S-variables for the positivity check

of matrix polynomials with matrix indeterminates

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Introduction

- Many mathematical tools in robust control for building LMI conditions
  - Lyapunov, S-procedure, KYP, DG-scaling, IQC, Quadratic Separation,
    Finsler lemma, S-variables, Positivstellensatz, SOS, Polyá...
    - Developed for specific uncertainties and view points
    - Conservative SDP relaxations, with various decision variables (certificates)
    - Hierarchies of relaxations with decreasing conservatism
  - This presentation : Attempt to establish links between these tools
    - Positivity of matrix polynomials with matrix indeterminates
    - Continuation of work of S-variable approach [Ebihara]
    - Strongly inspired by Quadratic Separation & Generalized KYP lemma [Iwasaki]
    - Connexions to be done with Generalized Frequency Variables [Hara]
    - Technicalities linking SOS and S-variables [Sato]

- The linear system $\dot{x} = Ax$ is stable

$\Leftrightarrow$ All eigenvalues of A have negative real part

$\Leftrightarrow$ A does not have eigenvalues in the closed right-half plane

$\Leftrightarrow$ $sI - A$ is non singular for all $s \in \overline{\mathbb{C}}_+$

$\Leftrightarrow$ $I - As^{-1}$ is non singular for all $s^{-1} \in \overline{\mathbb{C}}_+$

$\Leftrightarrow$ $\exists \epsilon \; : \; (I - As^{-1})^*(I - As^{-1}) \succeq \epsilon I \succ 0$ for all $s^{-1} + s^{-*} \geq 0$

- Matrix valued polynomial inequality (PMI) constrained by a polynomial inequality (PI)
- Indeterminate is complex-valued $s^{-1} \in \mathbb{C}$

$\Leftrightarrow$ $\exists P \succeq 0, \exists \epsilon > 0$ such that $A^*P + PA \preceq -\epsilon I$

- **Lossless** LMI formulation
- P is such that $P(s^{-1} + s^{-*}) \succeq 0$

- Polytopic constraints on matrices :

$$\mathcal{CO}\{A^{[1]}\dots A^{[\bar v]}\} = \mathcal{CO}\{\dots A^{[v]}\dots\} = \left\{ A = \sum_{v=1}^{\bar v} \xi_v A^{[v]} \;:\; \xi_v \geq 0,\ \sum_{v=1}^{\bar v} \xi_v = 1 \right\}$$

- The uncertain linear system $\dot x = Ax$ with $A \in \mathcal{CO}\{\dots A^{[v]}\dots\}$ is robustly stable

$\Leftrightarrow I - As^{-1}$ is non singular $\forall s^{-1} \in \overline{\mathbb{C}}_+, \forall A \in \mathcal{CO}\{\dots A^{[v]}\dots\}$

$\Leftrightarrow \begin{bmatrix} I & -s^{-1}I \\ -A & I \end{bmatrix}$ is non singular $\forall s^{-1} \in \overline{\mathbb{C}}_+, \forall A \in \mathcal{CO}\{\dots A^{[v]}\dots\}$

$$\Leftrightarrow \exists \epsilon \;:\; \begin{bmatrix} I & -s^{-1}I \\ -A & I \end{bmatrix}^* \begin{bmatrix} I & -s^{-1}I \\ -A & I \end{bmatrix} \succeq \epsilon I \succ 0 \quad \begin{array}{l} \forall s^{-1} + s^{-*} \geq 0 \\ \forall A \in \mathcal{CO}\{\dots A^{[v]}\dots\} \end{array}$$

- PMI with scalar/matrix indeterminates constrained by PI & polytopes
- Indeterminates are in independent rows and columns

Motivation - Lyapunov & S-variables

- The uncertain linear system $\dot{x} = Ax$ with $A \in CO\{\dots A^{[v]} \dots\}$ is robustly stable

$$\Leftrightarrow \exists \epsilon \ : \left[\begin{array}{cc} I & -s^{-1}I \\ -A & I \end{array}\right]^* \left[\begin{array}{cc} I & -s^{-1}I \\ -A & I \end{array}\right] \succeq \epsilon I \succ 0 \quad \begin{array}{l} \forall s^{-1} + s^{-*} \geq 0 \\ \forall A \in CO\{\dots A^{[v]} \dots\} \end{array}$$

- PMI with scalar/matrix indeterminates constrained by PI & polytopes

$\Leftarrow \exists S \ : \ \forall v = 1 \dots \bar{v}, \ \exists P^{[v]} \succeq 0$ such that

$$\left[\begin{array}{cc} \epsilon I & P^{[v]} \\ P^{[v]} & \epsilon I \end{array}\right] \preceq S \left[\begin{array}{cc} A^{[v]} & -I \end{array}\right] + \left(S \left[\begin{array}{cc} A^{[v]} & -I \end{array}\right]\right)^*$$

