Saturated control of infinite-dimensional systems

Christophe PRIEUR
CNRS, Gipsa-lab, Grenoble, France

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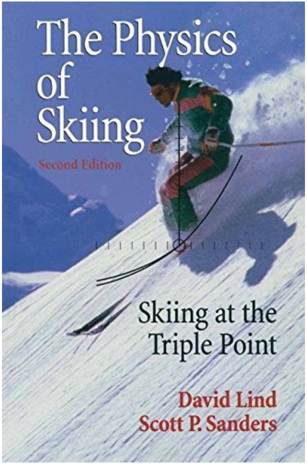

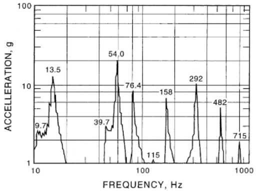

FIGURE 3.5. The frequency distribution of the normal vibration modes of a ski. The ski is clamped at the center to a shaker and driven. An output accelerometer located on the afterbody records the vibration response shown. [Reprinted with permission from R. L. Pizialli and C. D. Mote, Jr., "The Snow Ski as a Dynamic System," J. Dynamic Syst. Meas. Control, Trans. ASME **94**, 134 (1972).]

Page 63: Natural frequency with "good and bad vibrations"

[David A. Lind et Scott P. Sanders, The Physics of Skiing: Skiing at the Triple Point, 2nd edition; 2013]

One way to kill bad vibrations?

Control your skis with smart materials!

Use passively piezoelectric patches
[L. Rothemann, H. Schretter, Active vibration damping of the alpine ski; 2010]

How to do it actively?
Need to consider a distributed parameter systems:
How to control the flexible ski structure? Euler Bernoulli equation:

$$\rho \frac{\partial^2 w}{\partial t^2} + YI \frac{\partial^4 w}{\partial x^4} = \text{piezo force under control}$$

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FIGURE 2.1. This skier heads down the hill, his skis lubricated by a film of water that forms under his skis. In his thoughts he mulls over a mathematical formula that we will discuss later in Chapter 8 on snow friction processes. (Colbeck, 1992. Drawn by Marilyn Aber, CRREL.)

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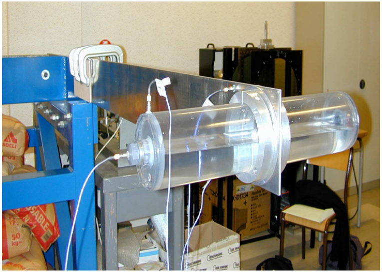
Another domain with large flexible structures:

- satellites with large flexible structures
- and large airplanes with flexible wings and fluid dynamics

Flexible structure+ sloshing modes



control of distributed parameters systems (PDE) with

- robustness
- experiments
- in-domain control

See [Robu, Baudouin, CP; 12], [Robu, Baudouin, CP, Arzelier; 12]

Can we use saturated control?

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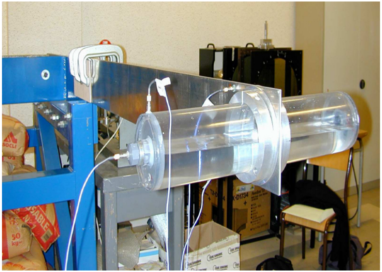

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Can we use saturated control?

Given a PDE, there exists now a large variety on methods to design linear controllers. It is well known that saturation can reduce the performance or even destabilize the system, even for finite-dimensional systems.
More precisely, even if

$$\dot{z} = Az + BKz \tag{1}$$

is asymp. stable, it may hold that

$$\dot{z} = Az + \mathtt{sat}(BKz) \tag{2}$$

is not globally asymptotically stable.
It may exist new equilibrium, new limit cycles...
See e.g. [Tarbouriech, Garcia, Gomes da Silva Jr., Queinnec; 2011]
Goal of this talk:
What happens if in (2), instead of matrices A, B..., we have operators? More precisely, what happens if A generates a semigroup and B is a bounded control operator? An example of such a nonlinear PDE given by (2):
Wave equation with saturating in-domain control

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Wave equation with saturating in-domain control

Two objectives

- Well-posedness
- Stability

of the wave equation in presence of a disturbed saturating control with a Lyapunov method.

Outline

1 Well-posedness and stability of linear wave equation with a saturated in-domain control

Lyapunov method, LaSalle invariance principle

2 Well-posedness and stability of linear abstract systems with a saturated in-domain control

strict Lyapunov method, robustness result

3 Numerical simulations on wave equation

effect of the saturation level

4 Conclusion

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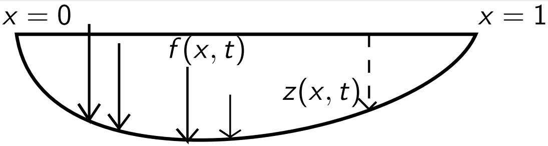

1D wave equation with in-domain control.

