# Two-player games between polynomial optimizers and semidefinite solvers

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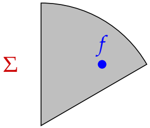
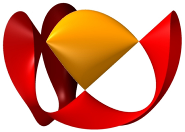
Séminaire MOSAR-SCA
Meeting Business Center Toulouse

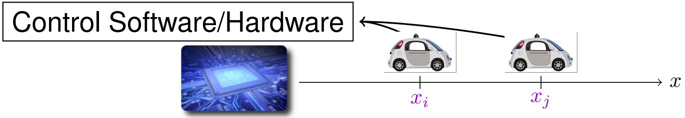



# Introduction

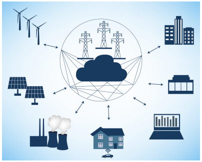
## Verification of nonlinear systems . . .

**SAFETY** of critical parts for **computing** $\bigoplus$ **physical** devices

**Cars**

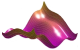



**Smart Grids**



**Space Systems**

. . . cast as Certified optimization $\rightsquigarrow$ Solve **OFFLINE**

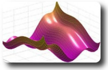
Input: linear  semidefinite  polynomial 

Output: value + numerical/symbolic/formal **certificate**

# SDP for Polynomial Optimization

**NP-hard NON CONVEX Problem** $f^\star = \inf f(\mathbf{x})$

**Theory**

| (Primal) | | (Dual) |
|---|---|---|
| $\inf \int f \, d\mu$ |  | $\sup \lambda$ |
| with $\mu$ proba $\Rightarrow$ | **INFINITE LP** | $\Leftarrow$ with $p - \lambda \geqslant 0$ |

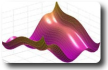
# SDP for Polynomial Optimization

**NP-hard NON CONVEX Problem** $f^\star = \inf f(\mathbf{x})$

**Practice**

(Primal **Relaxation**)

**moments** $\int \mathbf{x}^\alpha \, d\mu$



(Dual **Strengthening**)

$f - \lambda =$ **sum of squares**

**finite** number $\Rightarrow$ **SDP** $\Leftarrow$ **fixed** degree

LASSERRE'S HIERARCHY of **CONVEX PROBLEMS** $f_d^\star \uparrow f^*$ [Lasserre/Parrilo 01]

degree $d$
$n$ vars $\Longrightarrow \binom{n+d}{n}$ **SDP** VARIABLES

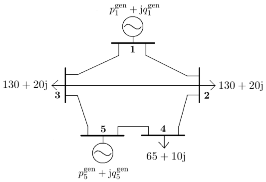
**Numeric** Solvers $\Longrightarrow$ **Approx** Certificate

# Success Stories: Lasserre's Hierarchy

MODELING POWER: **Cast** as $\infty$-*dimensional* LP over measures

STATIC **Polynomial Optimization**

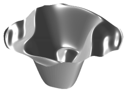
**Optimal Powerflow** $n \simeq \mathbf{10^3}$ [Josz et al 16]



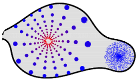

**Roundoff Error** $n \simeq \mathbf{10^2}$ [Magron et al 17]

DYNAMICAL **Polynomial Optimization**

**Regions of attraction** [Henrion et al 14]



**Reachable sets** [Magron et al 17]



**APPROXIMATE** OPTIMIZATION BOUNDS!

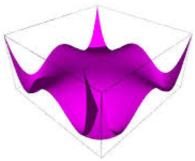
# Two-player Games: Optimizers vs Solvers

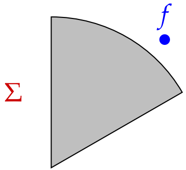
## MOTZKIN POLYNOMIAL

sums of squares $= \Sigma$

$$f = \frac{1}{27} + x^2y^4 + x^4y^2 - x^2y^2$$

$f \geqslant 0$ but $f \notin \Sigma$





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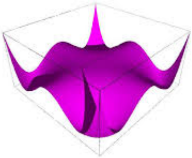

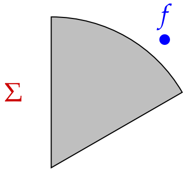


$$f^\star = \min_{(x,y)\in\mathbb{R}^2} f(x,y) = 0 \text{ for } |x^\star| = |y^\star| = \frac{\sqrt{3}}{3}$$

Lasserre's hierarchy:

- order 3 $\rightsquigarrow f_3^\star = -\infty$ unbounded SDP $\implies f \notin \Sigma$

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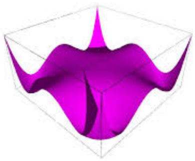

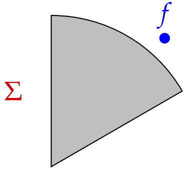


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- order 4 $\rightsquigarrow f_4^\star = -\infty$

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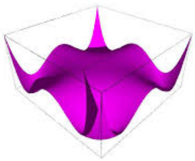
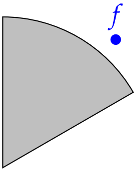


$\Sigma$

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Lasserre's hierarchy:

- order 3 $\rightsquigarrow f_3^\star = -\infty$ unbounded SDP $\Longrightarrow f \notin \Sigma$
- order 4 $\rightsquigarrow f_4^\star = -\infty$
- order 5 $\rightsquigarrow f_5^\star \simeq -0.4$

# Two-player Games: Optimizers vs Solvers

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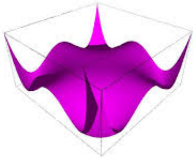

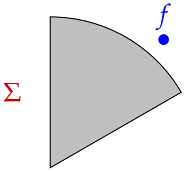


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Lasserre's hierarchy:

- order 3 $\rightsquigarrow f_3^\star = -\infty$ unbounded SDP $\implies f \notin \Sigma$
- order 4 $\rightsquigarrow f_4^\star = -\infty$
- order 5 $\rightsquigarrow f_5^\star \simeq -0.4$
- order 8 $\rightsquigarrow f_8^\star \simeq -10^{-8} \oplus$ extraction of $x^\star, y^\star$ **Paradox** ?!

