{bmatrix} \quad (2)$$

where

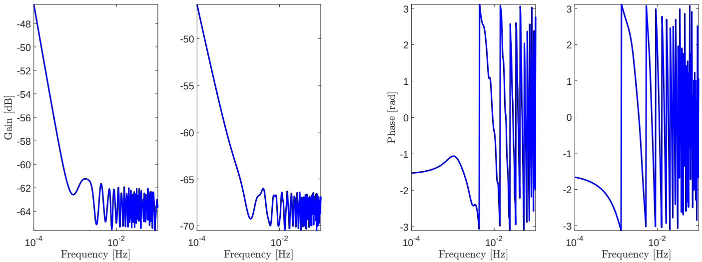
$$\begin{aligned} \mathbf{G_e}(s,x,Q_0) &= \frac{\lambda_1(s)e^{\lambda_2(s)L+\lambda_1(s)x} - \lambda_2(s)e^{\lambda_1(s)L+\lambda_2(s)x}}{B_0 s(e^{\lambda_1(s)L} - e^{\lambda_2(s)L})} \\ \mathbf{G_s}(s,x,Q_0) &= \frac{\lambda_1(s)e^{\lambda_1(s)x} - \lambda_2(s)e^{\lambda_2(s)x}}{B_0 s(e^{\lambda_1(s)L} - e^{\lambda_2(s)L})} \end{aligned} \quad (3)$$

- Irrational transfer function.
- infinite model order.

Benchmarks

St-Venant

$$\mathbf{H}(s,x,Q_0) = \begin{bmatrix}\mathbf{G_e}(s,x,Q_0) & -\mathbf{G_s}(s,x,Q_0)\end{bmatrix} \begin{bmatrix} q_e(s) \\ q_s(s) \end{bmatrix} \quad (2)$$



- Experience and simulations shows I/O-delay behavior.
- Try to search I/O delay approximation.

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Outlines

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Delay-free $\mathcal{H}_2$ model approximation problem formulation

Model approximation $\sim$ Mathematical optimization

Objectives: find a reduced order system $\hat{\mathbf{H}}$ for which:

- ✔ the approximation error is small;
- ✔ and the stability is preserved. . .

. . . based on a procedure computationally stable and efficient.

The quality of the approximation can be evaluated using some mathematical norms. For any given system $\mathbf{G}$ of order $N \in \mathbb{N}^\star$, let find $\hat{\mathbf{H}}$ defined by:

$$\hat{\mathbf{H}} := \begin{cases} \hat{E}\dot{x}(t) = \hat{A}x(t) + \hat{B}u(t) \\ y(t) = \hat{C}x(t) \end{cases}$$

s.t.:

$\mathcal{J} = \|\mathbf{G} - \hat{\mathbf{H}}\|^2$ is minimum $\rightarrow$ **optimisation problem to solve**

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Preliminaries on $\mathcal{H}_2$ model approximation

Delay-free $\mathcal{H}_2$ model approximation

Recall: $\langle \mathbf{G}, \hat{\mathbf{H}} \rangle_{\mathcal{H}_2} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \overline{\mathbf{G}(i\omega)} \hat{\mathbf{H}}(i\omega) d\omega$

Mathematical formulation

Find $\hat{\mathbf{H}}^\star$ of order $n << N$ which minimizes:

$$\hat{\mathbf{H}}^\star := \underset{\substack{\hat{\mathbf{H}} \in \mathcal{H}_2 \\ \dim(\hat{\mathbf{H}}) = n}}{\textbf{arg min}} ||\mathbf{G} - \hat{\mathbf{H}}||_{\mathcal{H}_2} \tag{3}$$

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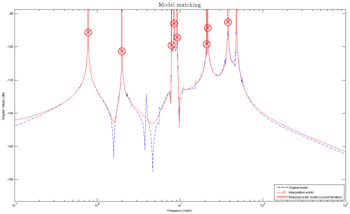
Rational Interpolation: Given shift points $\sigma_1, \ldots, \sigma_r \in \mathbb{C}$ find $\hat{\mathbf{H}} = (\hat{E}, \hat{A}, \hat{B}, \hat{C})$ s.t.

$$\begin{aligned} \mathbf{H}(\sigma_j) &= \hat{\mathbf{H}}(\sigma_j) \\ \frac{d}{ds}\mathbf{H}(s)\Big|_{s=\sigma_j} &= \frac{d}{ds}\hat{\mathbf{H}}(s)\Big|_{s=\sigma_j} \end{aligned}$$



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Preliminaries on $\mathcal{H}_2$ model approximation

$\mathcal{H}_2$ optimality conditions[3]

Let

$$\hat{\mathbf{H}}(s) = \sum_{k=1}^{n} \frac{\hat{\phi}_k}{s - \hat{\lambda}_k}$$

delay-free $\mathcal{H}_2$-optimality conditions (SISO)

If $\hat{\mathbf{H}}$ is a local optimum of $\mathcal{H}_2$ approximation problem, then

$$\begin{aligned} \hat{\mathbf{H}}(-\hat{\lambda}_k) &= \mathbf{G}(-\hat{\lambda}_k) \\ \hat{\mathbf{H}}'(-\hat{\lambda}_k) &= \mathbf{G}'(-\hat{\lambda}_k) \end{aligned} \quad (4)$$

for $k = 1, \ldots, n$.

[3] S. Gugercin, A.C. Antoulas and C. Beattie, *" $\mathcal{H}_2$ model reduction for large-scale linear dynamical systems"*, SIAM Journal on matrix analysis and applications, vol. 30, no. 2, pp. 609–638, 2008.

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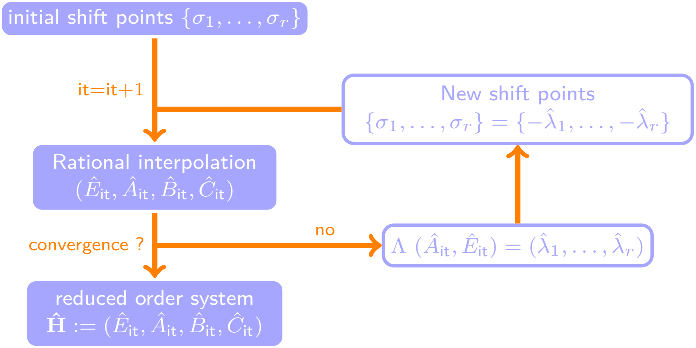
Preliminaries on $\mathcal{H}_2$ model approximation

$\mathcal{H}_2$ optimality conditions[4]

Given a system $\mathbf{G} \in \mathcal{H}_2$,



- Point-fixed iterative techniques: **IRKA**, **TF-IRKA**, ...
- Rational interpolation: **Krylov subspaces**, **Loewner framework**, ...