Dynamics of the vibration:

$$z_{tt}(x,t) = z_{xx}(x,t) + f(x,t),\ \forall x \in (0,1), t \geq 0, \tag{3}$$

Boundary conditions, $\forall t \geq 0$,

$$\begin{aligned} z(0,t) &= 0\ , \\ z(1,t) &= 0\ , \end{aligned} \tag{4}$$

and with the following initial condition, $\forall x \in (0,1)$,

$$\begin{aligned} z(x,0) &= z^0(x)\ , \\ z_t(x,0) &= z^1(x)\ , \end{aligned} \tag{5}$$

where z^0 and z^1 stand respectively for the initial deflection and the initial deflection speed.

When closing the loop with a linear control

Let us define the linear control by

$$f(x,t) = -az_t(x,t), x \in (0,1), \ \forall t \geq 0, \tag{6}$$

and consider

$$V_1 = \frac{1}{2}\int (z_x^2 + z_t^2)dx.$$

Formal computation. Along the solutions to (3), (4) and (6):

$$\begin{aligned} \dot{V}_1 &= \int_0^1 (z_x z_{xt} - az_t^2 + z_t z_{xx})dx \\ &= -\int_0^1 az_t^2 dx + [z_t z_x]_{x=0}^{x=1} \\ &= -\int_0^1 az_t^2 dx \end{aligned}$$

Thus, it $a > 0$, V_1 is a (non strict) Lyapunov function.

Using standard technics (Lumer-Philipps thereom (for the well-posedness) and Huang-Prüss theorem (for the exp. stability)):

Proposition

$\forall a > 0$, $\forall (z^0, z^1)$ in $H_0^1(0,1) \times L^2(0,1)$,
$\exists$! solution z: $[0,\infty) \to H_0^1(0,1) \times L^2(0,1)$ to (3)-(6). Moreover, $\exists$ C, $\mu > 0$, such that, for any initial condition $H_0^1(0,1) \times L^2(0,1)$, it holds, $\forall t \geq 0$,

$$\|z\|_{H_0^1(0,1)} + \|z_t\|_{L^2(0,1)} \leq Ce^{-\mu t}(\|z^0\|_{H_0^1(0,1)} + \|z^1\|_{L^2(0,1)}).$$

In the previous proposition:

- stability
- attractivity of the equilibrium
- with an exponential speed

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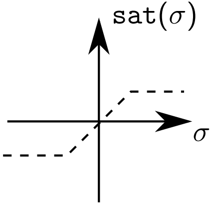
When closing the loop with a saturating control

Let us consider now the nonlinear control

$$f(x,t) = -\texttt{sat}(az_t(x,t)),\ x \in (0,1),\ \forall t \geq 0, \tag{7}$$

where sat is the localized saturated map:



$$\texttt{sat}(\sigma) = \begin{cases} \sigma & \text{if } |\sigma| < 1 \\ \texttt{sign}(\sigma) & \text{else} \end{cases}$$

Equation (3) in closed loop with the control (7) becomes

$$z_{tt} = z_{xx} - \texttt{sat}(az_t) \tag{8}$$

A formal computation gives, along the solutions to (8) and (4),

$$\dot{V}_1 = -\int_0^1 z_t \texttt{sat}(az_t) dx$$

which asks to handle the nonlinearity $z_t\texttt{sat}(az_t)$.

Remark: Choice of the saturation map

[Slemrod; 1989] and [Lasiecka and Seidman; 2003] deal with L^2 saturation:
Given $\sigma : [0,1] \to \mathbb{R}$, $\mathtt{sat}_2(\sigma)$ is the function defined by

$$\mathtt{sat}_2(\sigma)(x) = \begin{cases} \sigma(x) & \text{if } \|\sigma\|_{L^2(0,1)} < 1 \\ \frac{\sigma(x)}{\|\sigma\|_{L^2(0,1)}} & \text{else} \end{cases}$$

Here we consider **localized** saturation which is more physically relevant:

$$\mathtt{sat}(\sigma(x)) = \begin{cases} \sigma(x) & \text{if } |\sigma(x)| < 1 \\ \mathtt{sign}(\sigma(x)) & \text{else} \end{cases}$$

Well-posedness of this nonlinear PDE

Theorem 1 [CP, Tarbouriech, Gomes da Silva Jr; 2016]

$\forall a \geq 0$, for all (z^0, z^1) in $(H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1)$, there exists a unique solution z: $[0,\infty) \to H^2(0,1) \cap H_0^1(0,1)$ to (8) with the boundary conditions (4) and the initial condition (5).