# Two-player Games: Optimizers vs Solvers

## APPROXIMATE SOLUTIONS

sum of squares of $a^2 - 2ab + b^2$?

$(1.00001a - 0.99998b)^2$!



$$a^2 - 2ab + b^2 \simeq (1.00001a - 0.99998b)^2$$

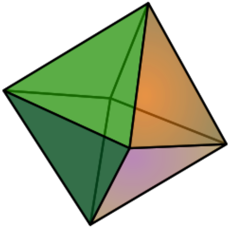
$$a^2 - 2ab + b^2 \neq 1.0000200001a^2 - 1.9999799996ab + 0.9999600004b^2$$

$\boxed{\simeq \quad \rightarrow \quad =}$ ?

# What is Semidefinite Optimization?

- Linear Programming (LP):

$$\min_{\mathbf{z}} \quad \mathbf{c}^\top \mathbf{z}$$
$$\text{s.t.} \quad \mathbf{A}\,\mathbf{z} \geqslant \mathbf{d}\ .$$



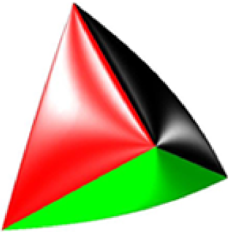
**Polyhedron**

  - Linear cost $\mathbf{c}$
  - Linear inequalities "$\sum_i A_{ij}\, z_j \geqslant d_i$"

# What is Semidefinite Optimization?

- Semidefinite Programming (SDP):

$$\min_{\mathbf{z}} \quad \mathbf{c}^\top \mathbf{z}$$
$$\text{s.t.} \quad \sum_i \mathbf{F}_i z_i \succcurlyeq \mathbf{F}_0 \ .$$



**Spectrahedron**

  - Linear cost $\mathbf{c}$
  - Symmetric matrices $\mathbf{F}_0, \mathbf{F}_i$
  - Linear matrix inequalities "$\mathbf{F} \succcurlyeq 0$" ($\mathbf{F}$ has nonnegative eigenvalues)

# What is Semidefinite Optimization?

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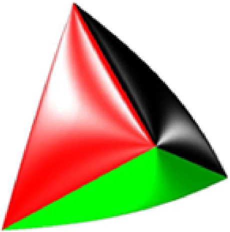

**Spectrahedron**

# Applications of SDP

- Combinatorial optimization
- Control theory
- Matrix completion
- Unique Games Conjecture (Khot '02) :
  "A *single concrete algorithm* provides **optimal guarantees** among all efficient algorithms for a large class of computational problems."
  (Barak and Steurer survey at ICM'14)
- Solving polynomial optimization (Lasserre '01)

# Lasserre's Hierarchy

- Prove **polynomial inequalities** with SDP:

$$f(a,b) := a^2 - 2ab + b^2 \geqslant 0 \ .$$

- Find $\mathbf{z}$ s.t. $f(a,b) = \begin{pmatrix} a & b \end{pmatrix} \underbrace{\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}}_{\succcurlyeq 0} \begin{pmatrix} a \\ b \end{pmatrix}$ .

- Find $\mathbf{z}$ s.t. $a^2 - 2ab + b^2 = z_1 a^2 + 2 z_2 ab + z_3 b^2 \qquad (\mathbf{A}\,\mathbf{z} = \mathbf{d})$

- $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F}_1} z_1 + \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\mathbf{F}_2} z_2 + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{F}_3} z_3 \succcurlyeq \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F}_0}$

# Lasserre's Hierarchy

- Choose a cost $\mathbf{c}$ e.g. $(1, 0, 1)$ and solve:

$$\min_{\mathbf{z}} \quad \mathbf{c}^\top \mathbf{z}$$
$$\text{s.t.} \quad \sum_i \mathbf{F}_i z_i \succcurlyeq \mathbf{F}_0 \,, \quad \mathbf{A}\,\mathbf{z} = \mathbf{d} \,.$$

- Solution $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \succcurlyeq 0$ (eigenvalues 0 and 2)

- $a^2 - 2ab + b^2 = \begin{pmatrix} a & b \end{pmatrix} \underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}_{\succcurlyeq 0} \begin{pmatrix} a \\ b \end{pmatrix} = (a - b)^2 \,.$

- Solving **SDP** $\implies$ Finding **SUMS OF SQUARES** certificates

# Lasserre's Hierarchy

**NP hard General Problem**: $f^* := \min_{\mathbf{x} \in \mathbf{K}} f(\mathbf{x})$

- Semialgebraic set
  $\mathbf{K} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geqslant 0, \ldots, g_m(\mathbf{x}) \geqslant 0\}$

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- $:= [0,1]^2 = \{\mathbf{x} \in \mathbb{R}^2 : x_1(1 - x_1) \geqslant 0, \quad x_2(1 - x_2) \geqslant 0\}$