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$\mathcal{H}_2$ optimal model reduction with I/O delay structure

Problem formulation and goals

Let $\hat{\mathbf{H}} = (\hat{E}, \hat{A}, \hat{B}, \hat{C}, 0)$ be defined as:

$$\begin{cases} \hat{E}\dot{x}(t) = \hat{A}x(t) + \hat{B}u(t) \\ y(t) = \hat{C}x(t) \end{cases} \quad (5)$$

whose transfer function is

$$\hat{\mathbf{H}}(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B} \quad (6)$$

$\mathcal{H}_2$ model approximation

Given a system $\mathbf{G} \in \mathcal{H}_2$, the goal is to find a system $\hat{\mathbf{H}}^\star$

$$\hat{\mathbf{H}}^\star := \underset{\hat{\mathbf{H}} \in \mathcal{H}_2, \dim(\hat{\mathbf{H}}) \leq r}{\textbf{arg min}} \|\mathbf{G} - \hat{\mathbf{H}}\|_{\mathcal{H}_2}.$$

$\mathcal{H}_2$ optimal model reduction with I/O delay structure

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Let $\hat{\mathbf{H}}_d = (\hat{E}, \hat{A}, \hat{B}, \hat{C}, \hat{\Delta}_i(s), \hat{\Delta}_o(s))$ be defined as:

$$\begin{cases} \hat{E}\dot{x}(t) = \hat{A}x(t) + \hat{B}\hat{\Delta}_i(u(t)) \\ y(t) = \hat{\Delta}_o(\hat{C}x(t)) \end{cases} \qquad (7)$$

whose transfer function is

$$\hat{\mathbf{H}}_d(s) = \hat{\Delta}_o(s)\hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B}\hat{\Delta}_i(s) \qquad (8)$$

where $\hat{\Delta}_i(s) = \textbf{diag}(e^{-s\hat{\tau}_1}, \ldots, e^{-s\hat{\tau}_{n_u}})$ and $\hat{\Delta}_o(s) = \textbf{diag}(e^{-s\hat{\gamma}_1}, \ldots, e^{-s\hat{\gamma}_{n_y}})$

I/O Delay $\mathcal{H}_2$ model approximation

Given a system $\mathbf{G} \in \mathcal{H}_2$, the goal is to find a system $\hat{\mathbf{H}}_d^\star = (\hat{E}, \hat{A}, \hat{B}, \hat{C}, \hat{\Delta}_i(s), \hat{\Delta}_o(s))$

$$\hat{\mathbf{H}}_d^\star := \underset{\hat{\mathbf{H}}_d \in \mathcal{H}_2, \dim(\hat{\mathbf{H}}_d) \leq r}{\textbf{arg min}} \|\mathbf{G} - \hat{\mathbf{H}}_d\|_{\mathcal{H}_2}.$$

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$\mathcal{H}_2$ optimal model reduction with I/O delay structure

Problem formulation and goals

Let $\hat{\mathbf{H}} = (\hat{E}, \hat{A}, \hat{B}, \hat{C}, 0)$ be defined as:

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$$\hat{\mathbf{H}}(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B} \tag{10}$$

$\mathcal{H}_2$ model approximation

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$$\hat{\mathbf{H}}^\star := \underset{\hat{\mathbf{H}} \in \mathcal{H}_2, \dim(\hat{\mathbf{H}}) \leq r}{\arg\min} \|\mathbf{G} - \hat{\mathbf{H}}\|_{\mathcal{H}_2}.$$

Let $\hat{\mathbf{H}}_d = (\hat{E}, \hat{A}, \hat{B}, \hat{C}, \tau)$ be defined as:

$$\begin{cases} \hat{E}\dot{x}(t) = \hat{A}x(t) + \hat{B}u(t-\tau) \\ y(t) = \hat{C}x(t) \end{cases} \tag{11}$$

whose transfer function is

$$\hat{\mathbf{H}}_\mathbf{d}(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B}e^{-\tau s} \tag{12}$$

(SISO) Input-Delay $\mathcal{H}_2$ model approximation

Given a system $\mathbf{G} \in \mathcal{H}_2$, the goal is to find a system $\hat{\mathbf{H}}_d = (\hat{E}, \hat{A}, \hat{B}, \hat{C}, \tau)$

$$\hat{\mathbf{H}}_\mathbf{d}^\star := \underset{\hat{\mathbf{H}}_\mathbf{d} \in \mathcal{H}_2, \dim(\hat{\mathbf{H}}_d) \leq r}{\arg\min} \|\mathbf{G} - \hat{\mathbf{H}}_d\|_{\mathcal{H}_2}.$$

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$\mathcal{H}_2$ optimal model reduction with I/O delay structure

Input delay $\mathcal{H}_2$ inner product

Given a stable N-th order system $\mathbf{G} \in \mathcal{H}_2$, find a reduced n-th order (such that $n \ll N$) stable input-delay delays model $\hat{\mathbf{H}}_d = (\hat{E}, \hat{A}, \hat{B}, \hat{C}, \hat{\tau})$ which minimizes

$$\mathcal{J}_2 = \|\mathbf{G} - \hat{\mathbf{H}}_d\|_{\mathcal{H}_2}^2. \tag{13}$$

Suppose that $\hat{\mathbf{H}}_d = \hat{\mathbf{H}}e^{-s\hat{\tau}}$, and both models have the pole residue decomposition

$$\mathbf{G}(s) = \sum_{j=1}^{N} \frac{\psi_j}{s - \mu_j} \text{ and } \hat{\mathbf{H}}(s) = \sum_{k=1}^{n} \frac{\hat{\phi}_k}{s - \hat{\lambda}_k}$$

Then,

$$\begin{aligned} \mathcal{J}_2 &= \|\mathbf{G} - \hat{\mathbf{H}}_d\|_{\mathcal{H}_2}^2 \\ &= \|\mathbf{G}\|_{\mathcal{H}_2}^2 - 2\langle \mathbf{G}, \hat{\mathbf{H}}e^{-s\tau}\rangle_{\mathcal{H}_2} + \|\hat{\mathbf{H}}e^{-s\tau}\|_{\mathcal{H}_2}^2. \end{aligned} \tag{14}$$

- Compute $\mathcal{H}_2$ inner product in the presence of a input delay.

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$\mathcal{H}_2$ optimal model reduction with I/O delay structure

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- Compute $\mathcal{H}_2$ inner product in the presence of a input delay.

$\mathcal{H}_2$ optimal model reduction with I/O delay structure

Input delay $\mathcal{H}_2$ inner product

Recall : $\langle \mathbf{G}, \hat{\mathbf{H}} \rangle_{\mathcal{H}_2} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \overline{\mathbf{G}(i\omega)} \hat{\mathbf{H}}(i\omega) d\omega.$

First, $\|\hat{\mathbf{H}} e^{-s\tau}\|_{\mathcal{H}_2}$?