Consider

$$A_1 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ u_{xx} - \mathtt{sat}(av) \end{pmatrix}$$

with the domain $D(A_1) = (H^2(0,1) \cap H^1_0(0,1)) \times H^1_0(0,1)$.
Let us use a generalization of Lumer-Phillips theorem which is the so-called Crandall-Liggett theorem, as given in [Barbu; 1976]. See also [Brezis; 1973] and [Miyadera; 1992].
Again two conditions

1. A_1 is dissipative, that is

$$\Re\left(\langle A_1 \begin{pmatrix} u \\ v \end{pmatrix} - A_1 \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \rangle\right) \leq 0$$

2. For all $\lambda > 0$, $D(A_1) \subset \mathtt{Ran}(I - \lambda A_1)$

First item: Easy step!

Instead of proving
$\Re\left(\langle A_1\begin{pmatrix} u \\ v \end{pmatrix} - A_1\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}\rangle\right) \leq 0$, let us check, for all $\begin{pmatrix} u \\ v \end{pmatrix} \in H_0^1(0,1) \times L^2(0,1)$:

$$\Re\left(\langle A_1\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\rangle\right) \leq 0$$

To do that, using the definition of A_1, and of the scalar product in $H_0^1(0,1) \times L^2(0,1)$, it is equal to:

$$\begin{aligned} &\int_0^1 v_x(x)\overline{u_x(x)}dx + \int_0^1 (u_{xx}(x) - \mathtt{sat}(a\,v(x)))\overline{v(x)}dx\ , \\ &= \int_0^1 v_x(x)\overline{u_x(x)}dx + \int_0^1 u_{xx}(x)\overline{v(x)}dx - \int_0^1 \mathtt{sat}(a\,v(x))\overline{v(x)}dx \\ &= [u_x(x)\overline{v(x)}]_{x=0}^{x=1} - \int_0^1 \mathtt{sat}(a\,v(x))\overline{v(x)}dx \leq 0 \end{aligned}$$

due to the boundary and since $a \geq 0$.

Second item asks to deal with a nonlinear ODE.

Let $\begin{pmatrix} u \\ v \end{pmatrix} \in H_0^1(0,1) \times L^2(0,1)$ we have to find $\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \in D(A_1)$ such that

$$(I - \lambda A_1)\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

that is

$$\begin{cases} \tilde{u} - \lambda \tilde{v} = u \ , \\ \tilde{v} - \lambda(\tilde{u}_{xx} - \mathtt{sat}(a\,\tilde{v})) = v \ , \end{cases}$$

In particular, we have to find $\tilde{u}$ such that

$$\tilde{u}_{xx} - \frac{1}{\lambda^2}\tilde{u} - \mathtt{sat}(\frac{a}{\lambda}(\tilde{u} - u)) = -\frac{1}{\lambda}v - \frac{1}{\lambda^2}u$$
$$\tilde{u}(0) = \tilde{u}(1) = 0$$

holds.

Nonhomogeneous nonlinear ODE with two boundary conditions

Lemma

If a is nonnegative and λ is positive, then there exists $\tilde{u}$ solution to

$$\begin{array}{c} \tilde{u}_{xx} - \frac{1}{\lambda^2}\tilde{u} - \mathtt{sat}(\frac{a}{\lambda}(\tilde{u}-u)) = -\frac{1}{\lambda}v - \frac{1}{\lambda^2}u \\ \tilde{u}(0) = \tilde{u}(1) = 0 \end{array} \qquad (9)$$

To prove this lemma, let us introduce the following map

$$\begin{array}{rrcl} \mathcal{T}: & L^2(0,1) & \to & L^2(0,1)\ , \\ & y & \mapsto & z = \mathcal{T}(y)\ , \end{array}$$

where $z = \mathcal{T}(y)$ is the unique solution to

$$\begin{array}{c} z_{xx} - \frac{1}{\lambda^2}z = -\frac{1}{\lambda}v - \frac{1}{\lambda^2}u + \mathtt{sat}(\frac{a}{\lambda}(y-u))\ , \\ z(0) = z(1) = 0\ . \end{array}$$

Prove that $\mathcal{T}$ is well defined and apply the Schauder fixed-point theorem (see e.g., [Coron; 2007]), to deduce that there exists y such that $\mathcal{T}(y) = y$

$\tilde{u} = y$ solves (9)

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Global asymptotic stability of this nonlinear PDE

