Opinion dynamic control with leadership

JAMACS Lille
16th November 2016

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Outline of the talk

Motivation

Controllability

- System dynamics
- Problem
- Rallying control
- Trajectory tracking

Time optimal control

- System dynamics
- Optimal control problem
- Pontryagin Maximum Principle
- Optimal controllers
- Examples

Conclusion

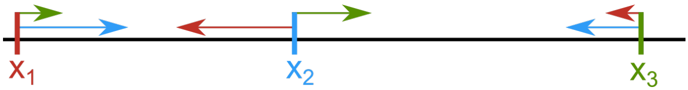
Opinion dynamic

What is an opinion ?

Consider a finite number of agents. The opinion of the *i*th agent is

$$x_i(t) \in \mathbb{R}.$$

Dynamics intuitive principle :



Motivations

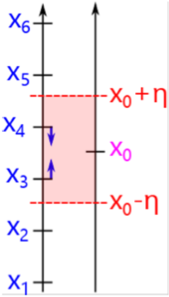
- Social networks, MOOC, multiagent system, ...
- Well known and interesting asymptotic behaviors : consensus, clusters,...

Main issue

Is it possible to influence opinions ? And how to act ?

We introduce a leader as an extra agent in the system $x_0(t) \in \mathbb{R}$:

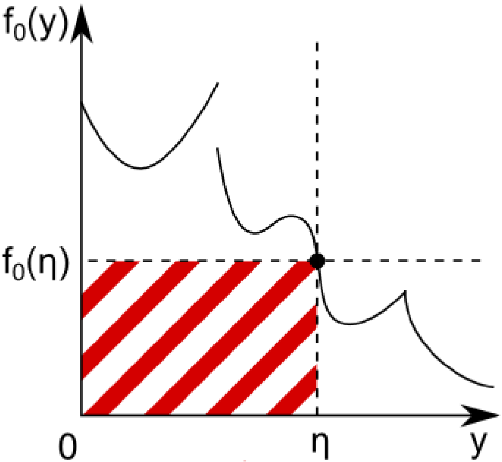
- The leader is controlled ;
- The leader can influence the other agents on a local range.



System dynamics

$$
\begin{cases}
\dot{x}_i(t) &= \displaystyle\sum_{j=1}^{n} f(|x_j(t) - x_i(t)|)(x_j(t) - x_i(t)) \\
& \quad + f_0(|x_0(t) - x_i(t)|)(x_0(t) - x_i(t)), \\
\dot{x}_0(t) &= u(t),\ |u(t)| \leq \sigma,\ \sigma > 0\ ,
\end{cases}
$$

Main assumption It exists $\eta > 0$ such that $\forall\, y \in [0;\eta]$ we have $f_0(y) \geq f_0(\eta)$.



Problem and idea of solution

Problem Given an arbitrary consensus value $\alpha \in \mathbb{R}$ and arbitrary fixed initial conditions $x_i(0) \in \mathbb{R}$, for $i \in \{1, \dots, n\}$ and $x_0(0) \in \mathbb{R}$. Find a control $u(t)$, for $t \in \mathbb{R}^+$ and with $|u(t)| < \sigma$, $\sigma > 0$ such that all x_i converge toward α, i.e. $\lim_{t \to +\infty} x_i(t) = \alpha$.

Decomposition into subproblems :

- Gathering all the agents at a distance less than η from the leader ;
- Maintaining the agent in the neighborhood of the leader and guide the leader to the consensus value α ;
- Ensuring the convergence of all the agents to the consensus value α.

Intuition ?

How to gather all the agents at a distance less than η from the leader ?

Theorem : Rallying control

Let $c \in (0, \sigma]$, $\kappa \in (0, \min\{\eta f_0(\eta), \sigma\}]$ and $\varepsilon \in (0, \eta)$.
For any initial conditions the following control

$$u(t) = \begin{cases} -c & \text{for } t \in [0, T_1), \\ \kappa & \text{for } t \in [T_1, T_2], \end{cases}$$

where

$$T_1 = \inf\{t \geq 0 \mid x_0(t) \leq x_1(t) + \varepsilon\},$$

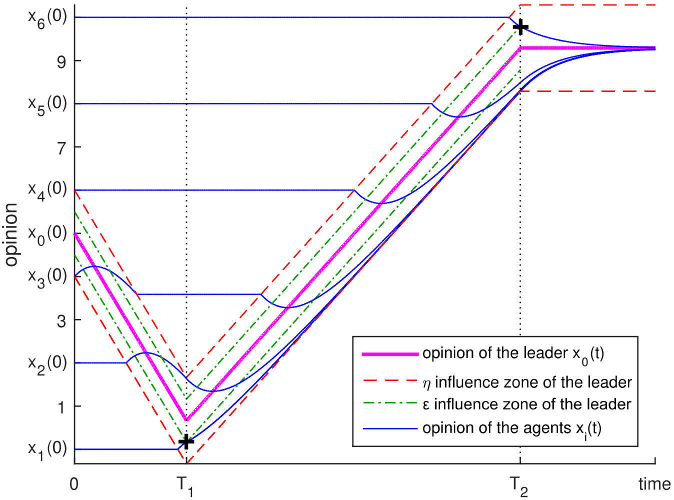
and

$$T_2 = \inf\{t \geq T_1 \mid x_0(t) \geq x_n(t) - \varepsilon\},$$

gather all the agents in the η neighbourhood of the leader.

