Static Anti-Windup Design for Discrete-Time Large-Scale Saturated Synchrotron System

Isabelle Queinnec, Sophie Tarbouriech, Sandira Gayadeen, Luca Zaccarian

LAAS-CNRS, France, Diamond Light Source, UK, LAAS-CNRS and University of Trento

Workshop LPV and AW, ONERA Toulouse, April 17, 2018

Outline

Problem Statement

What are cross-directional processes?

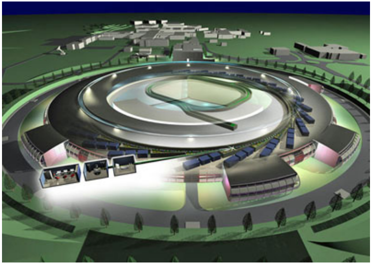
- Processes in which the variations of a variable in a profile orthogonal to the direction of propagation of the variable are controlled

- Typical examples:
  - rolling processes involved in paper machines
  - plastic film extrusion
  - metal forming

- Everything you want to know on such processes in a special issue of IEE Proc - Control Theory Appl. in 2002, vol.149(5) (Guest editor: S. Duncan)

What are synchrotrons?

- Synchrotron light: electromagnetic radiation emitted by charged particles (electrons) that move at high speeds and change direction
- Acceleration of the electrons in a circumference storage ring in which they are confined by magnetic fields
- Ring: succession of identical cells involving straight sections and bending magnets to curve the electron beam
- Synchrotrons are cross-directional plants



Control problems associated to synchrotron systems

- Electron beam subjected to disturbance
  - ▷ control by acting on magnet power
- Change in a single corrector magnet extends around the ring
  - ▷ strong interactions
- Actuator saturations may strongly deteriorate the closed-loop behavior
  - ▷ anti-windup strategies to mitigate those saturation effects

Static linear anti-windup scheme, acting on the controller state and output equations

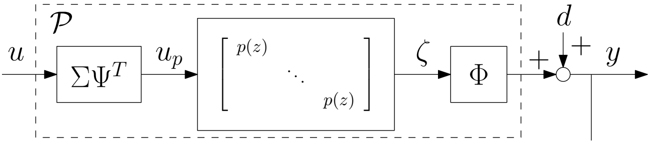
Basic description of the system

- N spatially distributed dynamic actuators/sensors form plant $\mathcal{P}$

$$y = p(z)Bu + d = B \begin{bmatrix} p(z) & & \\ & \ddots & \\ & & p(z) \end{bmatrix} \underbrace{\mathrm{sat}(y_c)}_{u} + d$$

- Using SVD of B matrix:

$$y = \Phi\Sigma\Psi^T p(z)u + d = \Phi(p(z) \otimes I)\Sigma\Psi^T u + d$$



- Allows to design a modular controller that works in the "modal space"

Controller $\mathcal{C}$ design reduces to linear SISO feedback

- Spatially distributed dynamic actuators/sensors form plant $\mathcal{P}$
- Using SVD:

$$y = B \begin{bmatrix} p(z) & & \\ & \ddots & \\ & & p(z) \end{bmatrix} \underbrace{\mathrm{sat}(y_c)}_{u} + d$$

$$= \Phi \Sigma \Psi^T p(z) u + d$$

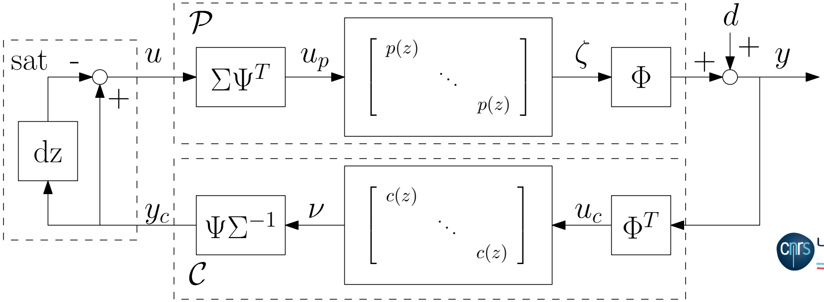
$$= \Phi (p(z) \otimes I) \Sigma \Psi^T u + d$$

- Controller $\mathcal{C}$ equally selected using SVD:

$$y_c = \Psi \Sigma^{-1} c(z) \Phi^T y$$

$$= \Psi \Sigma^{-1} \begin{bmatrix} c(z) & & \\ & \ddots & \\ & & c(z) \end{bmatrix} \Phi^T y$$