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$$\overbrace{x_1 x_2}^{f} + \frac{1}{8} = \overbrace{\frac{1}{2}\left(x_1 + x_2 - \frac{1}{2}\right)^2}^{\sigma_0} + \overbrace{\frac{1}{2}}^{\sigma_1} \overbrace{x_1(1 - x_1)}^{g_1} + \overbrace{\frac{1}{2}}^{\sigma_2} \overbrace{x_2(1 - x_2)}^{g_2}$$

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- Sums of squares (SOS) $\sigma_i$

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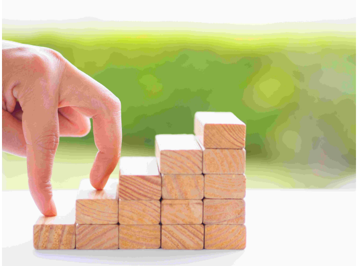
- Sums of squares (SOS) $\sigma_i$

- Bounded degree:
  $\mathcal{Q}_d(\mathbf{K}) := \left\{ \sigma_0 + \sum_{j=1}^m \sigma_j g_j, \text{ with } \deg \sigma_j \, g_j \leqslant 2d \right\}$

# Lasserre's Hierarchy

- **Hierarchy of SDP relaxations**:
  $\lambda_d := \sup_{\lambda} \left\{ \lambda : f - \lambda \in \mathcal{Q}_d(\mathbf{K}) \right\}$



- Convergence guarantees $\lambda_d \uparrow f^*$ [Lasserre 01]

- Can be computed with SDP solvers (CSDP, SDPA)

- **"No Free Lunch" Rule**: $\binom{n+2d}{n}$ SDP variables

SDP for Polynomial Optimization

Inaccurate SDP do Robust Optimization

`RealCertify`: Certify Non-negativity

# Inaccurate SDP do Robust Optimization

$$f = \sum_{\alpha} f_\alpha \, \mathbf{x}^\alpha$$

$$f^\star = \inf f(\mathbf{x})$$

**Moment matrix** $\quad \mathbf{M}_d(\mathbf{y})_{\alpha,\beta} = y_{\alpha+\beta}$

## Accurate SDP Relaxations

(Primal **Relaxation**)

$$\inf_{\mathbf{y}} \sum_{\alpha} f_\alpha \, y_\alpha$$
$$\text{s.t. } \mathbf{M}_d(\mathbf{y}) \succcurlyeq 0$$
$$y_0 = 1$$

(Dual **Strengthening**)

$$\sup_{\lambda,\sigma} \lambda$$
$$f - \lambda = \sigma$$
$$\sigma \in \Sigma_d$$

# Inaccurate SDP do Robust Optimization

$$f = \sum_\alpha f_\alpha \, \mathbf{x}^\alpha$$

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**Moment matrix** $\mathbf{M}_d(\mathbf{y})_{\alpha,\beta} = y_{\alpha+\beta}$

$$\mathbf{M}_d(\mathbf{y}) = \sum_\alpha \mathbf{B}_\alpha \, y_\alpha$$

## Accurate SDP Relaxations

| (Primal **Relaxation**) | (Dual **Strengthening**) |
| --- | --- |
| $\inf_{\mathbf{y}} \sum_\alpha f_\alpha \, y_\alpha$ | $\sup_{\lambda, \mathbf{Q}} \lambda$ |
| s.t. $\mathbf{M}_d(\mathbf{y}) \succcurlyeq 0$ | $f_\alpha - \lambda 1_{\alpha=0} = \langle \mathbf{B}_\alpha, \mathbf{Q} \rangle$ |
| $y_0 = 1$ | $\mathbf{Q} \succcurlyeq 0$ |

# Inaccurate SDP do Robust Optimization

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$$f^\star = \inf \; f(\mathbf{x})$$

**Moment matrix** $\quad \mathbf{M}_d(\mathbf{y})_{\alpha,\beta} = y_{\alpha+\beta}$

$$\mathbf{M}_d(\mathbf{y}) = \sum_\alpha \mathbf{B}_\alpha \, y_\alpha$$

## Inaccurate SDP Relaxations

(Primal **Relaxation**) (Dual **Strengthening**)

$$\sup_{\lambda, \mathbf{Q}} \; \lambda$$

$$| \, f_\alpha - \lambda 1_{\alpha=0} - \langle \mathbf{B}_\alpha, \mathbf{Q} \rangle \, | \leqslant \varepsilon$$

$$\mathbf{Q} \succcurlyeq -\eta \mathbf{I}$$

# Inaccurate SDP do Robust Optimization

$$f = \sum_\alpha f_\alpha \, \mathbf{x}^\alpha$$

$$f^\star = \inf f(\mathbf{x})$$

**Moment matrix** $\quad \mathbf{M}_d(\mathbf{y})_{\alpha,\beta} = y_{\alpha+\beta}$

$$\mathbf{M}_d(\mathbf{y}) = \sum_\alpha \mathbf{B}_\alpha \, y_\alpha$$

## Inaccurate SDP Relaxations

(Primal **Relaxation**)

$$\inf_{\mathbf{y}} \sum_\alpha f_\alpha \, y_\alpha + \eta \langle \mathbf{M}_d(\mathbf{y}), \mathbf{I} \rangle + \varepsilon \|\mathbf{y}\|_1$$

$$\text{s.t. } \mathbf{M}_d(\mathbf{y}) \succcurlyeq 0$$

$$y_0 = 1$$

(Dual **Strengthening**)

$$\sup_{\lambda, \mathbf{Q}} \ \lambda$$

$$| \, f_\alpha - \lambda 1_{\alpha=0} - \langle \mathbf{B}_\alpha, \mathbf{Q} \rangle \, | \leqslant \varepsilon$$

