Homogeneous polynomial Lyapunov functions for the admissibility analysis of uncertain descriptor systems

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Outline

Motivation

State of the art

Mathematical background
- HPLFs
- TV descriptor systems

Our contributions
- Admissibility condition
- LMI admissibility conditions

Numerical examples

Conclusions

Time-varying descriptor systems

Time-varying systems

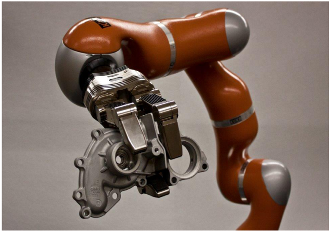
- Dynamics evolve with time
- Dynamics depend on variable physical quantities (parameters): *pressure, temperature, operation point*
- Parameters can be either measurable or uncertain

Descriptor systems

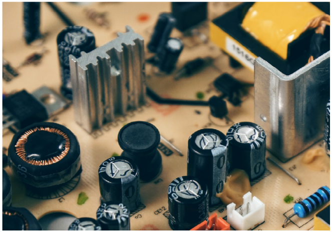
- Representation of algebraic constraints and non-causal phenomena

Applications

Aeronautics



Robotics



Electronics

Economics

Related literature

	$E\dot{x} = A(w)x$	$Ex(k+1) = A(w)x(k)$
TI	(Fang 2002) (Gao, Chen, and Sun 2003) (Sakuwa and Fujisaki 2005) (Yagoubi, Bouali, and Chevrel 2008) (Souza, Barbosa, and Fu 2008)	(Fang 2002) (Kuo and Fang 2003) (Bara 2011a)
TV	(Bara 2010) (Barbosa, Souza, and Coutinho 2013) (Bara 2011b)	(Bara 2011a) (Barbosa, Souza, and Coutinho 2012) (Santos Paulino and Bara 2017)

Analysis using Lyapunov functions

(Daafouz and Bernussou 2001)

(Bara 2011b)

Affine quadratic

(Blanchini 1995)

Polyhedral

(Chesi et al. 2007)

PD Homogeneous polynomial

Classic → **Universal**

Quadratic

(Oliveira, Bernussou, and Geromel 1999)

Biquadratic

(Barbosa, Souza, and Coutinho 2013)

Homogeneous polynomial

(Chesi et al. 2003)

Objective

Open fields

- Research on non-conservative approaches for *admissibility analysis* of uncertain TV descriptor systems
- Absence of literature using Homogeneous Polynomial LFs on descriptor systems

Our contribution

- A less restrictive approach for the admissibility analysis of uncertain TV CT descriptor systems
- Extend *Homogeneous Polynomial LFs* for descriptor systems

HPLFs for standard SS

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Standard state-space system

$$\frac{d}{dt}x(t) = A\big(w(t)\big)x(t) = \Big(A_0 + \sum_{i=1}^{s} w_i(t)A_i\Big)x(t)$$

Base vectors of homogeneous forms *Power transformation of degree q of x*

$$x_l^{[q]} = x_1^{i_1}x_2^{i_2}\dots x_n^{i_n}, \quad i_1 + i_2 + \dots + i_n = q$$
$$i_1,\, i_2 \dots,\, i_n \geq 0$$

Example:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \rightarrow \quad x^{[2]} = \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix}, \quad x^{[3]} = \begin{bmatrix} x_1^3 \\ x_1^2x_2 \\ x_1x_2^2 \\ x_2^3 \end{bmatrix}$$

HPLFs for standard SS

HPLFs of order $2q$

$$V_{2q}(x) = \sum_{i_1,\dots,\, i_n} p_{i_1 i_2 \dots i_n} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$$

$$i_1 + i_2 + \dots + i_n = 2q, \quad i_1,\, i_2 \dots,\, i_n \geq 0$$

Complete Square Matricial Representation (CSMR)

$$V_{2q}(x) = {x^{[q]}}^T (P + P_0) x^{[q]}.$$

$$P_0 \in \mathcal{P} = \{P_0 = P_0^T \mid {x^{[q]}}^T P_0 x^{[q]} = 0 \; \forall \; x \in \mathcal{R}^n\}.$$