Theorem 2

$\forall a > 0$, for all (z^0, z^1) in $(H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1)$, the solution to (8) with the boundary conditions (4) and the initial condition (5) satisfies the following stability property, $\forall t \geq 0$,

$$\|z(.,t)\|_{H_0^1(0,1)} + \|z_t(.,t)\|_{L^2(0,1)} \leq \|z^0\|_{H_0^1(0,1)} + \|z^1\|_{L^2(0,1)} ,$$

together with the attractivity property

$$\|z(.,t)\|_{H_0^1(0,1)} + \|z_t(.,t)\|_{L^2(0,1)} \to 0, \quad \text{as } t \to \infty .$$

Due to Theorem 1, the formal computation

$$\dot{V}_1 = -\int_0^1 z_t \mathtt{sat}(az_t)dx$$

makes sense. This is only a weak Lyapunov function $\dot{V}_1 \leq 0$
(the state is (z, z_t), and there is no $-z^2$).
To be able to apply LaSalle's Invariance Principle, we have to check that the trajectories are precompact
(see e.g. [Dafermos, Slemrod; 1973], [d'Andréa-Novel *et al*; 1994]).
It comes from:

Lemma

The canonical embedding from $D(A_1)$, equipped with the graph norm, into $H_0^1(0,1) \times L^2(0,1)$ is compact.

Sketch of the proof of

The canonical embedding from $D(A_1)$, equipped with the graph norm, into $H_0^1(0,1) \times L^2(0,1)$ is compact.

Consider a sequence $\begin{pmatrix} u_n \\ v_n \end{pmatrix}_{n\in\mathbb{N}}$ in $D(A_1)$, which is bounded with the graph norm, that is $\exists M > 0$, $\forall n \in \mathbb{N}$,

$$\begin{aligned} \left\| \begin{pmatrix} u_n \\ v_n \end{pmatrix} \right\|^2_{D(A_1)} &:= \left\| \begin{pmatrix} u_n \\ v_n \end{pmatrix} \right\|^2 + \left\| A_1 \begin{pmatrix} u_n \\ v_n \end{pmatrix} \right\|^2 , \\ &= \int_0^1 (\left|u_n'\right|^2 + \left|v_n\right|^2 + \left|v_n'\right|^2 \\ &\quad + \left|u_n'' - a\mathtt{sat}(v_n)\right|^2) dx < M \end{aligned}$$

From that, we deduce that $\int_0^1 (|v_n|^2 + |v_n'|^2) dx$ and $\int_0^1 (|u_n'|^2 + |u_n''|^2) dx$ are bounded.
Thus there exists a subsequence which converges in $H_0^1(0,1) \times L^2(0,1)$. $\square$

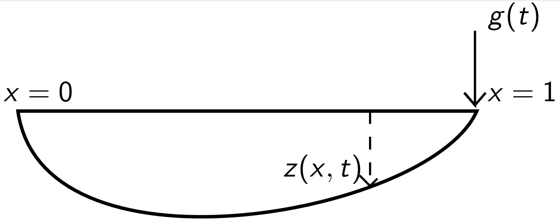
Using the dissipativity of A_1, and previous lemma the trajectory $\begin{pmatrix} z(.,t) \\ z_t(.,t) \end{pmatrix}$ is precompact in $H_0^1(0,1) \times L^2(0,1)$.

Moreover the ω-limit set $\omega \left[\begin{pmatrix} z(.,0) \\ z_t(.,0) \end{pmatrix} \right] \subset D(A_1)$, is not empty and invariant with respect to the nonlinear semigroup $T(t)$ (see [Slemrod; 1989]).

We now use LaSalle's invariance principle to show that $\omega \left[\begin{pmatrix} z(.,0) \\ z_t(.,0) \end{pmatrix} \right] = \{0\}$.

Therefore the convergence property holds. $\square$

Remark: Boundary control



1D wave equation with a boundary control.

Dynamics: $\forall x \in (0,1), t \geq 0$,

$$z_{tt}(x,t) = z_{xx}(x,t),$$

Boundary conditions: $\forall t \geq 0$,

$$\begin{aligned} z(0,t) &= 0\ , \\ z_x(1,t) &= -\mathtt{sat}(bz_t(1,t))\ , \end{aligned}$$

In the same work, stability proof using the sector condition + strict Lyapunov function.

▸ Wave equation and saturated boundary control

For the wave equation+ saturated in-domain control, a non-strict Lyapunov function has been computed. Thus:

- No robustness margin. What happens in presence of noise?
- For linear PDE, we have exponential convergence (see Proposition on Slide 10).
  Do we have exp. stability for the nonlinear PDE?