T_1 and T_2 are finite with this control law.

Example of rallying control



Theorem : trajectory tracking

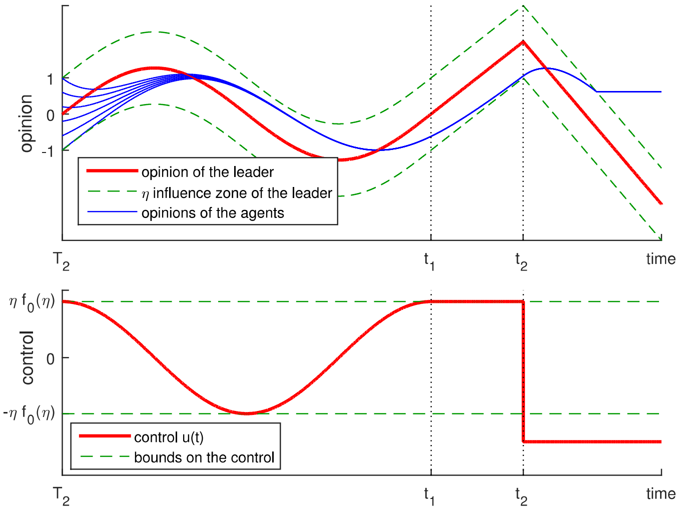
If at a time t_0 we have $\forall\, i \in \{1,\dots,n\},\ |x_i(t_0) - x_0(t_0)| < \eta$, and if

$$\forall\, t \geq t_0,\ |u(t)| \leq \min\{\eta\, f_0(\eta), \sigma\},$$

then $\forall\, t \geq t_0,\ \forall\, i \in \{1,\dots,n\},\ |x_i(t) - x_0(t)| < \eta$. If the control is

sufficiently small in norm and all agents are initially in the η influence zone of the leader, then all the agents remain at all time in the η influence zone of the leader.

Example of trajectory tracking



Outline of the talk

System definition

$$\begin{cases} \dot{x}_0(t) &= u(t) \\ \dot{x}_i(t) &= h_0(x_0(t) - x_i(t)), \quad \forall i \in \{1, \dots, n\} \end{cases}$$

where $u(t) \in [-\sigma, \sigma]$, $\sigma > 0$ and

$$h_0 \; : \; \begin{cases} \mathbb{R} \to \mathbb{R}^+ \\ y \mapsto y\, f_0(y) \end{cases}$$

No interactions among agents simplification with $f = 0$

Time optimal problem

Find an optimal control u^* to apply on the system in order to have all agents within an η radius of the leader in a minimal time t_f^*.

$$\min_{u \in \Omega_\sigma} t_f = \min_{u \in \Omega_\sigma} \int_0^{t_f} \mathrm{d}t$$

where $\Omega_\sigma = \{u : t \mapsto u(t) \mid \forall\, t \geq 0,\ |u(t)| \leq \sigma\}$ and

where $t_f = \inf\{t \geq 0 | \forall\, i \in \{1, \ldots, n\},\ x_i(t) \in [x_0(t) - \eta, x_0(t) + \eta]\}$.

Useful substitution

Substitution given by $y_0(t) = x_0(t)$ and $y_i(t) = x_i(t) - x_0(t)$

The system is rewritten as

$$\begin{cases} \dot{y}_0(t) & = & u(t) \\ \dot{y}_i(t) & = & -h_0(y_i(t)) - u(t) \end{cases}$$

Property

If for all $(i,j) \in \{1,\ldots,n\}^2$ such that $i < j$ we have $y_i(0) < y_j(0)$.

Then for all time we have $y_1(t) < y_2(t) < \ldots < y_n(t)$.

Pontryagin Maximum Principle

Definition of the Hamiltonian

$$H(Y(t), u(t), \lambda(t), t) = -1 + \lambda(t)^T \; \dot{Y}(t),$$

Necessary conditions for optimality :

$$\dot{\lambda}_i(t) = -\frac{\partial H}{\partial y_i}(Y(t), u(t), \lambda(t), t) = \lambda_i(t)\, h_0'(y_i(t)),$$

$$\dot{y}_i(t) = \frac{\partial H}{\partial \lambda_i}(Y(t), u(t), \lambda(t), t).$$

$$u^*(t) \in \mathrm{argmax}_{v \in [-\sigma, \sigma]} H(Y^*(t), v, \lambda^*(t), t).$$

In addition, transversality conditions related to the final interval condition defining t_f.