- **Conservative** LMI formulation
- $P(A) = \sum_{v=1}^{\bar{v}} \xi_v P^{[v]}$, parameter-dependent, s.t. $P(A)(s^{-1} + s^{-*}) \succeq 0$
- S-variable copes with the polytopic uncertainty

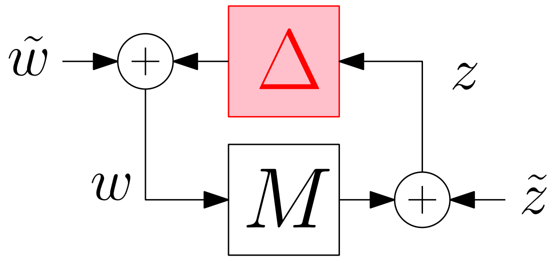
Motivation - DG-scalings

- Well-posedness of $\Delta \star M$

- $\Delta = \begin{bmatrix} \delta_1 I_{r_1} & & 0 \\ & \ddots & \\ 0 & & \Delta_{\bar{k}} \end{bmatrix}$

  - independent uncertainties
  - scalar repeated or matrix valued
  - real or complex
  - norm-bounded by 1 : $|\delta_k| \leq 1$ or $\|\Delta_k\| \leq 1$

- $\star$: feedback-loop



- M : Complex valued matrix

- Well-posedness : internal (w, z) bounded for all bounded disturbances $(\tilde{w}, \tilde{z})$

Motivation - DG-scalings

■ Well-posedness of $\Delta \star M$

● $\Delta = \begin{bmatrix} \delta_1 I_{r_1} & & 0 \\ & \ddots & \\ 0 & & \Delta_{\bar{k}} \end{bmatrix}$

- independent uncertainties
- scalar repeated or matrix valued
- real or complex
- norm-bounded by 1 : $|\delta_k| \leq 1$ or $\|\Delta_k\| \leq 1$

$$\delta_k \in \mathbb{C}, \quad |\delta_k| \leq 1 \quad \Leftrightarrow \quad 1 \geq {\delta_k}^* \delta_k$$

$$\Delta_k \in \mathbb{C}^{m_{1k}, m_{2k}}, \quad \|\Delta_k\| \leq 1 \quad \Leftrightarrow \quad I \succeq {\Delta_k}^* \Delta_k$$

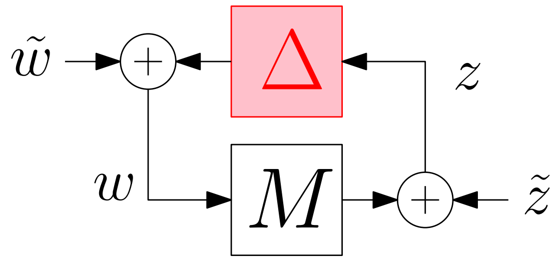
$$\delta_k \in \mathbb{R}, |\delta_k| \leq 1 \quad \Leftrightarrow \quad -j{\delta_k}^* + j\delta_k = 0, \quad 1 \geq {\delta_k}^* \delta_k$$

$$\delta_k \in \mathbb{R}, |\delta_k| \leq 1 \quad \Leftrightarrow \quad \delta_k \in CO\{-1, 1\}$$

- Complex matrix valued indeterminates Δ_k
- Constrained by polynomial inequalities (PMI), equalities (PME) & Polytopes
- Indeterminates are repeated $I_{r_k} \otimes \Delta_k$ (generalization of $\delta_1 I_{r_1}$ to matrices)

Motivation - DG-scalings

■ Well-posedness of $\Delta \star M$



$\Leftrightarrow$ internal (w, z) bounded for all bounded disturbances $(\tilde{w}, \tilde{z})$ and all Δ...

$\Leftrightarrow \exists \epsilon > 0 \ : (I_{m_2} - M\Delta)^*(I_{m_2} - M\Delta) \succeq \epsilon I_{m_2}$ for all Δ...

▼ PMI with matrix indeterminates constrained by PMIs, PMEs (& Polytopes)

▲ Indeterminates are in independent rows and columns (Δ block-diagonal)

$$\Leftarrow \exists D_k \succeq 0, G_k \quad : \quad \begin{bmatrix} I & M^* \end{bmatrix} \Theta(D_k, G_k) \begin{bmatrix} I \\ M \end{bmatrix} \succeq \epsilon I$$

▲ $\Theta(D_k, G_k)$: linear in the decision variables

▲ $D_k \succeq 0$ such that $D_k \otimes PMI(\Delta_k) \succeq 0$

▲ $G_k = G_k^*$ such that $G_k \otimes PME(\Delta_k) = 0$

Motivation - Proving positivity under constraints

- Robustness analysis of linear time-invariant systems

- Most problems can be recast as proving positivity of polynomials
  - ▼ matrix valued (semi-definite constraints)
  - ▼ indeterminates are matrices (or scalars), complex valued
  - ▼ constrained by PMIs, PMEs & Polytopes
  - ▲ LFT modeling allows to have indeterminates in separate rows/columns