HPLFs for standard SS

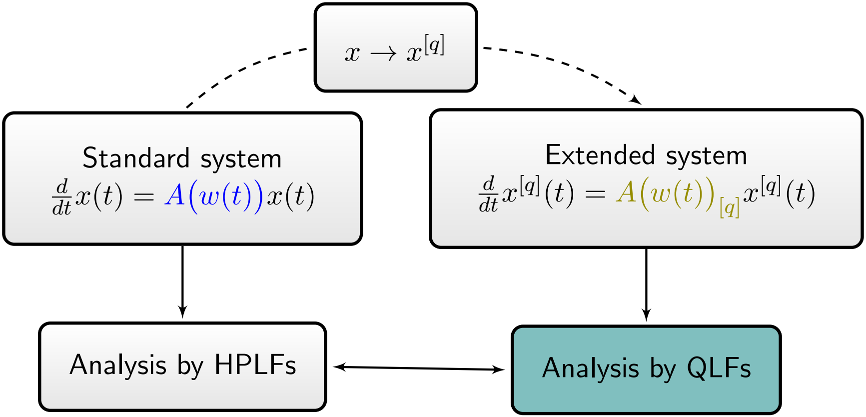
Extended system and extended matrix

$$\frac{d}{dt}x^{[q]}(t) = A\big(w(t)\big)_{[q]}x^{[q]}(t)$$
$$= \left(A_{0[q]} + \sum_{i=1}^{s} w_i(t)A_{i[q]}\right)x^{[q]}(t)$$

Remark

There exists a **linear mapping** between $A(w(t))$ and $A(w(t))_{[q]}$

HPLFs for standard SS



Admissibility of TV descriptor systems

System

$$E\dot{X}(t) = A(t)X(t)$$

$$E,\ A(t) \in \mathbb{R}^{(n+n_\xi)\times(n+n_\xi)},\ \operatorname{rank} E = n \leq n + n_\xi$$

Regularity

$$\det\Big(sE - A(t)\Big) \not\equiv 0$$

Causality

$\deg\Big(\det\Big(sE - A(t)\Big)\Big) = \operatorname{rank}(E)$ holds for all t

Stability

Asymptotic convergence of $X \to 0$

TV descriptor systems

SVD normal equivalent form

There exists a transformation

$$SET\dot{\bar{X}}(t) = SA(t)T\bar{X}(t)$$

$$\begin{bmatrix} I_n & \\ & 0_{n_\xi} \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{\xi}(t) \end{bmatrix} = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix} \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix} \begin{matrix} \textbf{Dynamical eq.} \\ \textbf{Algebraic eq.} \end{matrix}$$

Regularity and **Causality** hold for bounded and invertible $A_{22}(t)$ (Santos Paulino and Bara 2017)

Stability of the SVD equivalent form

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Rewriting dynamics - equivalent standard form

If $A_{22}(t)$ is bounded and invertible, we have:

$$\xi(t) = -A_{22}{}^{-1}(t)A_{21}(t)x(t)$$
$$\dot{x}(t) = \Big(A_{11}(t) - A_{12}(t)A_{22}{}^{-1}(t)A_{21}(t)\Big)x(t)$$

Stability

$\dot{x}(t) = \Big(A_{11}(t) - A_{12}(t)A_{22}{}^{-1}(t)A_{21}(t)\Big)x(t)$ stable for all $t \geq 0$

Admissibility of PD descriptor systems

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System

$$E\dot{X}(t) = A\big(w(t)\big)X(t)$$

$$E,\ A\big(w(t)\big) \in \mathbb{R}^{(n+n_\xi)\times(n+n_\xi)},\ \operatorname{rank} E = n \leq n + n_\xi$$

Regularity

$$\det\Big(sE - A\big(w(t)\big)\Big) \not\equiv 0$$

Causality

$\deg\Big(\det\Big(sE - A\big(w(t)\big)\Big)\Big) = \operatorname{rank}(E)$ holds for all $w(t) \in \mathcal{W}$

Stability

Stable finite modes for every possible trajectory of $w(t) \in \mathcal{W}$

PD descriptor systems

SVD normal equivalent form

There exists a transformation

$$SET\dot{\bar{X}}(t) = SA\big(w(t)\big)T\bar{X}(t)$$

$$\begin{bmatrix} I_n & \\ & 0_{n_\xi} \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{\xi}(t) \end{bmatrix} = \begin{bmatrix} A_{11}\big(w(t)\big) & A_{12}\big(w(t)\big) \\ A_{21}\big(w(t)\big) & A_{22}\big(w(t)\big) \end{bmatrix} \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}$$

Dynamical eq.

Algebraic eq.