$$\mathbf{Q} \succcurlyeq -\eta \mathbf{I}$$

# Priority to Trace Equalities: $\varepsilon = 0$

$$\tilde{f} = f + \eta \sum_{\beta} \mathbf{x}^{2\beta}$$

**Inaccurate SDP Relaxations**

| (Primal **Relaxation**) | (Dual **Strengthening**) |
|---|---|
| $\inf_{\mathbf{y}} \sum_{\alpha} f_\alpha y_\alpha + \eta \langle \mathbf{M}_d(\mathbf{y}), \mathbf{I} \rangle$ | $\sup_{\lambda, \mathbf{Q}} \lambda$ |
| s.t. $\mathbf{M}_d(\mathbf{y}) \succcurlyeq 0$ | $f_\alpha - \lambda 1_{\alpha=0} - \langle \mathbf{B}_\alpha, \mathbf{Q} \rangle = 0$ |
| $y_0 = 1$ | $\mathbf{Q} \succcurlyeq -\eta \mathbf{I}$ |

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$$\tilde{f} = f + \eta \sum_{\beta} \mathbf{x}^{2\beta}$$

**Inaccurate SDP Relaxations**

| (Primal **Relaxation**) | (Dual **Strengthening**) |
|---|---|
| $\inf_{\mathbf{y}} \sum_{\alpha} f_\alpha y_\alpha + \eta \langle \mathbf{M}_d(\mathbf{y}), \mathbf{I} \rangle$ | $\sup_{\lambda, \mathbf{Q}} \lambda$ |
| s.t. $\mathbf{M}_d(\mathbf{y}) \succcurlyeq 0$ | $f_\alpha - \lambda 1_{\alpha=0} - \langle \mathbf{B}_\alpha, \mathbf{Q} - \eta \mathbf{I} \rangle = 0$ |
| $y_0 = 1$ | $\mathbf{Q} \succcurlyeq 0$ |

# Priority to Trace Equalities: $\varepsilon = 0$

$$\tilde{f} = f + \eta \sum_{\beta} \mathbf{x}^{2\beta}$$

**Inaccurate SDP Relaxations**

(Primal **Relaxation**)

$$\inf_{\mathbf{y}} \sum_{\alpha} \tilde{f}_{\alpha} y_{\alpha}$$
$$\text{s.t. } \mathbf{M}_d(\mathbf{y}) \succcurlyeq 0$$
$$y_0 = 1$$

(Dual **Strengthening**)

$$\sup_{\lambda,\sigma} \lambda$$
$$\tilde{f} - \lambda = \sigma$$
$$\sigma \in \Sigma_d$$

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**Theorem (Lasserre 06)**

For fixed $n$, any $f \geqslant 0$ can be approximated arbitrarily closely by SOS polynomials.

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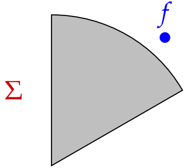
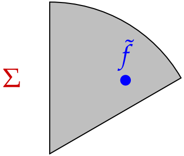



$$\tilde{f} = f + \eta \sum_{|\beta| \leqslant d} \mathbf{x}^{2\beta}$$



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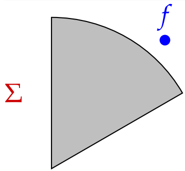


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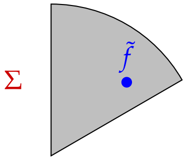
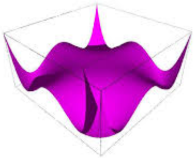



At fixed $\eta$, when $d \nearrow$, $\tilde{f} \in \Sigma$!



$$f + 10^{-7} \sum_{|\beta| \leqslant 4} \mathbf{x}^{2\beta} \in \Sigma$$

**Paradox Explanation**

# Priority to Trace Equalities: $\varepsilon = 0$

In the constrained case $\mathbf{K} := \{\mathbf{x} : g_j(\mathbf{x}) \geqslant 0\}$

$$\mathbf{B}_\infty(f, \mathbf{K}, \eta) := \{f + \theta \sum_j g_j(\mathbf{x}) \sum_\beta \mathbf{x}^{2\beta} :| \theta | \leqslant \eta\}$$

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**Theorem (Lasserre-Magron)**

Inaccurate SDP relaxations of the **robust** problem

$$\max_{\tilde{f} \in \mathbf{B}_\infty(f, \mathbf{K}, \eta)} \min_{\mathbf{x} \in \mathbf{K}} \tilde{f}(\mathbf{x})$$

# Priority to SDP Inequalities: $\eta = 0$

## Inaccurate SDP Relaxations

(Primal **Relaxation**)

$$\inf_{\mathbf{y}} \sum_{\alpha} f_\alpha y_\alpha + \varepsilon \|\mathbf{y}\|_1$$
$$\text{s.t. } \mathbf{M}_d(\mathbf{y}) \succcurlyeq 0$$
$$y_0 = 1$$

(Dual **Strengthening**)

$$\sup_{\lambda, \mathbf{Q}} \lambda$$
$$| f_\alpha - \lambda 1_{\alpha=0} - \langle \mathbf{B}_\alpha, \mathbf{Q} \rangle | \leqslant \varepsilon$$
$$\mathbf{Q} \succcurlyeq 0$$

# Priority to SDP Inequalities: $\eta = 0$

$$\mathbf{B}_\infty(f, \varepsilon) := \{\tilde{f} : \|\tilde{f} - f\|_\infty \leqslant \varepsilon\}$$

## Inaccurate SDP Relaxations

(Primal **Relaxation**)

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# A Two-player Game Interpretation