Let us start with the abstract control system

$$\begin{cases} \frac{d}{dt}z = Az + Bu, \\ z(0) = z_0, \end{cases} \tag{10}$$

where $A : D(A) \subset H \to H$ be a linear operator whose domain $D(A)$ is dense in H. Assume it generates a strongly continuous semigroup of contractions denoted by e^{tA}.
Let $B : U \to H$ be a bounded operator.
Wave equation with in-domain control applies !

A natural feedback law for (10) is $u = -B^\star z$.

Assumption 1: a linear feedback law is given

The linear closed-loop system

$$\begin{cases} \dfrac{d}{dt} z = (A - BB^\star) z, \\ z(0) = z_0. \end{cases} \tag{11}$$

globally exponentially stable.

Under Assumption 1, there exist a self-adjoint and definite positive operator $P \in \mathcal{L}(H)$ and a positive value C such that

$$\langle \tilde{A}z, Pz, \rangle_H + \langle Pz, \tilde{A}z \rangle_H \leq -C \|z\|_H^2, \quad \forall z \in D(\tilde{A}), \tag{12}$$

with $\tilde{A} = A - BB^\star$.

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In presence of saturated input and disturbances

Consider the saturated case

$$\begin{cases} \frac{d}{dt}z = Az - B\mathtt{sat}_U(B^\star z), \\ z(0) = z_0, \end{cases} \tag{13}$$

In presence of saturated input and disturbances

Consider the saturated case + disturbance

$$\begin{cases} \frac{d}{dt} z = Az - B\mathtt{sat}_U(B^\star z + d), \\ z(0) = z_0, \end{cases} \tag{13}$$

where $d : (0, \infty) \to U$ is a disturbance.

In presence of saturated input and disturbances

Consider the saturated case + disturbance

$$\begin{cases} \frac{d}{dt}z = Az - B\mathtt{sat}_U(B^\star z + d), \\ z(0) = z_0, \end{cases} \tag{13}$$

where $d : (0, \infty) \to U$ is a disturbance.

Recall the L^2 saturation: Given $u : [0, 1] \to \mathbb{R}$, $\mathtt{sat}_2(\sigma)$ is the function defined by $\mathtt{sat}_2(\sigma) = \begin{cases} \sigma & \text{if } \|\sigma\|_{L^2(0,1)} < 1 \\ \frac{\sigma}{\|\sigma\|_{L^2(0,1)}} & \text{else} \end{cases}$

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What can be said about the exp. stability when $d = 0$
and about the robustness in presence of d?

Input-to-State Stability definition

A positive definite function $V : H \to \mathbb{R}_{\geq 0}$ is said to be an ISS-Lyapunov function with respect to d if $\exists$ two class $\mathcal{K}_\infty$ functions α and ρ such that, for any solution to (13)

$$\frac{d}{dt} V(z) \leq -\alpha(\|z\|) + \rho(\|d\|_U).$$

Remark: Of course ISS Lyapunov function
$+ \, \exists$ two functions $\underline{\alpha}$ and $\overline{\alpha}$ of class $\mathcal{K}$ such that

$$\underline{\alpha}(\|z\|_H) \leq V(z) \leq \underline{\alpha}(\|z\|_H) \ , \forall z \in H$$

$\Rightarrow$ the origin of (13) with $d = 0$ is globally asymptotically stable.

ISS notion

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Input-to-state stability result

Theorem 3 [Marx, Chitour, CP; 2018]

Suppose that Assumption 1 holds and let $P \in \mathcal{L}(H)$ be a self-adjoint and positive operator satisfying (12). Then, there exists M such that

$$V(z) := \langle Pz, z \rangle_H + M \|z\|_H^3 \tag{14}$$

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The proof follows the finite-dimensional case considered in [Liu, Chitour, and Sontag; 1996].

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Let us consider the following candidate Lyapunov function

$$V(z) := \langle Pz, z\rangle_H + \tfrac{2M}{3}\|z\|_H^3 = V_1(z) + \tfrac{2M}{3}\|z\|_H^3, \qquad (15)$$

Along the strong solutions to (13), with $\tilde{A} = A - BB^\star$

$$\begin{aligned}\frac{d}{dt}V_1(z) = \langle Pz, Az\rangle_H + \langle PAz, z\rangle_H \\ + \langle PB(\mathtt{sat}_U(B^\star z) - \mathtt{sat}_U(B^\star z + d)), z\rangle_H \\ + \langle z, PB(\mathtt{sat}_U(B^\star z) - \mathtt{sat}_U(B^\star z + d))\rangle_H\end{aligned}$$

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\leq & -C\|z\|_H^2 + 2\|B^\star z\|_U \|P\|_{\mathcal{L}(H)} \|B^\star z - \mathtt{sat}_U(B^\star z)\|_U \\
& + 2k\|d\|_U \|B^\star\|_{\mathcal{L}(H,U)} \|P\|_{\mathcal{L}(H)} \|z\|_H,
\end{aligned}$$

using $\mathtt{sat}_U$ Lipchitz, Cauchy-Schwarz inequality and the fact that $B^\star$ is bounded.