- Design of $c(z)$ as SISO feedback ☺
- Saturation mixes up everything ☹



Controller $\mathcal{C}$ design reduces to linear SISO feedback

- Spatially distributed dynamic actuators/sensors form plant $\mathcal{P}$
- Using SVD:

$$y = {\color{red}B}\begin{bmatrix} p(z) & & \\ & \ddots & \\ & & p(z) \end{bmatrix} \underbrace{\mathrm{sat}(y_c)}_{u} + d$$

$$= {\color{red}\Phi\Sigma\Psi^T} p(z)u + d$$

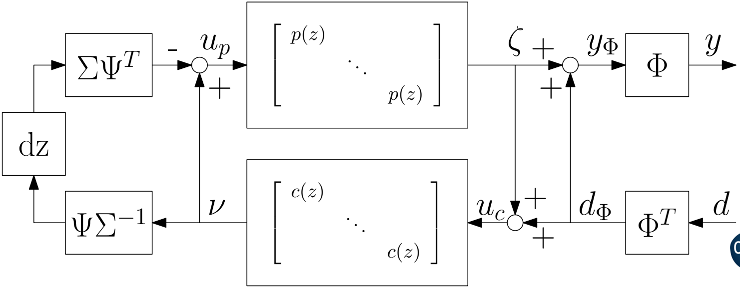
$$= {\color{red}\Phi}(p(z) \otimes I){\color{red}\Sigma\Psi^T} u + d$$

- Controller $\mathcal{C}$ equally selected using SVD:

$$y_c = {\color{red}\Psi\Sigma^{-1}} c(z){\color{red}\Phi^T} y$$

$$= {\color{red}\Psi\Sigma^{-1}} \begin{bmatrix} c(z) & & \\ & \ddots & \\ & & c(z) \end{bmatrix} {\color{red}\Phi^T} y$$

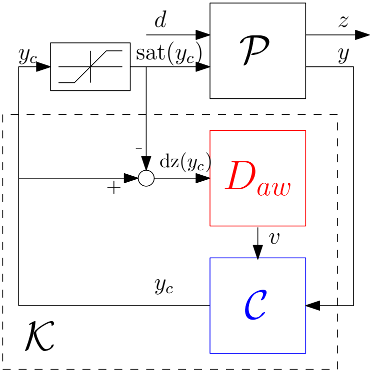
- Design of $c(z)$ as SISO feedback ☺
- Saturation mixes up everything ☹



Static Anti-Windup

Direct Linear static linear anti-windup design (LMI)

Mulder et al. [2001], Grimm et al. [2003], Gomes da Silva Jr and Tarbouriech [2005], Hu et al. [2008]



- Given $\mathcal{P}$ linear, $\mathcal{C}$ linear, **design** only
  - linear anti-windup gain $D_{aw} = \begin{bmatrix} D_{aw,1} \\ D_{aw,2} \end{bmatrix}$
- **Performance objective**:

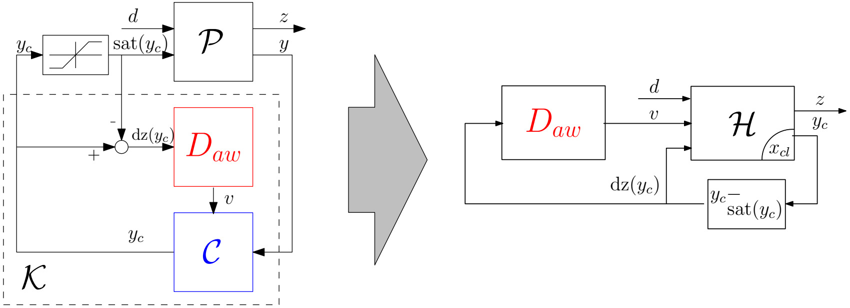
  given s, minimize $\gamma_{d \to z}(s)$ s.t.:

  $\|z\|_2 \leq \gamma \|d\|_2$ for all $\|d\|_2 \leq s$
- Linear **controller** $\mathcal{K}$ **equations**

  $$x_c^+ = Ax_c + By + D_{aw,1}(y_c - \mathrm{sat}(y_c))$$
  $$y_c = Cx_c + Dy + D_{aw,2}(y_c - \mathrm{sat}(y_c))$$
- LMI-based design Mulder et al. [2001], Grimm et al. [2003], Gomes da Silva Jr and Tarbouriech [2005], Hu et al. [2008]

- **Preserve** of *small signal* response (D_{aw} multiplies $\mathrm{dz}(y_c) = y_c - \mathrm{sat}(y_c)$)
  Asymptotically recover *large signal* response (global not always possible)