$\max - \min$ ROBUST OPTIMIZATION

Player 1 (solver) picks $\tilde{f} \in \mathbf{B}_\infty(f)$ ⇝ **SDP leads**

Player 2 (optimizer) picks an SOS ⇝ **User follows**

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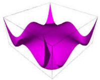
$\min - \max$ ROBUST OPTIMIZATION

Player 1 (robust optimizer) picks an SOS $\rightsquigarrow$ **User leads**

Player 2 (solver) picks $\tilde{f} \in \mathbf{B}_\infty(f) \rightsquigarrow$ **SDP follows**

# The Certification Game

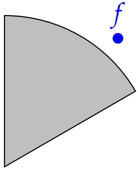


## NUMERICAL GAME

Interior-point solvers **OUTPUT** inaccurate certificates

$\implies$ extract solutions for $\tilde{f}$

$\Sigma$

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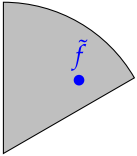

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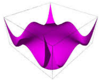

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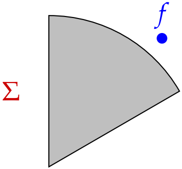


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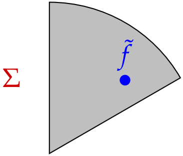


## Symbolic Game

Polynomial optimizer **INPUT** inaccurate $\tilde{f} = f - \eta \sum_{|\beta| \leqslant d} \mathbf{x}^{2\beta}$

$\Longrightarrow$ exact certificates for $f$

SDP for Polynomial Optimization

Inaccurate SDP do Robust Optimization

`RealCertify`: Certify Non-negativity

# RealCertify: Certify Non-negativity

$X = (X_1, \ldots, X_n)$

**co-NP hard problem: check $f \geqslant 0$ on $\mathbf{K}$**

$f \in \mathbb{Q}[X]$

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1. Unconstrained $\rightsquigarrow \mathbf{K} = \mathbb{R}^n$

$\boxed{n = 1}$ $f = 1 + X + X^2 + X^3 + X^4$

$\boxed{n > 1}$ $f = 4X_1^4 + 4X_1^3 X_2 - 7X_1^2 X_2^2 - 2X_1 X_2^3 + 10X_2^4$

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   $g_j \in \mathbb{Q}[X]$

   $f = -X_1^2 - 2X_1X_2 - 2X_2^2 + 6 \quad \mathbf{K} = \{1 - X_1^2 \geqslant 0, 1 - X_2^2 \geqslant 0\}$

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1. $f \in \Sigma$ = sums of squares (SOS)

   $f = \sigma = {h_1}^2 + \cdots + {h_p}^2 \geqslant 0$

2. Weighted SOS $\quad f = \sigma_0 + \sigma_1 g_1 + \cdots + \sigma_m g_m \geqslant 0$ on $\mathbf{K}$

# From Approximate to Exact Solutions

## Approximate solutions

sum of squares of $a^2 - 2ab + b^2$?

$(1.00001a - 0.99998b)^2$!



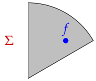
$$a^2 - 2ab + b^2 \simeq (1.00001a - 0.99998b)^2$$

$$a^2 - 2ab + b^2 \neq 1.0000200001a^2 - 1.9999799996ab + 0.9999600004b^2$$

$\boxed{\simeq \quad \rightarrow \quad =}$ ?

# From Approximate to Exact Solutions

Win TWO-PLAYER GAME

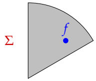



sum of squares of $f$?


$\simeq$ Output!

# From Approximate to Exact Solutions

Win TWO-PLAYER GAME




**Hybrid** Symbolic/Numeric Algorithms

sum of squares of $f + \varepsilon$?

≃ Output!


Error Compensation

$\simeq \quad \rightarrow \quad =$

# Rational SOS Decompositions

- Let $f \in \mathbb{R}[X]$ and $f \geqslant 0$ on $\mathbb{R}$ ($n = 1$)

**Theorem**

There exist $f_1, f_2 \in \mathbb{R}[X]$ s.t. $f = {f_1}^2 + {f_2}^2$.

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**Proof.**

$f = h^2(q + ir)(q - ir)$ □

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**Examples**

$$1 + X + X^2 = \left(X + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

$$1 + X + X^2 + X^3 + X^4 = \left(X^2 + \frac{1}{2}X + \frac{1+\sqrt{5}}{4}\right)^2 + \left(\frac{\sqrt{10+2\sqrt{5}} + \sqrt{10-2\sqrt{5}}}{4}X + \frac{\sqrt{10-2\sqrt{5}}}{4}\right)^2$$

# Rational SOS Decompositions

- $f \in \mathbb{Q}[X] \cap \mathring{\Sigma}[X]$ (interior of the SOS cone)

**Existence Question**

Does there exist $f_i \in \mathbb{Q}[X], c_i \in \mathbb{Q}^{>0}$ s.t. $f = \sum_i c_i f_i^2$?