H_2-norm input-delay invariance

Let $\hat{\mathbf{H}} \in \mathcal{H}_2$ and $\tau > 0$. Then $\hat{\mathbf{H}} e^{-s\tau} \in \mathcal{H}_2$ and

$$\|\hat{\mathbf{H}}\|_{\mathcal{H}_2} = \|\hat{\mathbf{H}} e^{-s\tau}\|_{\mathcal{H}_2}$$

Proof.

$$2\pi \|\mathbf{H} e^{-s\tau}\|^2_{\mathcal{H}_2} = \int_{-\infty}^{\infty} \overline{\mathbf{H}(j\omega)} e^{j\omega} \mathbf{H}(j\omega) e^{-j\omega} d\omega = \int_{-\infty}^{\infty} \overline{\mathbf{H}(j\omega)} \mathbf{H}(j\omega) d\omega = 2\pi \|\mathbf{H}\|^2_{\mathcal{H}_2}$$

□

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$\mathcal{H}_2$ optimal model reduction with I/O delay structure

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$\mathcal{H}_2$ optimal model reduction with I/O delay structure

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$\mathcal{H}_2$ optimal model reduction with I/O delay structure

Input delay $\mathcal{H}_2$ inner product

Recall : If both systems are real, $\langle \mathbf{G}, \hat{\mathbf{H}} \rangle_{\mathcal{H}_2} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \overline{\mathbf{G}(i\omega)} \hat{\mathbf{H}}(i\omega) d\omega.$

Delay-less $\mathcal{H}_2$-inner product expression

Let $\mathbf{G} \in \mathcal{H}_2$ to be a strictly proper real model, $\phi \in \mathbb{C}$ and $\hat{\lambda} \in \mathbb{C}^-$. Then

$$\langle \mathbf{G}, \frac{\phi}{s - \hat{\lambda}} \rangle_{\mathcal{H}_2} = \mathbf{G}(-\hat{\lambda})\phi.$$

$\mathcal{H}_2$ optimal model reduction with I/O delay structure

Input delay $\mathcal{H}_2$ inner product

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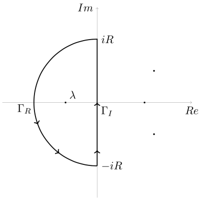
Proof Cauchy's residues theorem



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$\mathcal{H}_2$ optimal model reduction with I/O delay structure

Input delay $\mathcal{H}_2$ inner product

Input delay $\mathcal{H}_2$-inner product expression

Let $\mathbf{G} \in \mathcal{H}_2$ to be a strictly proper real model expressed by

$$\mathbf{G}(s) = \sum_{j=1}^{N} \frac{\psi_j}{s - \mu_j}.$$

Let $\tau > 0$ and $\hat{\lambda} \in \mathbb{C}^-$. Then

$$\langle \mathbf{G}, \frac{e^{-s\tau}}{s - \hat{\lambda}} \rangle_{\mathcal{H}_2} = \sum_{j=1}^{N} \frac{\psi_j e^{-\mu_j \tau}}{-\hat{\lambda} - \mu_j}. \quad (15)$$

- The expression depends on the pole residue decomposition of $\mathbf{G}$.
- The delay "break the structure" of $\mathbf{G}$.

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$\mathcal{H}_2$ optimal model reduction with I/O delay structure

Input delay $\mathcal{H}_2$ inner product

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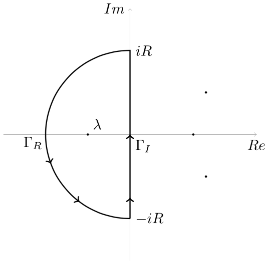
$\mathcal{H}_2$ optimal model reduction with I/O delay structure

Input delay $\mathcal{H}_2$ inner product

Proof Cauchy's theorem

- NOT POSSIBLE!! Because of exponential growth $e^{-s\tau}$

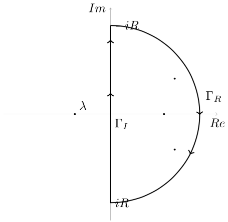


$\mathcal{H}_2$ optimal model reduction with I/O delay structure

Input delay $\mathcal{H}_2$ inner product

Proof Cauchy's theorem

- Use other contour encircling the poles of $\mathbf{G}$



$\mathcal{H}_2$ optimal model reduction with I/O delay structure

Finally we are able to characterize the inner product.

Input delay $\mathcal{H}_2$ inner product computation

Let $\mathbf{G}$, $\hat{\mathbf{H}}$ be two SISO systems in $\mathcal{H}_2$ whose respective transfer functions

$$\mathbf{G}(s) = \sum_{j=1}^{N} \frac{\psi_j}{s-\mu_j} \text{ and } \hat{\mathbf{H}}(s) = \sum_{k=1}^{n} \frac{\hat{\phi}_k}{s-\hat{\lambda}_k},$$

and let $\tau > 0$. Hence, if $\hat{\mathbf{H}}_d = \hat{\mathbf{H}}e^{-s\tau}$, the inner product $\langle \mathbf{G}, \hat{\mathbf{H}}_d \rangle_{\mathcal{H}_2}$ is given by:

$$\langle \mathbf{G}, \hat{\mathbf{H}}_d \rangle_{\mathcal{H}_2} = \sum_{j=1}^{N} \hat{\mathbf{H}}(-\mu_j)\psi_j e^{\tau\mu_j}. \tag{16}$$

$\mathcal{H}_2$ optimal model reduction with I/O delay structure

Input delay $\mathcal{H}_2$ inner product

Finally we are able to characterize the inner product.

Input delay $\mathcal{H}_2$ inner product computation

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and let $\tau > 0$. Hence, if $\hat{\mathbf{H}}_d = \hat{\mathbf{H}}e^{-s\tau}$, the inner product $\langle \mathbf{G}, \hat{\mathbf{H}}_d \rangle_{\mathcal{H}_2}$ is given by:

$$\begin{aligned} \langle \mathbf{G}, \hat{\mathbf{H}}e^{-s\tau} \rangle_{\mathcal{H}_2} &= \sum_{j=1}^{N} \hat{\mathbf{H}}(-\mu_j)\psi_j e^{\tau\mu_j} \\ &= \langle \tilde{\mathbf{G}}, \hat{\mathbf{H}} \rangle_{\mathcal{H}_2} \end{aligned} \tag{16}$$

where

$$\tilde{\mathbf{G}}(s) = \sum_{j=1}^{N} \frac{\psi_j e^{\tau\mu_j}}{s-\mu_j}$$

$\mathcal{H}_2$ optimal model reduction with I/O delay structure

Input delay $\mathcal{H}_2$ inner product

Let $\mathbf{G}(s) = \frac{1}{s+1} = \frac{\psi}{s-\mu}$ and $\mathbf{H}(s) = \frac{1}{s+2} = \frac{\phi}{s-\lambda}$.

- Delay-free case:

$$\langle \mathbf{G}, \mathbf{H} \rangle_{\mathcal{H}_2} = \psi \mathbf{H}(-\mu) = \psi \frac{\phi}{-\mu - \lambda} = \frac{1}{3} = \phi \frac{\psi}{-\lambda - \mu} = \phi \mathbf{G}(-\lambda) = \langle \mathbf{H}, \mathbf{G} \rangle_{\mathcal{H}_2}$$

$\mathcal{H}_2$ inner product can be computed using pole-residues decomposition of $\mathbf{G}$ or $\mathbf{H}$.