Moreover using $\|d\|_U \|z\|_H \leq \varepsilon \|d\|_U^2 + \frac{1}{\varepsilon}\|z\|_H^2$ and $\|B^\star z - \mathtt{sat}_U(B^\star z)\|_U \leq \langle \mathtt{sat}_U(B^\star z), B^\star z \rangle_U$, we get

$$\begin{aligned}\frac{d}{dt}V_1(z) \leq & - \left(C - \frac{\|B^\star\|^2_{\mathcal{L}(H,U)} \|P\|^2_{\mathcal{L}(H)}}{\varepsilon_1} \right) \|z\|_H^2 \\ & + 2\|B^\star\|_{\mathcal{L}(H,U)} \|P\|_{\mathcal{L}(H)} \|z\|_H \langle \mathtt{sat}_U(B^\star z), B^\star z \rangle_U \\ & + k^2 \varepsilon_1 \|d\|_U^2\end{aligned}$$

where ε_1 is a positive value that will be selected later.

Thus

$$\frac{d}{dt}V_1(z) \leq \text{good term} + \text{bad term} + d^2$$

Secondly, using the dissipativity of the operator $A_{\mathtt{sat}}$, $\langle \mathtt{sat}_U(B^\star z) - \mathtt{sat}_U(B^\star z + d), B^\star z\rangle_U \leq C_0\|d\|_U$, and $\|z\|_H\|d\|_U \leq \frac{1}{\varepsilon_2}\|z\|_H^2 + \varepsilon_2\|d\|_U^2$, one has

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\begin{aligned}
\frac{2M}{3}\frac{d}{dt}\|z\|_H^3 =& M\|z\|\big(\langle Az, z\rangle_H + \langle z, Az\rangle_H\big) \\
&- 2M\|z\|_H\langle B\mathtt{sat}_U(B^\star z + d), z\rangle_H \\
\leq& - 2M\|z\|_H\big(\langle \mathtt{sat}_U(B^\star z), B^\star z\rangle_U \\
&+ \langle \mathtt{sat}_U(B^\star z) - \mathtt{sat}_U(B^\star z + d), B^\star z\rangle_U\big) \\
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&+ 2MC_0\|z\|_H\|d\|_U \\
\leq& -2M\|z\|_H\langle \mathtt{sat}_U(B^\star z), B^\star z\rangle_U \\
&+ \frac{2MC_0}{\varepsilon_2}\|z\|_H^2 + 2MC_0\varepsilon_2\|d\|_U^2,
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where ε_2 is a positive value that has to be selected. For an appropriate choice of M, ε_1 and ε_2 we deduce the result. □

Secondly, using the dissipativity of the operator $A_{\mathtt{sat}}$, $\langle \mathtt{sat}_U(B^\star z) - \mathtt{sat}_U(B^\star z + d), B^\star z\rangle_U \leq C_0 \|d\|_U$, and $\|z\|_H \|d\|_U \leq \frac{1}{\varepsilon_2}\|z\|_H^2 + \varepsilon_2 \|d\|_U^2$, one has

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\leq & - 2M\|z\|_H\big(\langle \mathtt{sat}_U(B^\star z), B^\star z\rangle_U \\
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Consider again the wave equation
(with S. Tarbouriech and JM Gomes da Silva; 16):

$$z_{tt}(x,t) = z_{xx}(x,t) - \mathtt{sat}(az_t(x,t)),\ \forall x \in (0,1), t \geq 0,$$

Boundary conditions, $\forall t \geq 0$,

$$\begin{aligned} z(0,t) &= 0\ , \\ z(1,t) &= 0\ , \end{aligned}$$

and with the following initial condition, $\forall x \in (0,1)$,

$$\begin{aligned} z(x,0) &= z^0(x)\ , \\ z_t(x,0) &= z^1(x)\ , \end{aligned}$$

with $z^0(x) = \sin(2\pi x)$ and $z^1(x) = 0$, for all $x \in [0,1]$.

With the damping $a = 0.1$ and the level of the saturation $u_0 = 5$:

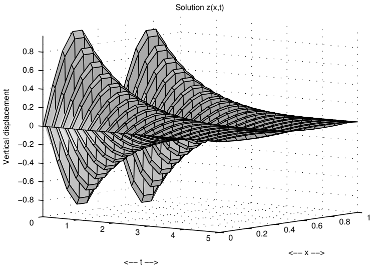

Figure: Time evolution of solution to nonlinear PDE with $u_0 = 5$.