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$$1 + X + X^2 = \left(X + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1\left(X + \frac{1}{2}\right)^2 + \frac{3}{4}(1)^2$$

$$1 + X + X^2 + X^3 + X^4 = \left(X^2 + \frac{1}{2}X + \frac{1+\sqrt{5}}{4}\right)^2 + \left(\frac{\sqrt{10+2\sqrt{5}} + \sqrt{10-2\sqrt{5}}}{4}X + \frac{\sqrt{10-2\sqrt{5}}}{4}\right)^2 = ???$$

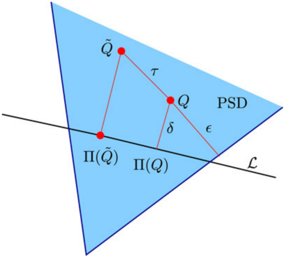
# Existing Frameworks

- project & round [Peyrl-Parrilo 08] [Kaltofen-Yang-Zhi 08]
  $f \in \mathring{\Sigma}[X]$ with $\deg f = 2D$

$f(X) \simeq \mathbf{v}_D{}^T(X)\, \tilde{\mathbf{Q}}\, \mathbf{v}_D(X) \quad \tilde{\mathbf{Q}} \succ 0$

$\mathbf{v}_D(X)$: vector of monomials of $\deg \leqslant D$

Find $\tilde{\mathbf{Q}}$ with semidefinite programming




$f(X) = \mathbf{v}_D{}^T(X) \prod(\mathbf{Q})\, \mathbf{v}_D(X)$

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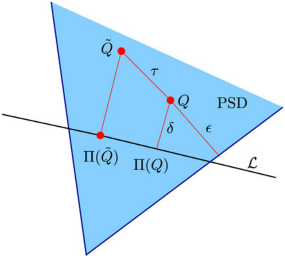


$f(X) = \mathbf{v}_D{}^T(X) \prod(\mathbf{Q})\, \mathbf{v}_D(X)$

- RAGLib (critical points) [Safey El Din]

- SamplePoints (CAD) [Moreno Maza-Alvandi et al.]

# One Answer when $\mathbf{K} = \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geqslant 0\}$

Hybrid **SYMBOLIC/NUMERIC** methods

Magron-Allamigeon-Gaubert-Werner 14

$$f \simeq \tilde{\sigma}_0 + \tilde{\sigma}_1 g_1 + \cdots + \tilde{\sigma}_m g_m$$

$$u = f - \tilde{\sigma}_0 + \tilde{\sigma}_1 g_1 + \cdots + \tilde{\sigma}_m g_m$$

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$f \simeq \tilde{\sigma}_0 + \tilde{\sigma}_1 \, g_1 + \cdots + \tilde{\sigma}_m \, g_m$

Compact $\mathbf{K} \subseteq [0,1]^n$

$u = f - \tilde{\sigma}_0 + \tilde{\sigma}_1 \, g_1 + \cdots + \tilde{\sigma}_m \, g_m$

$\boxed{\simeq \quad \to \quad =}$

$\forall \mathbf{x} \in [0,1]^n, u(\mathbf{x}) \leqslant -\varepsilon$

$\min_{\mathbf{K}} f \geqslant \varepsilon$ when $\varepsilon \to 0$

**COMPLEXITY?**

# Modules & Install

`gricad-gitlab:RealCertify`

Depends on Maple &

`univsos` $\boxed{n = 1}$

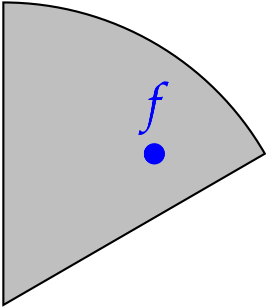
- Square free decomposition with `sqrfree`
- PARI/GP for complex zero isolation

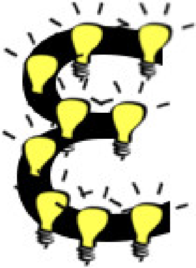
`multivsos` $\boxed{n > 1}$

- arbitrary precision SDP solver SDPA-GMP [Nakata 10]
- Newton Polytope with `convex` package [Franz 99]
- Cholesky's decomposition with `LUDecomposition`

# intsos with $n \geqslant 1$: Perturbation

$\Sigma$



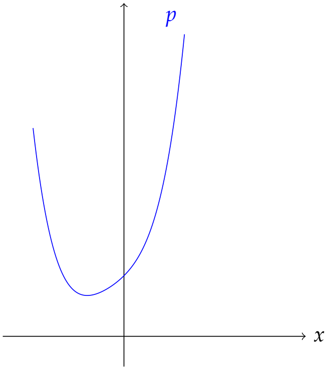
**PERTURBATION** idea

Approximate SOS Decomposition

$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

# intsos with $n = 1$ [Chevillard et. al 11]

$f \in \mathbb{Q}[X]$, $\deg f = d = 2k$, $f > 0$

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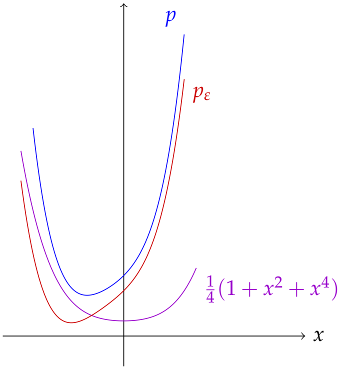
💡 **PERTURB**: find $\varepsilon \in \mathbb{Q}$ s.t.

$$f_\varepsilon := f - \varepsilon \sum_{i=0}^{k} X^{2i} > 0$$




$$f = 1 + X + X^2 + X^3 + X^4$$

$$\varepsilon = \frac{1}{4}$$

$$f > \frac{1}{4}(1 + X^2 + X^4)$$

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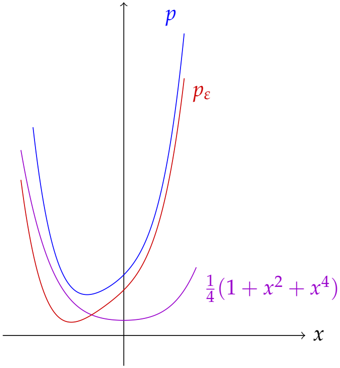
$$f_\varepsilon := f - \varepsilon \sum_{i=0}^{k} X^{2i} > 0$$