Regularity and **Causality** hold for invertible $A_{22}\big(w(t)\big)$

Stability of the SVD equivalent form

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Rewriting dynamics - equivalent standard form

If $A_{22}(w(t))$ is invertible, we have:

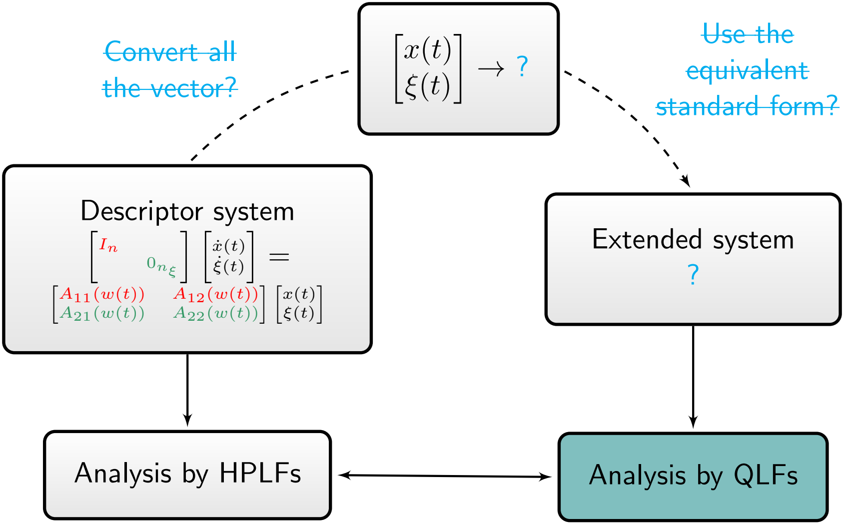
$$\xi(t) = -A_{22}{}^{-1}(w(t))A_{21}(w(t))x(t)$$
$$\dot{x}(t) = \Big(A_{11}(w(t)) - A_{12}(w(t))A_{22}{}^{-1}(w(t))A_{21}(w(t))\Big)x(t)$$

Stability

$$\dot{x}(t) = \Big(A_{11}(w(t)) - A_{12}(w(t))A_{22}{}^{-1}(w(t))A_{21}(w(t))\Big)x(t)$$

stable for all possible uncertain parameters trajectories

Open question



Linear mapping

Remark

There exists a **linear mapping** between $A(w(t))$ and $A(w(t))_{[q]}$

Our formulation

There exist suitable matrices M_i and N_i such that

$$A(w(t))_{[q]} = \sum_{i=1}^{d_{q-1}} M_i A(w(t)) N_i$$

Admissibility condition

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An equivalence

The uncertain descriptor system

$$\begin{bmatrix} I_n & \\ & 0_{n_\xi} \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{\xi}(t) \end{bmatrix} = \begin{bmatrix} A_{11}(w(t)) & A_{12}(w(t)) \\ A_{21}(w(t)) & A_{22}(w(t)) \end{bmatrix} \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}$$

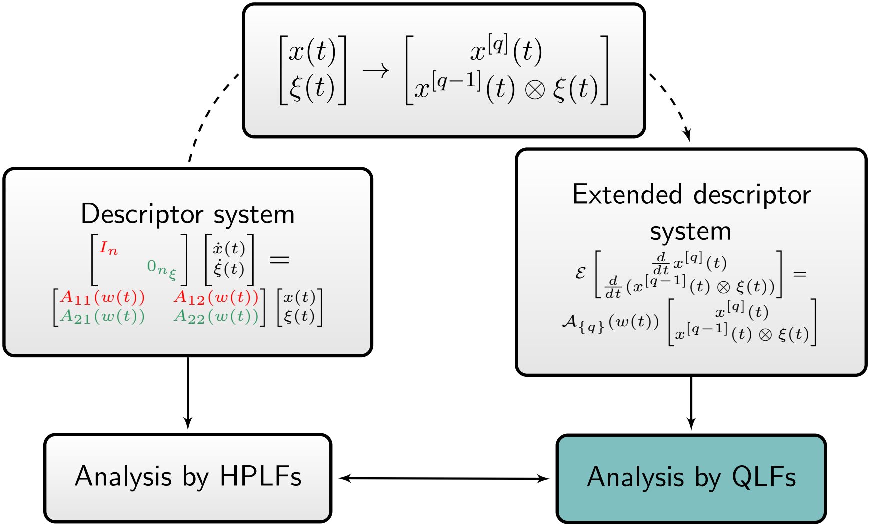
is robustly admissible with a HPLF of degree $2q$ if the *extended descriptor system*