Transversality constraints

$$\forall\, i \in \{1, \ldots, n\},\ y_i(t_f) \in [-\eta, \eta],$$

Equivalent to the following $2n$ inequalities

$$\left\{ \begin{array}{rcccl} a_{i1}(y_i(t_f)) & \triangleq & \eta + y_i(t_f) & \geq & 0, \\ a_{i2}(y_i(t_f)) & \triangleq & \eta - y_i(t_f) & \geq & 0, \end{array} \right.$$

Lagrangian multipliers

$$\begin{array}{c} \alpha_{i1}(\eta + y_i(t_f)) = 0, \quad \alpha_{i1} \geq 0, \\ \alpha_{i2}(\eta - y_i(t_f)) = 0, \quad \alpha_{i2} \geq 0. \end{array}$$

Coupling to the co-states by the transversality condition

$$\lambda_i(t_f) \quad = \quad \alpha_{i1} \frac{\partial a_{i1}(y_i(t_f))}{\partial y_i(t_f)} + \alpha_{i2} \frac{\partial a_{i2}(y_i(t_f))}{\partial y_i(t_f)} \quad = \quad \alpha_{i1} \ - \ \alpha_{i2},$$

Simplification property

$$\forall\, i \in \{2, \ldots, n-1\},\ \forall\, t \in [0, t_f],\ \lambda_i(t) = 0,$$

which allows to rewrite the Hamiltonian as

$$H(Y(t), u(t), \lambda(t), t) = -1 - (\lambda_1(t) + \lambda_n(t))\, u(t) \\ - \lambda_1(t)\, h_0(y_1(t)) - \lambda_n(t)\, h_0(y_n(t)),$$

and the terminal constraints become

$$(y_1(t_f), y_n(t_f)) \in [-\eta, \eta]^2.$$

Only consider the extreme agents

Property of the co-states

$\forall i \in \{1, n\}$ and $\forall t \in [0, t_f]$ the quantity $\lambda_i(t)$ keeps the same sign

Property of the control

Switching function $\phi : t \mapsto \lambda_1(t) + \lambda_n(t)$

Let $t \in [0, t_f]$, we have

$$\phi(t) \neq 0 \implies u^*(t) = -\sigma \operatorname{sign}(\phi(t)).$$

When $\phi(t) = 0$, $u^*(t)$ is a priori undefined. Other relations are required.

Uniformly saturated controls

If the final configuration is

$$\begin{aligned} y_1(t_f) = -\eta \quad &\text{and} \quad y_n(t_f) \in (-\eta, \eta), \text{ or} \\ y_1(t_f) \in (-\eta, \eta) \quad &\text{and} \quad y_n(t_f) = \eta \end{aligned}$$

then

$$\forall t \in [0, t_f],\ u(t) = \sigma \operatorname{sign}(y_i(t_f)),$$

is the unique control law satisfying the necessary conditions for optimality, where $i \in \{1, n\}$ is such that $|y_i(t_f)| = \eta$.

Case $y_1(t_f) = -\eta$ and $y_n(t_f) = +\eta$

$u^*(t)$ depends on the value of $\phi(t)$:

- $\phi(t) \neq 0$, $u^*(t) = -\sigma\, \mathrm{sign}(\phi(t))$.
- $\phi(t) = 0$ at only a point t (not on an interval) : $u^*(t)$ switches and the value at time t does not influence the trajectories.
- $\phi(t) = 0$ on an interval : Obtained by deriving twice ϕ.
  Let $T \subset [0, t_f]$ an interval such that for all $t \in T$

$$\phi(t) = 0,\ \dot{\phi}(t) = 0, \text{ and } \ddot{\phi}(t) = 0,$$

  then $\forall t \in T$, the singular control is $u^*(t) = sat(u_0(t))$ where

$$u_0(t) \triangleq \frac{h_0''(y_1(t))\, h_0(y_1(t)) - h_0''(y_n(t))\, h_0(y_n(t))}{h_0''(y_n(t)) - h_0''(y_1(t))}.$$

Difficulties : $\lambda_1(t_f)$, $\lambda_n(t_f)$ are not imposed. We face a generic nonlinear Two Boundary Problems with unknown values of the costate at t_f, needing numerical schema.

Strategies

Sum-up

Uniformly saturated controls :

- $S^+ : \forall\, t \geq 0,\; u(t) = +\sigma$.
- $S^- : \forall\, t \geq 0,\; u(t) = -\sigma$.

Known expression $\implies$ integration possible

S^0 : possibly singular control.