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$\mathcal{H}_2$ optimal model reduction with I/O delay structure

Input delay $\mathcal{H}_2$ inner product

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$\mathcal{H}_2$ inner product can be computed using pole-residues decomposition of $\mathbf{G}$ or $\mathbf{H}$.

- Input-delay case: Let $\tau = 1$. By noticing that, $\langle \mathbf{G}, \hat{\mathbf{H}} e^{-s} \rangle_{\mathcal{H}_2} = \langle \mathbf{G} e^{s}, \hat{\mathbf{H}} \rangle_{\mathcal{H}_2}$, one apply the symmetric version as follows :

$$\frac{1}{3} e^{-1} = \langle \mathbf{G}, \hat{\mathbf{H}} e^{-s} \rangle_{\mathcal{H}_2} = \underbrace{\langle \mathbf{G} e^{s}, \hat{\mathbf{H}} \rangle_{\mathcal{H}_2} \neq \hat{\phi} \mathbf{G}(-\lambda) e^{-\tau \hat{\lambda}}}_{\text{incorrect symmetric version}} = \hat{\phi} \frac{\psi}{-\hat{\lambda}-\mu} e^{-\lambda\tau} = \frac{1}{3} e^{2}.$$

Symmetric version of the $\mathcal{H}_2$ inner product does not provide the same result any more.

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$\mathcal{H}_2$ optimal model reduction with I/O delay structure

Input delay $\mathcal{H}_2$ inner product

Let $\mathbf{G}(s) = \frac{1}{s+1} = \frac{\psi}{s-\mu}$ and $\mathbf{H}(s) = \frac{1}{s+2} = \frac{\phi}{s-\lambda}$.

- Delay-free case:

$$\langle \mathbf{G}, \mathbf{H} \rangle_{\mathcal{H}_2} = \psi \mathbf{H}(-\mu) = \psi \frac{\phi}{-\mu - \lambda} = \frac{1}{3} = \phi \frac{\psi}{-\lambda - \mu} = \phi \mathbf{G}(-\lambda) = \langle \mathbf{H}, \mathbf{G} \rangle_{\mathcal{H}_2}$$

$\mathcal{H}_2$ inner product can be computed using pole-residues decomposition of $\mathbf{G}$ or $\mathbf{H}$.

- Input-delay case: Let us compute $\mathcal{H}_2$ inner product between $\mathbf{H}e^{-s}$ and $\mathbf{G}$ using the extended formula:

$$\langle \mathbf{G}, \mathbf{H}e^{-s\tau} \rangle_{\mathcal{H}_2} = \psi \mathbf{H}(-\mu) e^{\tau\mu} = \psi \frac{\phi}{-\mu - \lambda} e^{\tau\mu} = \frac{1}{3} e^{-1}.$$

Which modifies the optimality conditions.

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$\mathcal{H}_2$ optimal model reduction with I/O delay structure

Input delay $\mathcal{H}_2$ Optimality conditions

Recall: $\mathbf{G}(s) = \sum_{k=1}^{N} \frac{\psi_k}{s-\mu_k}$ and $\hat{\mathbf{H}}(s) = \sum_{k=1}^{n} \frac{\hat{\phi}_k}{s-\hat{\lambda}_k}$

delay-free $\mathcal{H}_2$-optimality conditions

If $\hat{\mathbf{H}}$ is a local optimum of $\mathcal{H}_2$ problem, then (interpolation condition on $\tilde{\mathbf{G}}$).

$$\begin{aligned} \hat{\mathbf{H}}(-\hat{\lambda}_k) &= \mathbf{G}(-\hat{\lambda}_k) \\ \hat{\mathbf{H}}'(-\hat{\lambda}_k) &= \mathbf{G}'(-\hat{\lambda}_k) \end{aligned} \quad (17)$$

for $k = 1, \ldots, n$.

$\mathcal{H}_2$ optimal model reduction with I/O delay structure

Input delay $\mathcal{H}_2$ Optimality conditions

$\mathbf{G}(s) = \sum_{k=1}^{N} \frac{\psi_k}{s-\mu_k}$ and $\hat{\mathbf{H}}(s) = \sum_{k=1}^{n} \frac{\hat{\phi}_k}{s-\hat{\lambda}_k}$[5]

Input-delay $\mathcal{H}_2$-optimality conditions

If $\hat{\mathbf{H}}_d = \hat{\mathbf{H}} e^{-s\hat{\tau}}$ is a local optimum of input delay $\mathcal{H}_2$ problem, then (interpolation condition on $\tilde{\mathbf{G}}$)

$$\begin{array}{rcl} \hat{\mathbf{H}}(-\hat{\lambda}_k) & = & \tilde{\mathbf{G}}(-\hat{\lambda}_k) \\ \hat{\mathbf{H}}'(-\hat{\lambda}_k) & = & \tilde{\mathbf{G}}'(-\hat{\lambda}_k) \end{array} \quad (18)$$

for $k = 1, \ldots, n$ where $\tilde{\mathbf{G}}(s)$ is given by

$$\tilde{\mathbf{G}}(s) = \sum_{k=1}^{N} \frac{\psi_k e^{\mu_k \tau}}{s - \mu_k}$$

and (delay condition)

$$\sum_{k=1}^{N} \mu_k \psi_k \left(\sum_{j=1}^{n} \frac{\phi_j}{\mu_k + \hat{\lambda}_j} \right) e^{\mu_k \tau} = 0. \quad (19)$$

[5] I. Pontes Duff Pereira, C. Poussot-Vassal and C. Seren, *"Optimal $\mathcal{H}_2$ model approximation based on multiple input/output delays systems"*, arXiv preprint arXiv:1511.05252.

$\mathcal{H}_2$ optimal model reduction with I/O delay structure

Input delay $\mathcal{H}_2$ Optimality conditions

$\mathbf{G}(s) = \sum_{k=1}^{N} \frac{\psi_k}{s-\mu_k}$ and $\hat{\mathbf{H}}(s) = \sum_{k=1}^{n} \frac{\hat{\phi}_k}{s-\hat{\lambda}_k}$ [5]

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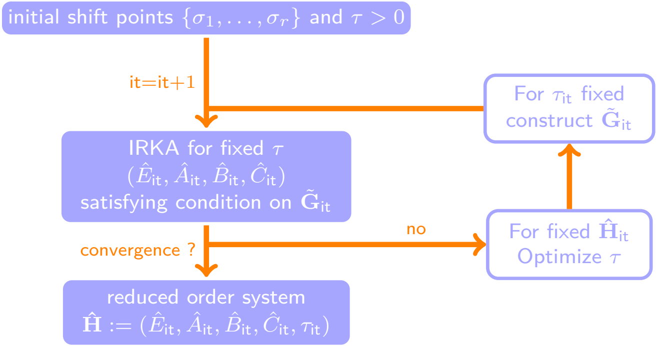
$\mathcal{H}_2$ optimal model reduction with I/O delay structure