SDP Approximation:

$$f - \varepsilon \sum_{i=0}^{k} X^{2i} = \tilde{\sigma} + u$$

**ABSORB**: small enough $u_i$

$\implies$ $\varepsilon \sum_{i=0}^{k} X^{2i} + u$ SOS




$$f = 1 + X + X^2 + X^3 + X^4$$

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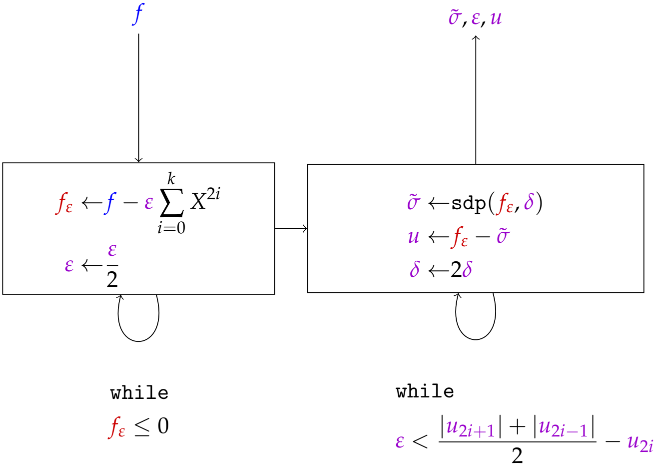
$$f > \frac{1}{4}(1 + X^2 + X^4)$$

# intsos with $n = 1$ and SDP Approximation

- **Input**: $f \geqslant 0 \in \mathbb{Q}[X]$ of degree $d \geqslant 2$, $\varepsilon \in \mathbb{Q}^{>0}$, $\delta \in \mathbb{N}^{>0}$
- **Output**: SOS decomposition with coefficients in $\mathbb{Q}$



# `intsos` with $n = 1$: Absorbtion

- $X = \frac{1}{2}\left[(X+1)^2 - 1 - X^2\right]$
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$$u_{2i+1} X^{2i+1} = \frac{|u_{2i+1}|}{2}\left[(X^{i+1} + \operatorname{sgn}(u_{2i+1}) X^i)^2 - X^{2i} - X^{2i+2}\right]$$

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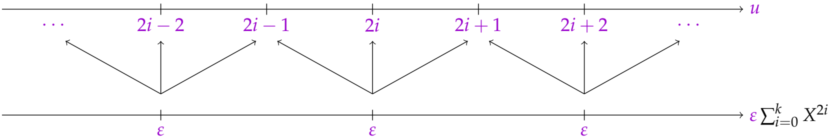
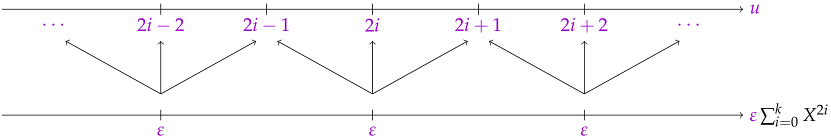


# intsos with $n = 1$: Absorbtion

$$X = \frac{1}{2}\left[(X+1)^2 - 1 - X^2\right]$$

$$-X = \frac{1}{2}\left[(X-1)^2 - 1 - X^2\right]$$

$$u_{2i+1} X^{2i+1} = \frac{|u_{2i+1}|}{2}\left[\left(X^{i+1} + \operatorname{sgn}(u_{2i+1}) X^i\right)^2 - X^{2i} - X^{2i+2}\right]$$

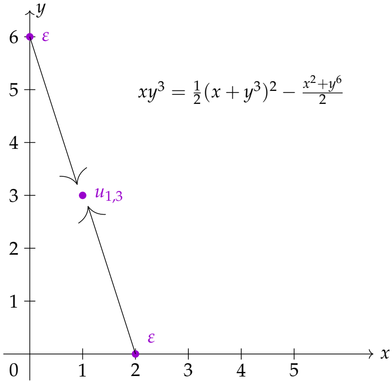



$$\varepsilon \geqslant \frac{|u_{2i+1}| + |u_{2i-1}|}{2} - u_{2i} \implies \varepsilon \sum_{i=0}^{k} X^{2i} + u \quad \text{SOS}$$

# intsos with $n \geqslant 1$: Absorbtion

$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

Choice of $\mathcal{P}$?



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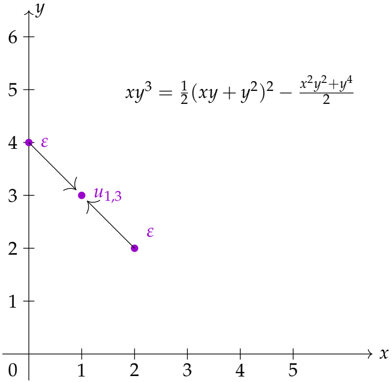

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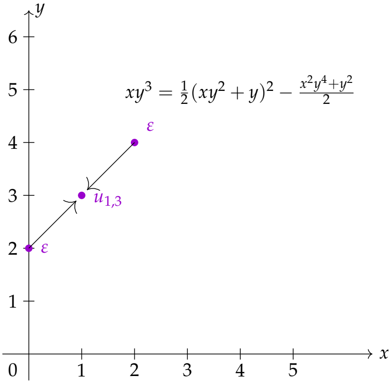


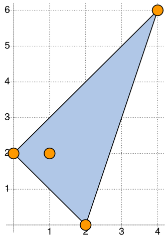
# intsos with $n \geqslant 1$: Absorbtion

$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

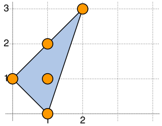
Choice of $\mathcal{P}$?