$$\begin{bmatrix} I_{d_q} & \\ & 0_{d_{q-1} n_\xi} \end{bmatrix} \begin{bmatrix} \frac{d}{dt} x^{[q]}(t) \\ \frac{d}{dt}(x^{[q-1]}(t) \otimes \xi(t)) \end{bmatrix} =$$

$$\left[\begin{array}{c|ccc} A_{11_{[q]}}(w) & M_1 A_{12}(w) & \dots & M_{d_{q-1}} A_{12}(w) \\ \hline A_{21}(w) N_1 & A_{22}(w) & & \\ \vdots & & \ddots & \\ A_{21}(w) N_{d_{q-1}} & & & A_{22}(w) \end{array}\right] \begin{bmatrix} x^{[q]}(t) \\ x^{[q-1]}(t) \otimes \xi(t) \end{bmatrix}$$

is robustly admissible with a quadratic LF.

Admissibility condition



Admissibility condition

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LMI admissibility conditions

The given PD descriptor system is robustly admissible based on HPLFs if there exist $P = \begin{bmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{bmatrix}$, and $\gamma_{ij}, \ \ i = 1, \dots, N, \ \ j = 1, \dots, d_{\mathcal{P}}$, satisfying the following LMI conditions:

$$P_{11} > 0$$

$$\mathcal{A}_{\{q\}}(w^{(i)})^T P + P\mathcal{A}_{\{q\}}(w^{(i)}) + \sum_{j=1}^{d_{\mathcal{P}}} \gamma_{ij} \begin{bmatrix} P_{0j} & \\ & 0 \end{bmatrix} < 0$$

for $i = 1, \dots, N$

where $w^{(i)}$ are the vertices of $\mathcal{W}$ and matrices P_{0j} are a base of the linear space $\mathcal{P}$.

Example 1

ℓ_∞ $2q$-HPLF asymmetric admissibility margins

$$E\begin{bmatrix}\dot{x}(t)\\ \dot{\xi}(t)\end{bmatrix} = (A_0 + w(t)A_1)\begin{bmatrix}x(t)\\ \xi(t)\end{bmatrix}, \quad 0 \le w(t) \le \kappa$$

$$E = \begin{bmatrix} I_3 & \\ & 0\end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & 1 & 0 & 0.5\\ 0 & 0 & 1 & 1\\ -1 & -2 & -4 & 0\\ 1 & 0 & 0 & 1\end{bmatrix}, \quad A_1 = \begin{bmatrix} -2 & 0 & -1 & 0\\ 1 & -10 & 3 & 1\\ 3 & -4 & 2 & 0\\ -1 & 0 & 0 & 0.2\end{bmatrix}$$

$$\kappa^*_{2q+} = sup\{\kappa : \exists V_{2q}, w(t) \in \bar{\mathcal{B}}_{\kappa+}\},$$
$$\bar{\mathcal{B}}_{\kappa+} = \{b \in I\!R^m :\ 0 \le b_i \le \kappa,\ i = 1, \ldots, m\}$$

$$\kappa^*_{2+} = 4.6667 \quad \kappa^*_{4+} = 16.3943$$

Example 2

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ℓ_∞ $2q$-HPLF symmetric admissibility margins

$$E\begin{bmatrix}\dot{x}(t)\\ \dot{\xi}(t)\end{bmatrix} = (A_0 + w(t)A_1)\begin{bmatrix}x(t)\\ \xi(t)\end{bmatrix}, \quad -\kappa \le w(t) \le \kappa$$

$$E = \begin{bmatrix} I_2 & \\ & 0\end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & 1 & 0.2\\ -6 & -1 & 0\\ 0.1 & 0.2 & 1\end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0.1\end{bmatrix}$$

$$\kappa^*_{2q|\cdot|} = sup\{\kappa : \exists V_{2q}, w(t) \in \bar{\mathcal{B}}_{\kappa|\cdot|}\},$$
$$\bar{\mathcal{B}}_{\kappa|\cdot|} = \{b \in I\!R^m : \ |b_i| \le \kappa, \ i = 1, \dots, m\}.$$

$2q$	2	4	6	8	10	12	14
$\kappa^*_{2q\|\cdot\|}$	2.572	3.451	3.650	3.715	3.817	3.820	3.866

Conclusions

- A less restrictive approach for the admissibility analysis of uncertain TV CT descriptor systems
- *Homogeneous Polynomial LFs* for descriptor systems
  - Power transformation wrt. descriptor systems
  - Definition of an extended descriptor system
- LMI admissibility conditions

Thanks for your attention!

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