IO-dIRKA Algorithm

IO-dIRKA Given $\mathbf{G}$, compute pole residue decomposition. Then,



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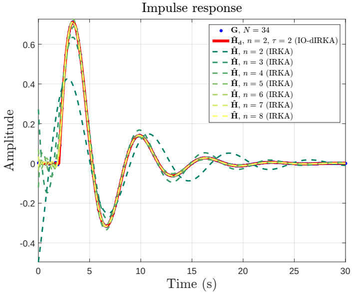
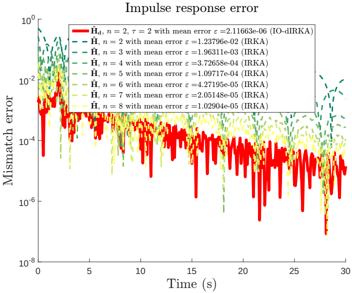
$\mathcal{H}_2$ optimal model reduction with I/O delay structure

IO-dIRKA Algorithm

- Take $\mathbf{G}_{delay}(s) = \dfrac{\psi}{s^2 + 2\xi\omega_0 s + \omega_0^2} e^{-\tau s}$, where $\tau = 2, \omega_0 = 1$ and $\xi = 1/4$.
- Loewner framework for uniformly spaced interpolation points $i\omega_k$, $k = 1, \ldots, 100$ $\Rightarrow \mathbf{G} = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B}$ of order $N = 34$, a delay-free model interpolating $\mathbf{G}_{delay}$.
- **IRKA** for $n = 2, \ldots, 8$; **I/O IRKA** for $n = 2$.





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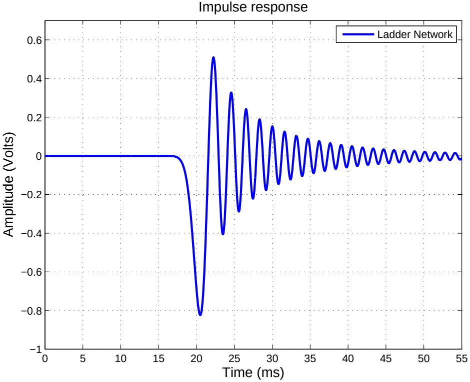
$\mathcal{H}_2$ optimal model reduction with I/O delay structure

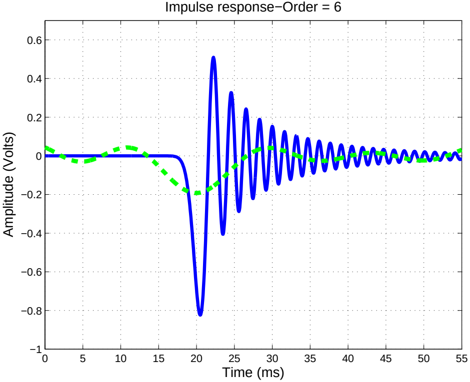
Ladder network benchmark



- Original model of order $=100$.
- $\mathcal{H}_2$ optimal delay-free approximations $n = 6, 12$ and 20.

$\mathcal{H}_2$ optimal model reduction with I/O delay structure

Ladder network benchmark



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$\mathcal{H}_2$ optimal model reduction with I/O delay structure

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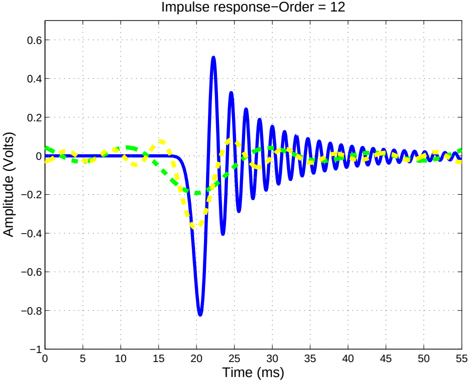

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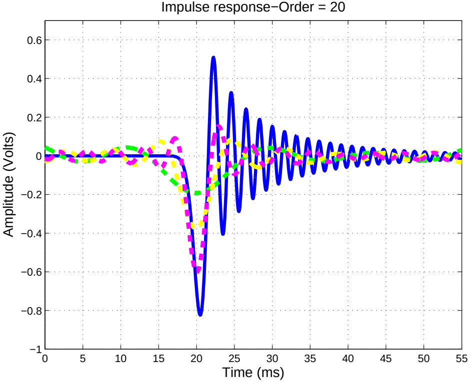
$\mathcal{H}_2$ optimal model reduction with I/O delay structure

Ladder network benchmark



- Original model of order $=100$.
- $\mathcal{H}_2$ optimal delay-free approximations $n = 6, 12$ and 20.

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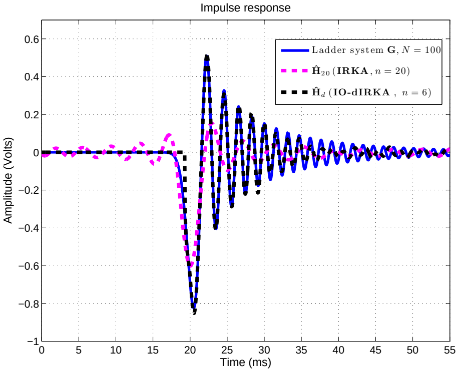
$\mathcal{H}_2$ optimal model reduction with I/O delay structure

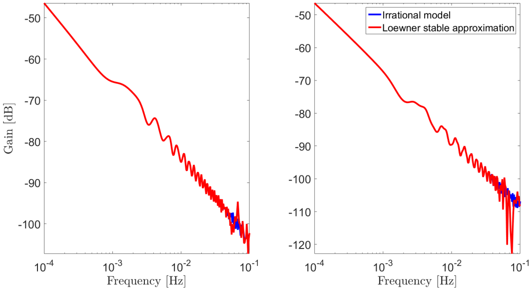
Ladder network benchmark



- Original model of order $=100$.
- $\mathcal{H}_2$ optimal delay-free approximations $n = 20$.
- $\mathcal{H}_2$ optimal input-delay approximations $n = 6$, $\tau_{opt} = 19.27$s.

$\mathcal{H}_2$ optimal model reduction with I/O delay structure

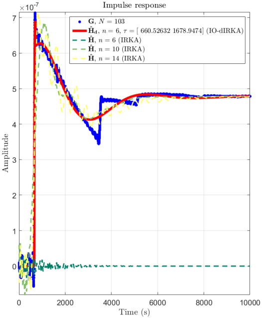
EDF Rhin flow benchmark

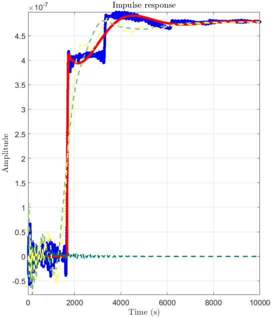


- Irrational model filtered.
- Loewner exact interpolation $n = 103$ (filtered and stable).

$\mathcal{H}_2$ optimal model reduction with I/O delay structure

EDF Rhin flow benchmark





- Impulse response of the models.
- This example illustrates the benefit of delay structured reuced order models for specific transport phenomena.