Compact representation of the closed-loop system



$$
\mathcal{H}:\begin{cases}
x_{cl}^{+} = A_{cl}x_{cl} + B_{cl,s}(y_c - \mathrm{sat}(y_c)) + B_{cl,v}v + B_{cl,d}d \\
y_c = C_{cl,u}x_p + D_{cl,us}(y_c - \mathrm{sat}(y_c)) + D_{cl,uv}v + D_{cl,ud}d \\
z = C_{cl,z}x_p + D_{cl,zs}\underbrace{(y_c - \mathrm{sat}(y_c))}_{\mathrm{dz}(y_c)} + D_{cl,zv}v + D_{cl,zd}d,
\end{cases}
$$

Quadratic analysis conditions are convex

Mulder et al. [2001], Gomes da Silva Jr and Tarbouriech [2005], Hu et al. [2008]

Proposition 1A: Given the above description and $s > 0$, if the LMI problem

$$\hat{\gamma}^2(s) = \min_{\{\gamma^2, Q, Y, U\}} \gamma^2 \text{ subject to } Q = Q^T > 0,\ U > 0 \text{ diagonal},$$

$$\mathrm{He}\begin{bmatrix} \frac{A_{cl}^T Q A_{cl} - Q}{2} & B_{cl,s}U + B_{cl,v}D_{aw}U + Y^T & B_{cl,d} & 0 \\ C_{cl,u}Q & (D_{cl,us} - I)U + D_{cl,uv}D_{aw}U & D_{cl,ud} & 0 \\ 0 & 0 & -I/2 & 0 \\ C_{cl,z}Q & D_{cl,zs}U + D_{cl,zv}D_{aw}U & D_{cl,zd} & -\frac{\gamma^2}{2}I \end{bmatrix} \prec 0, \quad \begin{bmatrix} Q & Y_{[k]}{}^T \\ Y_{[k]} & \bar{u}^2/s^2 \end{bmatrix} \succeq 0, \quad k = 1, \ldots, N$$

is feasible, then the following holds for the saturated closed-loop:

1. **[Stab]** the origin is **locally exponentially stable** (**LES**) with region of attraction (RA) containing the set $\mathcal{E}(Q, s) := \{x :\ x^T Q^{-1} x \leq s^2\}$;
2. **[Reach]** the **reachable set** from $x(0) = 0$ with $\|d\|_2 \leq s$ is contained in $\mathcal{E}(Q, s)$;
3. **[ℓ_2Perf]** for each d such that $\|d\|_2 \leq s$, the zero state solution satisfies the ℓ_2 gain bound:

$$\|z\|_2 \leq \hat{\gamma}(s)\|d\|_2$$

Quadratic analysis conditions easily lead to synthesis

Mulder et al. [2001], Gomes da Silva Jr and Tarbouriech [2005], Hu et al. [2008]

Proposition 1B: Given the above description and $s > 0$, if the LMI problem

$$\hat{\gamma}^2(s) = \min_{\{\gamma^2, Q, Y, U\}} \gamma^2 \text{ subject to } Q = Q^T > 0, \; U > 0 \text{ diagonal},$$

$$\text{He}\begin{bmatrix} \frac{A_{cl}^T Q A_{cl} - Q}{2} & B_{cl,s}U + B_{cl,v}D_{aw}U + Y^T & B_{cl,d} & 0 \\ C_{cl,u}Q & (D_{cl,us} - I)U + D_{cl,uv}D_{aw}U & D_{cl,ud} & 0 \\ 0 & 0 & -I/2 & 0 \\ C_{cl,z}Q & D_{cl,zs}U + D_{cl,zv}D_{aw}U & D_{cl,zd} & -\frac{\gamma^2}{2}I \end{bmatrix} \prec 0, \quad \begin{bmatrix} Q & Y_{[k]}{}^T \\ Y_{[k]} & \bar{u}^2/s^2 \end{bmatrix} \succeq 0, \quad k = 1, \ldots, N$$

is feasible, then the following holds for the saturated closed-loop:

1. **[Stab]** the origin is **locally exponentially stable** with region of attraction containing the set $\mathcal{E}(Q, s) := \{x : \; x^T Q^{-1} x \leq s^2\}$;
2. **[Reach]** the **reachable set** from $x(0) = 0$ with $\|d\|_2 \leq s$ is contained in $\mathcal{E}(Q, s)$;
3. **[ℓ_2Perf]** for each d such that $\|d\|_2 \leq s$, the zero state solution satisfies the ℓ_2 gain bound:

$$\|z\|_2 \leq \hat{\gamma}(s)\|d\|_2$$

Quadratic synthesis conditions are convex

Mulder et al. [2001], Gomes da Silva Jr and Tarbouriech [2005], Hu et al. [2008]