$f = 4x^4y^6 + x^2 - xy^2 + y^2$
spt($f$) = $\{(4,6),(2,0),(1,2),(0,2)\}$

Newton Polytope $\mathcal{P} = \text{conv}\,(\text{spt}(f))$

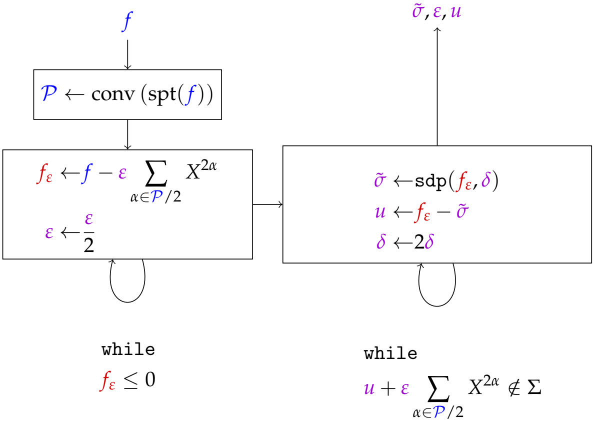



Squares in SOS decomposition $\subseteq \frac{\mathcal{P}}{2} \cap \mathbb{N}^n$
[Reznick 78]



# Algorithm `intsos`

- **Input**: $f \in \mathbb{Q}[\mathbf{X}] \cap \mathring{\Sigma}[X]$ of degree $d$, $\varepsilon \in \mathbb{Q}^{>0}$, $\delta \in \mathbb{N}^{>0}$
- **Output**: SOS decomposition with coefficients in $\mathbb{Q}$



# Algorithm `Putinarsos`

Assumption: $\exists i$ s.t. $g_i = 1 - \|X\|_2^2$

$f > 0$ on $\mathbf{K} := \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geqslant 0\}$ has **Putinar**'s representation:

$$f = \sigma_0 + \sum_j \sigma_j g_j \quad \text{with } \sigma_j \in \Sigma[X], \deg \sigma_j \leqslant 2D$$

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**ABSORBTION** as in Algorithm `intsos`: $u = f_\varepsilon - \tilde{\sigma}_0 - \sum_j \tilde{\sigma}_j g_j$

# Unconstrained Benchmarks

| Id | $n$ | $d$ | multivsos | | RoundProject | | RAGLib | CAD |
|---|---|---|---|---|---|---|---|---|
| | | | $\tau_1$ (bits) | $t_1$ (s) | $\tau_2$ (bits) | $t_2$ (s) | $t_3$ (s) | $t_4$ (s) |
| $f_{20}$ | 2 | 20 | 745 419 | 110. | 78 949 497 | 141. | 0.16 | 0.03 |
| $M$ | 3 | 8 | 17 232 | 0.35 | 18 831 | 0.29 | 0.15 | 0.03 |
| $f_2$ | 2 | 4 | 1 866 | 0.03 | 1 031 | 0.04 | 0.09 | 0.01 |
| $f_6$ | 6 | 4 | 56 890 | 0.34 | 475 359 | 0.54 | 598. | – |
| $f_{10}$ | 10 | 4 | 344 347 | 2.45 | 8 374 082 | 4.59 | – | – |

# Constrained Benchmarks

| Id | $n$ | $d$ | multivsos $D$ | multivsos $\tau_1$ (bits) | multivsos $t_1$ (s) | RAGLib $t_2$ (s) | CAD $t_3$ (s) |
|---|---|---|---|---|---|---|---|
| $f_{260}$ | 6 | 3 | 2 | 114 642 | 2.72 | 4.19 | – |
| $f_{491}$ | 6 | 3 | 2 | 108 359 | 9.65 | 0.01 | 0.05 |
| $f_{752}$ | 6 | 2 | 2 | 10 204 | 0.26 | 0.07 | – |
| $f_{859}$ | 6 | 7 | 4 | 6 355 724 | 303. | 0.05 | – |
| $f_{863}$ | 4 | 2 | 1 | 5 492 | 0.14 | 0.01 | 0.01 |
| $f_{884}$ | 4 | 4 | 3 | 300 784 | 25.1 | 113. | – |
| butcher | 6 | 3 | 2 | 247 623 | 1.32 | 231. | – |
| heart | 8 | 4 | 2 | 618 847 | 2.94 | 24.7 | – |

# Conclusion and Perspectives

Input $f$ on $\mathbf{K}$ with $\deg f = d$ and bit size $\tau$

| Algo | Input | $\mathbf{K}$ | Output Bit Size |
|---|---|---|---|
| `intsos` | $\mathring{\Sigma}$ | $\mathbb{R}^n$ | $\tau\, d^{\mathcal{O}(n)}$ |
| `Putinarsos` | $> 0$ | $\{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geqslant 0\}$ compact | $\mathcal{O}(2^{\tau\, d^{n^{C_{\mathbf{K}}}}})$ |

💡 How to handle degenerate situations?

# Conclusion and Perspectives

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- How to handle degenerate situations?
- Better arbitrary-precision SDP solvers
- Extension to other relaxations, sums of hermitian squares

**Crucial need for polynomial systems certification**
**Available PhD/Postdoc Positions**

# End

Thank you for your attention!

gricad-gitlab:RealCertify

https://homepages.laas.fr/vmagron


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