Unknown expression $\implies$ numerical optimisation scheme.
Final state forced at $(y_1(t_f), y_n(t_f)) = (-\eta, \eta)$.

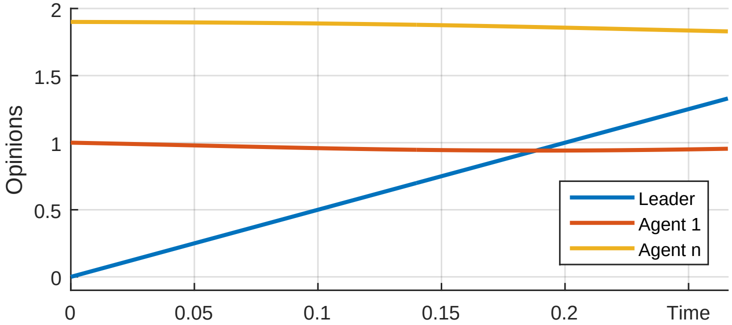
Methodology

For any initial condition $(y_1(0), y_n(0))$:

- Test S^+ and $S^- \implies t_f^+$ and t_f^-
- Numerical optimisation for $S^0 \implies t_f^0$
- Optimal strategy has the smallest t_f

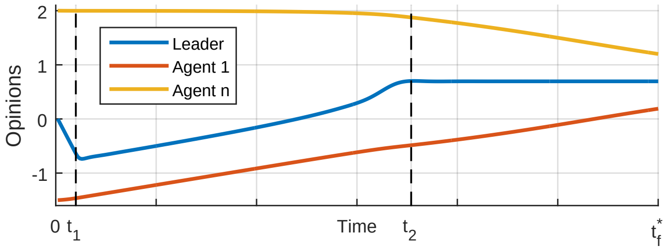
Example of optimal uniformly saturated control

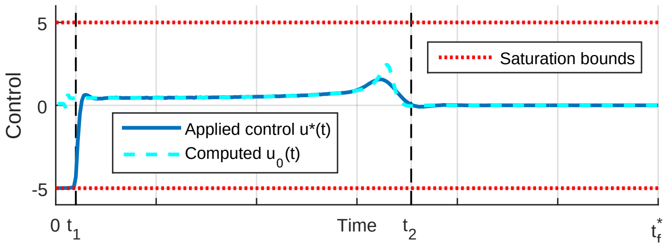
$t_f^+ = 0.27\,;\, t_f^- = +\infty\,;\, t_f^0 = 0.7$



$\eta = 0.5\,;\, f_0 = x \mapsto \exp(-x^2)\,;\, \sigma = 5$

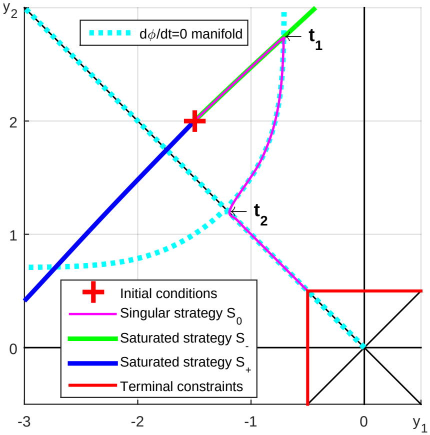
Example of singular control





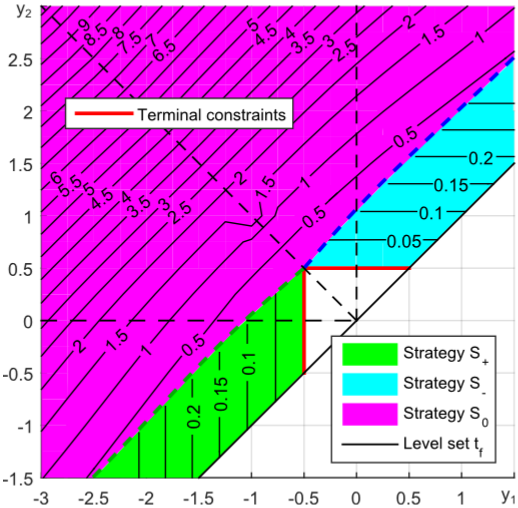
Example of singular control

$$t_f^+ = +\infty\,;\, t_f^- = +\infty \implies S^0 \text{ optimal}$$



$$\eta = 0.5\,;\, f_0 = x \mapsto \exp(-x^2)\,;\, \sigma = 5$$

Choice of the optimal control strategy



Conclusion

Conclusion

- Opinion dynamics with a leader has been investigated.
- Possibility of gathering all the opinions in the neighborhood of the leader has been emphasized.
- The time optimal control has been provided.

Perspectives

- Extend the result of time optimal control in the case with interactions between the agents.
- Consider other models of opinion dynamics.

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