Proposition 1C: Given the above description and $s > 0$, if the LMI problem

$$\hat{\gamma}^2(s) = \min_{\{\gamma^2, Q, Y, U, X\}} \gamma^2 \text{ subject to } Q = Q^T > 0,\ U > 0 \text{ diagonal},$$

$$\text{He}\begin{bmatrix} \frac{A_{cl}^T Q A_{cl} - Q}{2} & B_{cl,s}U + B_{cl,v}X + Y^T & B_{cl,d} & 0 \\ C_{cl,u}Q & (D_{cl,us} - I)U + D_{cl,uv}X & D_{cl,ud} & 0 \\ 0 & 0 & -I/2 & 0 \\ C_{cl,z}Q & D_{cl,zs}U + D_{cl,zv}X & D_{cl,zd} & -\frac{\gamma^2}{2}I \end{bmatrix} \prec 0, \quad \begin{bmatrix} Q & Y_{[k]}{}^T \\ Y_{[k]} & \bar{u}^2/s^2 \end{bmatrix} \succeq 0, \quad k = 1, \ldots, N$$

is feasible, then, selecting the static AW gain as

$$D_{aw} = XU^{-1}$$

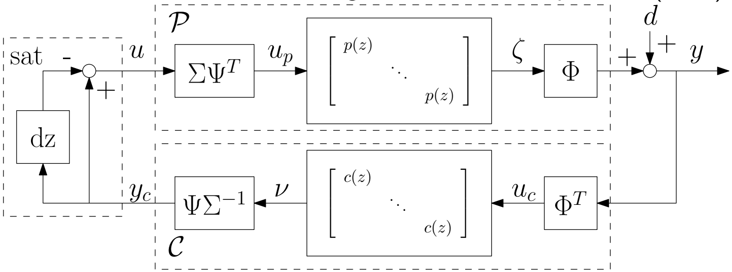
1. **[Stab]** the origin is **locally exponentially stable** with region of attraction containing the set $\mathcal{E}(Q, s) := \{x : \ x^T Q^{-1} x \leq s^2\}$;
2. **[Reach]** the **reachable set** from $x(0) = 0$ with $\|d\|_2 \leq s$ is contained in $\mathcal{E}(Q, s)$;
3. **[ℓ_2Perf]** for each d such that $\|d\|_2 \leq s$, the zero state solution satisfies the ℓ_2 gain bound:

$$\|z\|_2 \leq \hat{\gamma}(s)\|d\|_2$$

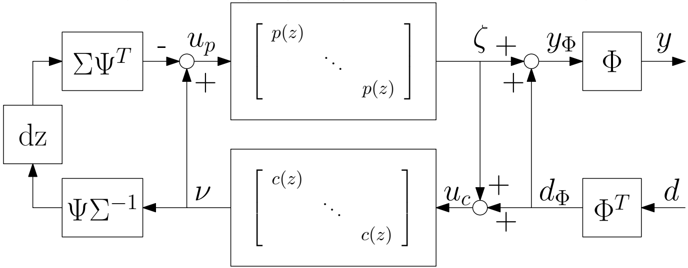
Cross-directional static anti-windup

Recall two equivalent closed-loop schemes

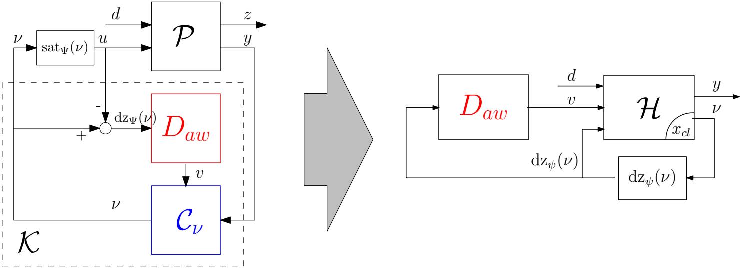
▷ Model is based on a suitable Singular Value Decomposition (SVD)