Outlines

$\mathcal{H}_2$ optimality conditions for reduced state-delay systems

Problem formulation and goals

Let $\hat{\mathbf{H}} = (\hat{E}, \hat{A}, \hat{B}, \hat{C}, 0)$ be defined as:

$$\begin{cases} \hat{E}\dot{x}(t) = \hat{A}x(t) + \hat{B}u(t) \\ y(t) = \hat{C}x(t) \end{cases} \tag{20}$$

whose transfer function is

$$\hat{\mathbf{H}}(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B} \tag{21}$$

$\mathcal{H}_2$ model approximation

Given a system $\mathbf{G} \in \mathcal{H}_2$, the goal is to find a system $\hat{\mathbf{H}}$

$$\hat{\mathbf{H}}^\star := \underset{\hat{\mathbf{H}} \in \mathcal{H}_2, \dim(\hat{\mathbf{H}}) \leq r}{\textbf{arg min}} \|\mathbf{G} - \hat{\mathbf{H}}\|_{\mathcal{H}_2}.$$

$\mathcal{H}_2$ optimality conditions for reduced state-delay systems

Problem formulation and goals

Let $\hat{\mathbf{H}} = (\hat{E}, \hat{A}, \hat{B}, \hat{C}, 0)$ be defined as:

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$$\hat{\mathbf{H}}^\star := \underset{\hat{\mathbf{H}} \in \mathcal{H}_2, \dim(\hat{\mathbf{H}}) \leq r}{\arg\min} \|\mathbf{G} - \hat{\mathbf{H}}\|_{\mathcal{H}_2}.$$

Let $\hat{\mathbf{H}}_d = (\hat{E}, \hat{A}, \hat{B}, \hat{C}, \tau)$ be defined as:

$$\begin{cases} \hat{E}\dot{x}(t) = \hat{A}x(t-\tau) + \hat{B}u(t) \\ y(t) = \hat{C}x(t) \end{cases} \tag{22}$$

whose transfer function is

$$\hat{\mathbf{H}}_d(s) = \hat{C}(s\hat{E} - \hat{A}e^{-\tau s})^{-1}\hat{B} \tag{23}$$

Single-state delay $\mathcal{H}_2$ model approximation

Given a system $\mathbf{G} \in \mathcal{H}_2$, the goal is to find a system $\hat{\mathbf{H}}_d = (\hat{E}, \hat{A}, \hat{B}, \hat{C}, \tau)$

$$\hat{\mathbf{H}}_d^\star := \underset{\hat{\mathbf{H}}_d \in \mathcal{H}_2, \dim(\hat{\mathbf{H}}_d) \leq r}{\arg\min} \|\mathbf{G} - \hat{\mathbf{H}}_d\|_{\mathcal{H}_2}.$$

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$\mathcal{H}_2$ optimality conditions for reduced state-delay systems

Problem formulation and goals

One dimension single-state delay problem

Given $\mathbf{G} \in \mathcal{H}_2$ and a fixed $\tau > 0$, find a reduced order single state delay model order 1,

$$\hat{\mathbf{H}}_d := \begin{cases} \dot{x}(t) &= \hat{\alpha}x(t-\tau) + \hat{\phi}u(t) \\ y(t) &= x(t) \end{cases}$$

whose transfer function is given by

$$\hat{\mathbf{H}}_d(s) = \frac{\hat{\phi}}{s - \hat{\alpha}e^{-s\tau}} \in \mathcal{H}_2. \quad (24)$$

such that:

$$\|\mathbf{G} - \hat{\mathbf{H}}_d\|_{\mathcal{H}_2} = \min_{(\hat{\phi},\hat{\alpha})\in\mathbb{R}^2} \|\mathbf{G} - \frac{\hat{\phi}}{s - \hat{\alpha}e^{-s\tau}}\|_{\mathcal{H}_2} \quad (25)$$

- Reduced order system is defined only by two real parameters.
- It has infinitely many poles.
- What are the optimality conditions ? Are they interpolation conditions ?

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$\mathcal{H}_2$ optimality conditions for reduced state-delay systems

Problem formulation and goals

Recall: if $\mathbf{G}$ & $\hat{\mathbf{H}}$ have semi-simple poles, they read s.t.:

$$\hat{\mathbf{H}}(s) = \sum_{i=1}^{n} \frac{\hat{\phi}_i}{s - \hat{\lambda}_i} \tag{26}$$

$\mathcal{H}_2$-optimality conditions

If both $\mathbf{G}$ and $\hat{\mathbf{H}}$ are $\in \mathcal{H}_2$ and $\hat{\mathbf{H}}$ is a local minimum of the $\mathcal{H}_2$ approximation problem, then the following interpolation equalities hold:

$$\begin{cases} \mathbf{G}(-\hat{\lambda}_i) = \hat{\mathbf{H}}(-\hat{\lambda}_i) \\ \mathbf{G}'(-\hat{\lambda}_i) = \hat{\mathbf{H}}'(-\hat{\lambda}_i) \end{cases}, \ \forall i = 1 \ldots r \tag{27}$$

In Equation (27), $\hat{\lambda}_i$ corresponds to the i^{th} pole of $\hat{\mathbf{H}}$.

- What are the poles of $\hat{\mathbf{H}}_d$?
- From now on, the model $\hat{\mathbf{H}}_d$ will be decomposed as

$$\hat{\mathbf{H}}_d(s) = \hat{\phi}\mathbf{P}_\tau(s) = \frac{\hat{\phi}}{s - \hat{\alpha}e^{-s\tau}}$$

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$\mathcal{H}_2$ optimality conditions for reduced state-delay systems

Spectral decomposition of $\hat{\mathbf{H}}_d(s)$[6]

The Lambert function $\mathbf{W_k}$

The Lambert function $\mathbf{W_k}(s)$ is a multivalued (except at 0) complex function associating for the k^{th} complex branch, a complex number $\mathbf{W_k}(s)$ such that :

$$s = \mathbf{W_k}(s)e^{\mathbf{W_k}(s)}, \quad k \in \mathbb{Z} \tag{28}$$

i.e., given a $s \in \mathbb{C}$, for each complex branch. Equation (28) has one solution in the k-th complex branch, namely $\mathbf{W_k}(s)$.

[6] Corless, Robert M., et al., *"On the LambertW function"*, Advances in Computational mathematics 5.1 (1996): 329-359..

$\mathcal{H}_2$ optimality conditions for reduced state-delay systems

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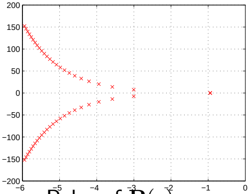
Example Poles of $\mathbf{P}(s) = \frac{1}{s+e^{-s}}$:

$$\begin{aligned} & \lambda_k + e^{-\lambda_k} = 0 \\ \Leftrightarrow \quad & \lambda_k e^{\lambda_k} = -1 \\ \Leftrightarrow \quad & \lambda_k = \mathbf{W_k}(-1) \text{ for } k \in \mathbb{Z} \end{aligned}$$



Poles of $\mathbf{P}(s)$.