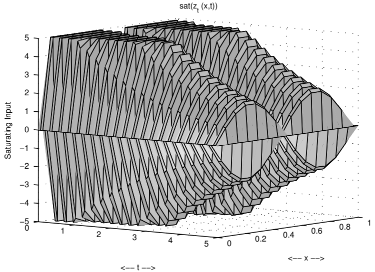

Figure: Time evolution of the saturating input with $u_0 = 5$.

Let us now select a lower saturation level: $u_0 = 1$.
It takes a longer time to converge, but still converging!

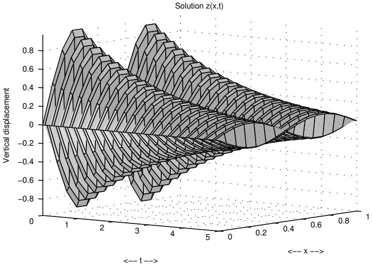

Figure: Time evolution of solution to nonlinear PDE with $u_0 = 1$.

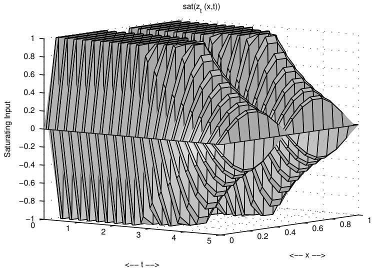

Figure: Time evolution of the saturating input with $u_0 = 1$.

Conclusion

- Well-posedness and global asymptotic stability for the nonlinear PDEs:

$$\begin{cases} z_{tt} = z_{xx} - \mathtt{sat}(az_t) \\ z(0,t) = z(1,t) = 0 \end{cases} \qquad \begin{cases} z_{tt} = z_{xx} \\ z(0,t) = 0,\ z_x(1,t) = -\mathtt{sat}(bz(1,t)) \end{cases}$$

- control abstract problems and ISS results
- distributed/localized (localized/L^2) saturating control
- strict and non-strict Lyapunov functions have been used

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Under actual investigation

Under actual investigation #1

ISS stability results for saturated boundary control?

Under actual investigation #2

Other PDE with saturated input? Beam equation?
See also [Tarbouriech, CP, and Gomes da Silva Jr.; 2005] for anti-windup and (discretized) beam equation.

Under actual investigation #3

Anti-windup design to improve the performance?

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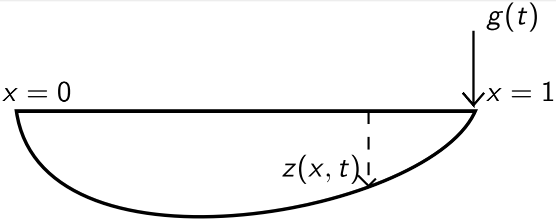
Anti-windup design to improve the performance?

Bonus – Wave equation with a boundary control



1D wave equation with a boundary control.

Dynamics:

$$z_{tt}(x,t) = z_{xx}(x,t),\ \forall x \in (0,1), t \geq 0, \tag{16}$$

Boundary conditions, $\forall t \geq 0$,

$$\begin{aligned} z(0,t) &= 0\ , \\ z_x(1,t) &= g(t)\ , \end{aligned} \tag{17}$$

and with the same initial condition, $\forall x \in (0,1)$,

$$\begin{aligned} z(x,0) &= z^0(x)\ , \\ z_t(x,0) &= z^1(x)\ . \end{aligned} \tag{18}$$

Let us define the linear control by

$$g(t) = -bz_t(1,t),\ x \in (0,1),\ \forall t \geq 0 \qquad (19)$$

and consider

$$V_2 = \frac{1}{2}\int (e^{\mu x}(z_t + z_x)^2 dx + \int (e^{-\mu x}(z_t - z_x)^2 dx,$$

Formal computation. Along the solutions to (16), (17) and (19):

$$\dot{V}_2 \;=\; -\mu V_2 + \tfrac{1}{2}\left(e^{\mu}(1-b)^2 - e^{-\mu}(1+b)^2\right) z_t^2(1,t)$$

Assuming $b > 0$ and letting $\mu > 0$ such that $e^{\mu}(1-b)^2 \leq e^{-\mu}(1+b)^2$, it holds $\dot{V}_2 \leq -\mu V_2$ and thus V_2 is a strict Lyapunov function and thus (16)-(19) is exponentially stable.