▷ Equivalent dynamics highlights generalized nonlinearity



Modal space representation of the closed-loop system



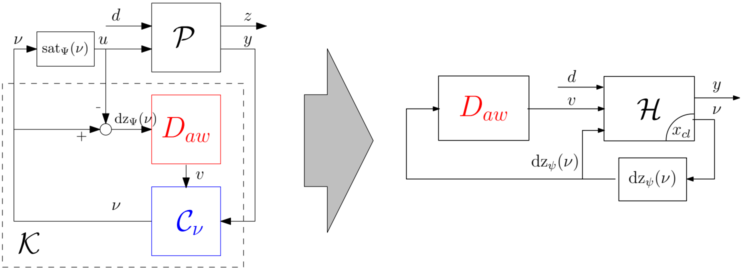
$$
\begin{aligned}
x_{cl}^{+} &= (A_{cl}\otimes I_N)x_{cl} + (B_{cl,u}\otimes I_N)\mathrm{dz}_\Psi(\nu) + (B_{cl,v}\otimes I_N)v + (B_{cl,d}\otimes I_N)\Phi^T d\\
\nu &= (C_{cl,\nu}\otimes I_N)x_{cl} + (D_{cl,\nu u}\otimes I_N)\mathrm{dz}_\Psi(\nu) + (D_{cl,\nu v}\otimes I_N)v + (D_{cl,\nu d}\otimes I_N)\Phi^T d\\
y &= \Phi\left((C_{cl,y}\otimes I_N)x_{cl} + (D_{cl,yu}\otimes I_N)\mathrm{dz}_\Psi(\nu) + (D_{cl,yv}\otimes I_N)v + (D_{cl,yd}\otimes I_N)\Phi^T d\right) + d
\end{aligned}
$$

with $\mathrm{dz}_\Psi(\nu) := \Sigma\Psi^T\mathrm{dz}(\Psi\Sigma^{-1}\nu)$.

▷ Select anti-windup action as:

$$v = D_{aw}\mathrm{dz}_\Psi(\nu) = D_{aw}\Sigma\Psi^T\mathrm{dz}(\Psi\Sigma^{-1}\nu)$$

Modal space representation of the closed-loop system



$$
\begin{aligned}
x_{cl}^{+} &= \bar{A}_{cl} x_{cl} + \bar{B}_{cl,u} \mathrm{dz}_{\Psi}(\nu) + \bar{B}_{cl,v} v + \bar{B}_{cl,d} \Phi^T d \\
\nu &= \bar{C}_{cl,\nu} x_{cl} + \bar{D}_{cl,\nu u} \mathrm{dz}_{\Psi}(\nu) + \bar{D}_{cl,\nu v} v + \bar{D}_{cl,\nu d} \Phi^T d \\
y &= \Phi \left(\bar{C}_{cl,y} x_{cl} + \bar{D}_{cl,yu} \mathrm{dz}_{\Psi}(\nu) + \bar{D}_{cl,yv} v + \bar{D}_{cl,yd} \Phi^T d \right)
\end{aligned}
$$

with $\mathrm{dz}_{\Psi}(\nu) := \Sigma \Psi^T \mathrm{dz}(\Psi \Sigma^{-1} \nu)$ and the anti-windup action

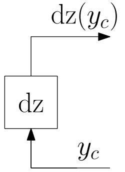
$$
v = D_{aw} \mathrm{dz}_{\Psi}(\nu) = D_{aw} \Sigma \Psi^T \mathrm{dz}(\Psi \Sigma^{-1} \nu)
$$

Key lemma for static AW: generalized sector condition

▷ Original lemma stated regardless of SVD:

Lemma 1: Given diagonal $W > 0$, for any $y_c, h \in \mathbb{R}^N$:

$$\mathrm{dz}(h) = 0 \;\Rightarrow\; \mathrm{dz}(y_c)^T W (y_c - \mathrm{dz}(y_c) + h) \geq 0$$



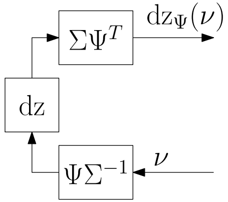
▷ A transformed version based on the SVD is useful:

Lemma 2: Given orthogonal matrix Ψ, diagonal matrices $W > 0$ and $\Sigma > 0$, denote $\bar{W} := \Sigma^{-1}\Psi^T W \Psi \Sigma^{-1}$ and

$$\mathrm{dz}_\Psi(\nu) := \Sigma \Psi^T \mathrm{dz}(\Psi \Sigma^{-1} \nu)$$

Then for any $\nu, h \in \mathbb{R}^N$:

$$\mathrm{dz}_\Psi(h) = 0 \;\Rightarrow\; \mathrm{dz}_\Psi(\nu)^T \bar{W} (\nu - \mathrm{dz}_\Psi(\nu) + h) \geq 0$$



▷ A "decentralized" selection is $W = w_0 I \;\Rightarrow\; \bar{W} = w_0 \Sigma^{-2}$