[6] Corless, Robert M., et al., *"On the LambertW function"*, Advances in Computational mathematics 5.1 (1996): 329-359..

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$\mathcal{H}_2$ optimality conditions for reduced state-delay systems

Spectral decomposition of $\hat{\mathbf{H}}_d(s)$

Spectral decomposition of $\mathbf{H}_d$

The model $\hat{\mathbf{H}}_d$ has infinite poles which can be computed using the Lambert function as follows :

$$\lambda_k = \frac{1}{\tau}\mathbf{W}_{\mathbf{k}}(\tau\hat{\alpha}), \quad \text{for} \quad k \in \mathbb{Z}. \tag{29}$$

Moreover, if $\hat{\mathbf{H}}_d = \hat{\phi}\mathbf{P}_\tau$, the infinite partial fraction decomposition of $\mathbf{P}_\tau = \frac{1}{s-\hat{\alpha}e^{-s\tau}}$ is given by

$$\mathbf{P}_\tau(s) \quad = \quad \sum_{k=-\infty}^{\infty} \phi_k \frac{1}{s-\lambda_k} \text{ where } \phi_k = \frac{1}{1+\tau\lambda_k}. \tag{30}$$

- $\mathcal{H}_\infty$ convergence[7], and $\mathcal{H}_2$-weak convergence.

[7] Partington, JR and Glover, K and Zwart, HJ and Curtain, Ruth F, *"L_∞ approximation and nuclearity of delay systems"*, Systems & control letters 10,1 (1988): 59-65..

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$\mathcal{H}_2$ optimality conditions for reduced state-delay systems

Spectral decomposition of $\hat{\mathbf{H}}_d(s)$

Spectral decomposition of $\mathbf{P}_\tau^2$.

Let $\mathbf{P}_\tau = \frac{1}{s-\hat{\alpha}e^{-s\tau}} \in \mathcal{H}_2$. Then

$$\mathbf{P}_\tau^2(s) = \sum_{k=-\infty}^{\infty} \psi_k \frac{1}{(s-\lambda_k)^2} + \rho_k \frac{1}{s-\lambda_k} \tag{31}$$

where

$$\psi_k = \frac{1}{(1+\tau\lambda_k)^2} \text{ and } \rho_k = \frac{2\tau^2\lambda_k}{(1+\tau\lambda_k)^3}.$$

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$\mathcal{H}_2$ optimality conditions for reduced state-delay systems

Spectral $\mathcal{H}_2$-inner product :

Let $\mathbf{F} \in \mathcal{H}_2$ and $\mathbf{P}_\tau = \frac{1}{s-\hat{\alpha}e^{-s\tau}} \in \mathcal{H}_2$. Then :

$$\langle \mathbf{F}, \mathbf{P}_\tau \rangle_{\mathcal{H}_2} = \sum_{k=-\infty}^{\infty} \phi_k \mathbf{F}(-\lambda_k), \tag{32}$$

where $\phi_k = \frac{1}{1+\tau\lambda_k}$.
In addition,

$$\langle \mathbf{F}, {\mathbf{P}_\tau}^2 \rangle_{\mathcal{H}_2} = \sum_{k=-\infty}^{\infty} \rho_k \mathbf{F}(-\lambda_k) - \psi_k \mathbf{F}'(-\lambda_k) \tag{33}$$

where $\psi_k = \frac{1}{(1+\tau\lambda_k)^2}$ and $\rho_k = \frac{2\tau^2\lambda_k}{(1+\tau\lambda_k)^3}$.

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$\mathcal{H}_2$ optimality conditions for reduced state-delay systems

Single-state delay $\mathcal{H}_2$-optimality conditions:

Let $\mathcal{E}(\hat{\phi}, \hat{\alpha})$ be the $\mathcal{H}_2$ error, *i.e.*,

$$\mathcal{E}(\hat{\phi}, \hat{\alpha}) = \|\mathbf{G} - \hat{\mathbf{H}}_{\mathbf{d}}\|^2_{\mathcal{H}_2} = \langle \mathbf{G} - \hat{\mathbf{H}}_d, \mathbf{G} - \hat{\mathbf{H}}_{\mathbf{d}} \rangle_{\mathcal{H}_2}$$

Partial derivatives:

The partial derivative of the $\mathcal{H}_2$ error $\mathcal{E}$ with respect to the parameters are given analytically by :

$$\begin{cases} \frac{\partial \mathcal{E}}{\partial \hat{\phi}} & = & -2\langle \mathbf{G} - \hat{\mathbf{H}}_d, \mathbf{P}_\tau \rangle_{\mathcal{H}_2} \\ \frac{\partial \mathcal{E}}{\partial \hat{\alpha}} & = & \frac{2\hat{\phi}}{\hat{\alpha}\tau} \langle \mathbf{G} - \hat{\mathbf{H}}_d, {\mathbf{P}_\tau}' + {\mathbf{P}_\tau}^2 \rangle_{\mathcal{H}_2} \end{cases}$$

Proof Sketch:

$$\frac{\partial \mathcal{E}}{\partial \Theta} = -2\langle \mathbf{G} - \hat{\mathbf{H}}_d, \frac{\partial \hat{\mathbf{H}}_{\mathbf{d}}}{\partial \Theta} \rangle_{\mathcal{H}_2}. \quad (34)$$

$\mathcal{H}_2$ optimality conditions for reduced state-delay systems

Single-state delay $\mathcal{H}_2$-optimality conditions:

Single-state delay $\mathcal{H}_2$-optimality conditions version 1

Let $\hat{\mathbf{H}}_d = \frac{\hat{\phi}}{s-\hat{\alpha}e^{-s\tau}} \in \mathcal{H}_2$ and $\mathbf{G} \in \mathcal{H}_2$. Let us suppose also that $\mathbf{G}' \in \mathcal{H}_2$. If $\hat{\mathbf{H}}_d$ is the best $\mathcal{H}_2$ approximation of $\mathbf{G}$, then :

$$\langle \mathbf{G}, \mathbf{P}_\tau \rangle_{\mathcal{H}_2} = \langle \hat{\mathbf{H}}_d, \mathbf{P}_\tau \rangle_{\mathcal{H}_2} \quad (35)$$

$$\langle \mathbf{G}, {\mathbf{P}_\tau}' + {\mathbf{P}_\tau}^2 \rangle_{\mathcal{H}_2} = \langle \hat{\mathbf{H}}_d, {\mathbf{P}_\tau}' + {\mathbf{P}_\tau}^2 \rangle_{\mathcal{H}_2} \quad (36)$$

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$\mathcal{H}_2$ optimality conditions for reduced state-delay systems

Single-state delay $\mathcal{H}_2$-optimality conditions:[8]