When closing the loop with a linear boundary control

Let us define the linear control by

$$g(t) = -bz_t(1,t),\ x \in (0,1),\ \forall t \geq 0 \qquad (19)$$

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When closing the loop with a saturating control

Let us consider now the nonlinear control $g(t) = -\mathtt{sat}(bz_t(1,t)), \forall t \geq 0$. The boundary conditions become:

$$z(0,t) = 0 \ , \quad z_x(1,t) = -\mathtt{sat}(bz_t(1,t)) \ . \qquad (20)$$

Theorem 3

$\forall b > 0$, for all (z^0, z^1) in $\{(u,v),\ (u,v) \in H^2(0,1) \times H^1_{(0)}(0,1),\ u_x(1) + \mathtt{sat}(bv(1)) = 0,\ u(0) = 0\}$, the solution to (16) with the boundary conditions (20) and the initial condition (5) satisfies the following stability property, $\forall t \geq 0$,

$$\|z(.,t)\|_{H^1_{(0)}(0,1)} + \|z_t(.,t)\|_{L^2(0,1)} \leq \|z^0\|_{H^1_{(0)}(0,1)} + \|z^1\|_{L^2(0,1)} \ ,$$

together with the attractivity property

$$\|z(.,t)\|_{H^1_{(0)}(0,1)} + \|z_t(.,t)\|_{L^2(0,1)} \to 0, \ \text{ as } t \to \infty \ .$$

To prove the well-posedness of the Cauchy problem we prove that A_2 defined by

$$A_2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ u'' \end{pmatrix}$$

with the domain $D(A_2) = \{(u, v),\ (u, v) \in H^2(0,1) \times H^1_{(0)}(0,1),\ u'(1) + \mathtt{sat}(bv(1)) = 0,\ u(0) = 0\}$ is a semigroup of contraction.

The global stability property comes directly from the dissipativity of A_2.

The global attractivity property comes from the following lemma:

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The global stability property comes directly from the dissipativity of A_2.

The global attractivity property comes from the following lemma:

Lemma (semi-global exponential stability)

For all $r > 0$, there exists $\mu > 0$ such that, for all initial condition satisfying

$$\|z^{0\prime\prime}\|^2_{L^2(0,1)} + \|z^1\|^2_{H^1_{(0)}(0,1)} \leq r^2 \ , \tag{21}$$

it holds

$$\dot{V}_2 \leq -\mu V_2$$

along the solutions to (16) with the boundary conditions (20).

Sketch of the proof of this lemma

First note that by dissipativity of A_2, it holds that

$$t \mapsto \left\| A_2 \begin{pmatrix} z(.,t) \\ z_t(.,t) \end{pmatrix} \right\|$$

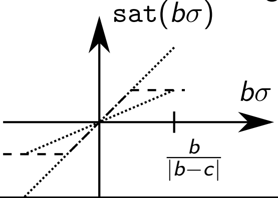
is a non-increasing function. Thus, for all $t \geq 0$,

$$|z_t(1,t)| \leq \left\| A_2 \begin{pmatrix} z(.,0) \\ z_t(.,0) \end{pmatrix} \right\| . \tag{22}$$

Now for all initial conditions satisfying (21), there exists $c \neq b$ such that, for all $t \geq 0$,

$$(b-c)|z_t(1,t)| \leq 1$$

and thus the following local sector condition holds:



Letting $\sigma = z_t(1,t)$, it holds
$(\mathtt{sat}(b\sigma) - b\sigma)(\mathtt{sat}(b\sigma) - (b-c)\sigma) \leq 0$

We come back to the Lyapunov function candidate V_2. Given $b > 0$, using the previous inequality, we compute

$$\begin{aligned}
\dot{V}_2 &= -\mu V_2 + e^{\mu}(\sigma - \mathtt{sat}(b\sigma))^2 - e^{-\mu}(\sigma + \mathtt{sat}(b\sigma))^2 \\
&\leq -\mu V_2 + \begin{pmatrix} \sigma \\ \mathtt{sat}(b\sigma) \end{pmatrix}^{\top} \begin{pmatrix} e^{\mu} - e^{-\mu} - b^2(b-c) & -e^{\mu} - e^{-\mu} + b + b(b-c) \\ -e^{\mu} - e^{-\mu} + b + b(b-c) & -1 + e^{\mu} - e^{-\mu} \end{pmatrix} \\
&\qquad \times \begin{pmatrix} \sigma \\ \mathtt{sat}(b\sigma) \end{pmatrix} \\
&\leq -\mu V_2
\end{aligned}$$

with a suitable choice of constant values μ and c.
The semi-global exponential stability follows. □

Back to the wave equation with in-domain control

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▸ Back to the wave equation with in-domain control