Quadratic synthesis conditions are still convex

Theorem 3. Given the above description and $s > 0$, If the LMI problem

$$\hat{\gamma}^2(s) = \min_{\{\gamma^2, Q, Y, U, X\}} \gamma^2 \text{ subject to } Q = Q^T > 0,\ U > 0 \text{ diagonal},$$

$$\mathrm{He}\begin{bmatrix} \frac{\bar{A}_{cl}^T Q \bar{A}_{cl} - Q}{2} & \bar{B}_{cl,u}\Sigma\Psi^T U \Psi\Sigma + \bar{B}_{cl,v}X + Y^T & \bar{B}_{cl,d} & 0 \\ \bar{C}_{cl,\nu}Q & (\bar{D}_{cl,\nu u} - I)\Sigma\Psi^T U\Psi\Sigma + \bar{D}_{cl,\nu v}X & \bar{D}_{cl,\nu d} & 0 \\ 0 & 0 & -I/2 & 0 \\ \bar{C}_{cl,y}Q & \bar{D}_{cl,yu}\Sigma\Psi^T U\Psi\Sigma + \bar{D}_{cl,yv}X & \bar{D}_{cl,yd} & -\frac{\gamma^2}{2}I \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} Q & Y_{[k]}{}^T \\ Y_{[k]} & \bar{u}^2/s^2 \end{bmatrix} \succeq 0, \text{ for all } k = 1, \dots, N$$

is feasible, then, selecting the static AW gain as $D_{aw} = X\Sigma^{-1}\Psi^T U^{-1}\Psi\Sigma^{-1}$

1. **[Stab]** the origin is **LES** with RA containing $\mathcal{E}(Q, s)$;
2. **[Reach]** the **reachable set** from $x(0) = 0$ with $\|d\|_2 \le s$ is contained in $\mathcal{E}(Q, s)$;
3. **[ℓ_2Perf]** for each d such that $\|d\|_2 \le s$, the zero state solution satisfies the ℓ_2 gain bound $\|z\|_2 \le \hat{\gamma}(s)\|d\|_2$

Conservatively reduce the number of constraints

Theorem 2. Given the above description and $s > 0$, If the LMI problem

$$\hat{\gamma}^2(s) = \min_{\{\gamma^2, Q, Y, U, X\}} \gamma^2 \text{ subject to } Q = Q^T > 0,\ U > 0 \text{ diagonal},$$

$$\mathrm{He}\begin{bmatrix} \frac{\bar{A}_{cl}^T Q \bar{A}_{cl} - Q}{2} & \bar{B}_{cl,u}\Sigma\Psi^T U \Psi\Sigma + \bar{B}_{cl,v} X + Y^T & \bar{B}_{cl,d} & 0 \\ \bar{C}_{cl,\nu} Q & (\bar{D}_{cl,\nu u} - I)\Sigma\Psi^T U \Psi\Sigma + \bar{D}_{cl,\nu v} X & \bar{D}_{cl,\nu d} & 0 \\ 0 & 0 & -I/2 & 0 \\ \bar{C}_{cl,y} Q & \bar{D}_{cl,yu}\Sigma\Psi^T U \Psi\Sigma + \bar{D}_{cl,yv} X & \bar{D}_{cl,yd} & -\frac{\gamma^2}{2} I \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} Q & Y^T \\ Y & \bar{u}^2/s^2 I \end{bmatrix} \succeq 0, \text{ (a single constraint now)}$$

is feasible, then, selecting the static AW gain as $D_{aw} = X\Sigma^{-1}\Psi^T U^{-1}\Psi\Sigma^{-1}$

1. **[Stab]** the origin is **LES** with RA containing $\mathcal{E}(Q, s)$;
2. **[Reach]** the **reachable set** from $x(0) = 0$ with $\|d\|_2 \leq s$ is contained in $\mathcal{E}(Q, s)$;
3. **[ℓ_2Perf]** for each d such that $\|d\|_2 \leq s$, the zero state solution satisfies the ℓ_2 gain bound $\|z\|_2 \leq \hat{\gamma}(s)\|d\|_2$

Comments

- Proof of Theorem3:
  - Direct use of Proposition 1C and the generalized sector condition on the SVD
  - AW conditions are considered with $d_\Phi = \Phi d$ and $y_\Phi = \Phi^T y$
  - Since Φ is an orthogonal matrix, one obtains:

$$|d_\Phi|^2 = d^T \Phi \Phi^T d = d^T d = |d|^2, \text{ and similarly } |y_\Phi|^2 = |y|^2$$