Single-state delay $\mathcal{H}_2$-optimality conditions version 2

Let $\hat{\mathbf{H}}_d = \frac{\hat{\phi}}{s-\hat{\alpha}e^{-s\tau}} \in \mathcal{H}_2$ and $\mathbf{G} \in \mathcal{H}_2$. Let us suppose also that $\mathbf{G}' \in \mathcal{H}_2$. If $\hat{\mathbf{H}}_d$ is the best $\mathcal{H}_2$ approximation of $\mathbf{G}$, then :

$$\sum_{k=-\infty}^{\infty} \mathbf{G}(-\lambda_k)\phi_k = \sum_{k=-\infty}^{\infty} \hat{\mathbf{H}}_{\mathbf{d}}(-\lambda_k)\phi_k, \quad (37)$$

$$\begin{array}{ccc} \sum_{k=-\infty}^{\infty} \mathbf{G}'(-\lambda_k)(\phi_k - \psi_k) & + & \sum_{k=-\infty}^{\infty} \mathbf{G}(-\lambda_k)\rho_k = \\ \sum_{k=-\infty}^{\infty} \hat{\mathbf{H}}'_d(-\lambda_k)(\phi_k - \psi_k) & + & \sum_{k=-\infty}^{\infty} \hat{\mathbf{H}}_d(-\lambda_k)\rho_k \end{array} \quad (38)$$

where λ_k, for $k \in \mathbb{Z}$, are the poles of $\hat{\mathbf{H}}_d$, $\phi_k = \frac{1}{1+\tau\lambda_k}$, $\psi_k = \frac{1}{(1+\tau\lambda_k)^2}$ and $\rho_k = \frac{2\tau^2\lambda_k}{(1+\tau\lambda_k)^3}$.

- Generalized interpolation conditions.
- if $\tau = 0$, $\mathbf{G}(-\hat{\alpha}) = \hat{\mathbf{H}}(-\hat{\alpha})$ and $\mathbf{G}'(-\hat{\alpha}) = \hat{\mathbf{H}}'(-\hat{\alpha})$.

8 Pontes Duff, I., Gugercin, S., Beattie, C., Poussot-Vassal, C. and Seren, C. , *"$\mathcal{H}_2$-optimality conditions for reduced time-delay systems of dimension one"*, in Proceedings of the 13th IFAC Workshop on Time Delay Systems, 2016 .

$\mathcal{H}_2$ optimality conditions for reduced state-delay systems

Application

Let

$$\mathbf{G}(s) = \frac{10}{s^2 + 11s + 10}.$$

Find $\hat{\phi} \in \mathbb{R}$ and $\hat{\alpha} \in (-\pi/2, 0)$ which minimizes :

$$\mathcal{E}(\hat{\phi}, \hat{\alpha}) = \|\mathbf{G} - \mathbf{H}_d\|_{\mathcal{H}_2}^2 = \|\mathbf{E}(\hat{\phi}, \hat{\alpha})\|_{\mathcal{H}_2}^2$$

where $\mathbf{H}_d(s) = \dfrac{\hat{\phi}}{s - \hat{\alpha}e^{-s}}$.

$\mathcal{H}_2$ optimality conditions for reduced state-delay systems

Application

Hence,

$$\mathbf{E}(\hat{\phi},\hat{\alpha}) := \begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{A}_\tau \mathbf{x}(t-\tau) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}\mathbf{x}(t) \end{cases}$$

where

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_\tau = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \hat{\alpha} \end{bmatrix}$$

$B = \begin{bmatrix} 10/9 \\ -10/9 \\ -\hat{\phi} \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$.

- Use delay Lyapunov equations[9] to compute the Norm
- MATLAB function `fminunc` $\Rightarrow \hat{\alpha}^* \approx -0.5371 \quad \hat{\phi}^* \approx 0.4986$.
- Verify generalized interpolation condition by truncation.

9 Jarlebring, E., Vanbiervliet, J., and Michiels, W. , *"Characterizing and computing the H2 norm of time-delay systems by solving the delay lyapunov equation."*, Automatic Control, IEEE Transactions on, 56(4), 814– 825.

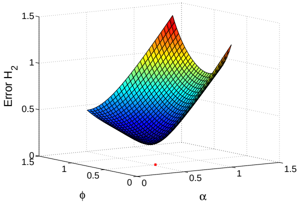
$\mathcal{H}_2$ optimality conditions for reduced state-delay systems

Application

N	$S_{1,\mathbf{G},N}$	$S_{1,\hat{\mathbf{H}}_\mathbf{d},N}$	$S_{2,\mathbf{G},N}$	$S_{2,\hat{\mathbf{H}}_\mathbf{d},N}$
2	0.80890	0.80326	0.15280	0.15361
6	0.81620	0.81234	0.15302	0.15310
10	0.81656	0.81410	0.15306	0.15308
200	0.81667	0.81655	0.15307	0.15307

$$S_{1,\mathbf{G},N} = \sum_{k=-N}^{N-1} \mathbf{G}(-\lambda_k)\phi_k \approx \sum_{k=-N}^{N-1} \hat{\mathbf{H}}_\mathbf{d}(-\lambda_k)\phi_k = S_{1,\hat{\mathbf{H}}_\mathbf{d},N}$$

$$\begin{aligned} S_{2,\mathbf{G},N} &= \sum_{k=-N}^{N-1} \mathbf{G}'(-\lambda_k)(\phi_k - \psi_k) + \sum_{k=-N}^{N-1} \mathbf{G}(-\lambda_k)\rho_k \approx \\ S_{2,\hat{\mathbf{H}}_\mathbf{d},N} &= \sum_{k=-N}^{N-1} \hat{\mathbf{H}}'_\mathbf{d}(-\lambda_k)(\phi_k - \psi_k) + \sum_{k=-N}^{N-1} \hat{\mathbf{H}}_\mathbf{d}(-\lambda_k)\rho_k . \end{aligned}$$



Outlines

Introduction

Preliminaries on $\mathcal{H}_2$ model approximation

$\mathcal{H}_2$ optimal model reduction with I/O delay structure

$\mathcal{H}_2$ optimality conditions for reduced state-delay systems

Conclusions and perspectives

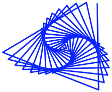
Conclusions and perspectives

Scientific contributions

- **$\mathcal{H}_2$ optimal model reduction with I/O structure :** I/O $\mathcal{H}_2$ inner product expression, interpolation conditions on $\tilde{\mathbf{G}}$, I/O IRKA algorithm, application to industrial benchmark.
- **Single-state delay $\mathcal{H}_2$-optimality conditions :** $\mathcal{H}_2$ spectral inner product expression, generalized interpolation conditions.

Future work

- **I/O delay** 1)Isometry structure (In progress - conjoint work with Christoph Zimmer -TU Berlin)
- **state-delay** 1) Generalize interpolation conditions to general framework. (In progress (conjoint work with S. Gugercin and C. Beattie - VT)) 2) Find a way to deal with them.



$\mathcal{H}_2$ optimal model approximation by structured time-delay reduced order models

I. Pontes Duff, Charles Poussot-Vassal, C. Seren

THE FRENCH AEROSPACE LAB

27th September, GT MOSAR and GT SAR meeting