- Proof of Theorem 2: The same + notice that

$$\begin{bmatrix} Q & Y^T \\ Y & \bar{u}^2/s^2 I \end{bmatrix} \succeq 0, \quad \Rightarrow \begin{bmatrix} Q & Y_{[k]}{}^T \\ Y_{[k]} & \bar{u}^2/s^2 \end{bmatrix} \succeq 0, \text{ for all } k = 1, \ldots, N$$

Decentralized design for centralized compensation

Theorem 1. Given the above description and $s > 0$, If the LMI problem

$$\hat{\gamma}^2(s) = \min_{\{\gamma^2, Q_0, Y_0, u_0, X\}} \gamma^2 \text{ subject to } Q_0 = Q_0{}^T > 0,\ u_0 > 0,$$

$$\mathrm{He}\begin{bmatrix} \frac{A_{cl}^T Q_0 A_{cl} - Q_0}{2} & B_{cl,u} u_0 + B_{cl,v} X_0 + Y_0{}^T & B_{cl,d} & 0 \\ C_{cl,\nu} Q_0 & (D_{cl,\nu v} - I) u_0 + D_{cl,\nu v} X_0 & D_{cl,\nu d} & 0 \\ 0 & 0 & -I/2 & 0 \\ C_{cl,y} Q_0 & D_{cl,yu} u_0 + D_{cl,yv} X_0 & D_{cl,yd} & -\frac{\sigma_m^2 \gamma^2}{2\sigma_M^2} I \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} Q_0 & Y_0{}^T \\ Y_0 & \bar{u}^2/(\sigma_M^2 s^2) \end{bmatrix} \succeq 0, \text{ (it is a single input small system)}$$

is feasible, then, selecting the static AW gain as $D_{aw} = u_0^{-1} X_0 \otimes I_n$

1. **[Stab]** the origin is **LEQ** with RA containing the set $\mathcal{E}(Q, s)$;
2. **[Reach]** the **reachable set** from $x(0) = 0$ with $\|d\|_2 \leq s$ is contained in $\mathcal{E}(Q, s)$;
3. **[ℓ_2Perf]** for each d such that $\|d\|_2 \leq s$, the zero state solution satisfies the ℓ_2 gain bound $\|z\|_2 \leq \hat{\gamma}(s)\|d\|_2$

Proof of Theorem 1

- AW conditions are considered with $\bar{d} = \Sigma^{-1} d_\Phi$ and $\bar{y} = \Sigma^{-1} y_\Phi$
- The solution Q_0, Y_0, X_0, u_0 of Theorem 1 allows to build $Q = Q_0 \otimes \Sigma^2$, $U = u_0 \otimes I_N$, $Y = Y_0 \otimes \Sigma^2$; which allows to show that the Lyapunov condition of Theorem 1 implies that one of Theorem 3.
- Similarly, for the sector condition, one uses the fact that

$$\begin{bmatrix} Q_0 \otimes \Sigma^2 & Y_0^T \otimes \Sigma^2 \\ Y_0 \otimes \Sigma^2 & \frac{\bar{u}}{s^2} \otimes I_N \end{bmatrix} \geq \begin{bmatrix} Q_0 \otimes \Sigma^2 & Y_0^T \otimes \Sigma^2 \\ Y_0 \otimes \Sigma^2 & \frac{\bar{u}}{s^2} \otimes \sigma_M^{-2} \Sigma^2 \end{bmatrix}$$
$$= \begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma \end{bmatrix} \begin{bmatrix} Q_0 & Y_0^T \\ Y_0 & \frac{\bar{u}}{(s\sigma_M)^2} \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma \end{bmatrix} \geq 0$$

- $|y|^2 = |y_\Phi|^2 \leq |\Sigma^2||\bar{y}|^2 \leq |\Sigma|^2 \bar{\gamma}^2 |\bar{d}|^2$
$$\leq |\Sigma^2| \frac{\sigma_m^2}{\sigma_M^2} \gamma^2 |\Sigma^{-2}||d_\Phi|^2 = \gamma^2 |d_\Phi|^2 = \gamma^2 |d|$$

Performance and complexity discussed on an example

Performance level and computational complexity

▷ Hierarchical relation among the three results:

Proposition. The optimal values of the three optimization problems in Theorems 1, 2 and 3 satisfy:

$$\gamma_1^2 \geq \gamma_2^2 \geq \gamma_3^2.$$

▷ Comparative computational complexity among the three anti-windup constructions

Result	Number of variables	Number of lines
Thm 1	$\frac{(n_p+n_c)(n_p+n_c+3)}{2} + n_c + 3$	$2(n_p + n_c) + 5$
Thm 2	$\frac{N(n_p+n_c)(N(n_p+n_c)+2N+1)}{2} + N^2(n_c + 1) + N + 1$	$N(2(n_p + n_c) + 5)$
Thm 3	$\frac{N(n_p+n_c)(N(n_p+n_c)+2N+1)}{2} + N^2(n_c + 1) + N + 1$	$N((1 + N)(n_p + n_c) + 5)$

where N = number of sensors/actuators, n_p, n_c = plant, controller order

Sample numerical application to a synchrotron model
Gayadeen and Duncan [2013]

- Plant $p(z)$ is order 8: first order response of the power supply units for the corrector magnet and delay in the sensor data acquisition and processing

$$p(z) = \frac{0.3558}{z^8 - 0.6442 z^7}$$

- (IMC-based) controller $c(z)$ is order 9

$$c(z) = \frac{q(z)}{1 - p(z)q(z)}, \quad q(z) = 0.4 \frac{0.3741z - 0.241}{z - 0.8669}$$

- Bound on the magnitude saturation and maximum size of the disturbance set as

$$\overline{u} = 1 \quad , \quad s = 10$$

- Static map from the N actuators to the N sensors position issued from a real machine from Diamond Light Source (small case with $N = 4$)

$$B = \begin{bmatrix} 0.5984 & 0.9022 & -0.9192 & 1.0138 \\ 1.5123 & 0.7611 & -0.7681 & 0.7771 \\ -1.4066 & -0.7489 & 0.6329 & -0.6318 \\ 0.6533 & 0.2355 & -0.1403 & 0.5424 \end{bmatrix}$$

Comparisons of the theorems (complexity)

- Comparative synthesis computational time among the three anti-windup constructions for various values of actuators/sensors configurations.

		Theorem 1	Theorem 2	Theorem 3
N=2	nb var	182	706	706
	nb lines	39	78	112
	time (s)	< 1	12	13
N=3	nb var	182	1573	1573
	nb lines	39	117	219
	time (s)	< 1	648	810
N=4	nb var	182	2783	2783
	nb lines	39	156	360
	time (s)	< 1	5068	8102
N=5	nb var	182	4336	4336
	nb lines	39	195	535
	time (s)	< 1	26364	46445

Comparisons of the theorems (performance)

- Computation of the anti-windup gains

$$D_{aw1} = \begin{bmatrix} D_{aw11} & 0 & 0 & 0 \\ 0 & D_{aw11} & 0 & 0 \\ 0 & 0 & D_{aw11} & 0 \\ 0 & 0 & 0 & D_{aw11} \end{bmatrix}, \; D_{aw2,3} = \begin{bmatrix} D_{awi1} & \bullet & \bullet & \bullet \\ \bullet & D_{awi2} & \bullet & \bullet \\ \bullet & \bullet & D_{awi3} & \bullet \\ \bullet & \bullet & \bullet & D_{awi4} \end{bmatrix}$$

- Use of Theorem 3 to compute the performance index of the three anti-windup gains issued from Theorems 1, 2 and 3

Anti-windup	$D_{aw} = 0$	D_{aw1}	D_{aw2}	D_{aw3}
γ_3	30.1605	6.8568	4.4839	4.4743

Summary

Summary

- LMI-based (Direct Linear) anti-windup can exploit special structure of cross-directional control systems

- Three approaches proposed, only one is numerically reasonable for synchrotron models

- Suitable characterizations of performance levels versus computational complexity has been established

S. Gayadeen and S. Duncan. Anti-windup compensation for electron beam stabilisation control systems on synchrotrons with rate constrained actuators. In *2013 European Control Conference*, pages 2752–2757, 2013.

J.M. Gomes da Silva Jr and S. Tarbouriech. Anti-windup design with guaranteed regions of stability: an LMI-based approach. *IEEE Transactions on Automatic Control*, 50(1):106–111, 2005.

G. Grimm, J. Hatfield, I. Postlethwaite, A.R. Teel, M.C. Turner, and L. Zaccarian. Antiwindup for stable linear systems with input saturation: an lmi-based synthesis. *IEEE Transactions on Automatic Control*, 48(9): 1509–1525, 2003.

T. Hu, A.R. Teel, and L. Zaccarian. Anti-windup synthesis for linear control systems with input saturation: achieving regional, nonlinear performance. *Automatica*, 44(2):512–519, 2008.

E.F. Mulder, M.V. Kothare, and M. Morari. Multivariable anti-windup controller synthesis using linear matrix inequalities. *Automatica*, 37(9): 1407–1416, September 2001.