- Many LMI results in the litterature,
  - ▼ in general **conservative** (problems are NP-hard)
  - ▲ some results are proved to be less conservative
  - ▲ on examples conservatism may vanish
  - ▲ duality of SDPs can extract worst case indeterminates (prove conservatism vanishes)
  - ▼ Numerical issues : limit size of LMIs using the structure of the data

Introduction

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  Finsler lemma, S-variables, **Positivstellensatz, SOS**, Pólya...

  - Developed separately for specific uncertainties and view points
  - Conservative SDP relaxations
  - Hierarchies of relaxations with decreasing conservatism

- This presentation : Attempt to establish links between these tools

  - Positivity of matrix polynomials with matrix indeterminates
  - Continuation of work of S-variable approach [Ebihara]
  - Strongly inspired by Quadratic Separation [Hara, Iwasaki]
  - Technicalities linking SOS and S-variables [Sato]

Sum-Of-Squares

- [Lasserre], [Parillo], [Scherer], [Chesi] ...

- Goal : proving a PMI $F(\delta) \succeq 0$
  - with (scalar) indeterminates $\delta \in \mathbb{R}^{\bar{k}}$
  - constrained by (scalar) polynomial inequalities $f_i(\delta) \geq 0$.

- Key steps for solving the problem using SDPs
  - Polynomials modeled as quadratic functions of monomials
  - Positivstellensatz
  - SDP relaxation
  - Hierarchies
  - (Moment problem)

- Case of matrix valued indeterminates : non-commutative polynomials [Helton]...
  - Each step may be much more complicated

Quadratic functions of monomials

- Goal : proving $F(\Delta) = F(I_{r_1} \otimes \Delta_1, \dots, I_{r_{\bar{k}}} \otimes \Delta_{\bar{k}}) \succeq 0$
- Indeterminates Δ_k are in independent columns
- Constrained by PMIs, PMEs & Polytopes

$$F_{ik}(\Delta_k) \succeq 0, \quad F_{ek}(\Delta_k) = 0, \quad \Delta_k \in CO\{\dots \Delta_k^{[v_k]} \dots\}$$

- Polynomials modeled as quadratic functions of monomials
- Real scalar : $1 + 2\delta^2 = \begin{pmatrix} 1 \\ \delta \end{pmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{pmatrix} 1 \\ \delta \end{pmatrix} = \begin{pmatrix} 1 \\ \delta \\ \delta^2 \end{pmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ \delta \\ \delta^2 \end{pmatrix}$
- Extension to $\Delta_k \in \mathbb{C}^{m_{1k}, m_{2k}}$ that could be non-square ?
- Our suggestion : monomials composed of

$$I_{m_{2k}}, \quad \Delta_k, \quad \Delta_k^* \Delta_k, \quad \Delta_k \Delta_k^* \Delta_k \dots$$

Quadratic functions of monomials

- Monomials with matrix indeterminates $\Delta_k \in \mathbb{C}^{m_{1k}\times m_{2k}}$

$$\Delta_k{}^{\{0\}} = I_{m_{2k}} \qquad \Delta_k{}^{\{1\}} = \Delta_k, \qquad \Delta_k{}^{\{2\}} = \Delta_k^*\Delta_k,$$
$$\Delta_k{}^{\{3\}} = \Delta_k\Delta_k^*\Delta_k, \quad \Delta_k{}^{\{4\}} = \Delta_k^*\Delta_k\Delta_k^*\Delta_k, \quad \dots$$

- Matrix with monomials from degree 0 to degree p_k :

$$\Delta_k{}^{\{0:p_k\}} = \begin{bmatrix} \Delta_k{}^{\{0\}} \\ \vdots \\ \Delta_k{}^{\{p_k\}} \end{bmatrix} = \begin{bmatrix} I_{m_{2k}} \\ \Delta_k \\ \Delta_k^*\Delta_k \\ \vdots \end{bmatrix}, \quad (I_{r_k}\otimes\Delta_k)^{\{0:p_k\}} = \begin{bmatrix} I_{r_k}\otimes\Delta_k{}^{\{0\}} \\ \vdots \\ I_{r_k}\otimes\Delta_k{}^{\{p_k\}} \end{bmatrix}$$

$$\Delta^{\{0:p\}} = \begin{bmatrix} (I_{r_1}\otimes\Delta_1)^{\{0:p_1\}} & & 0 \\ & \ddots & \\ 0 & & (I_{r_{\bar{k}}}\otimes\Delta_{\bar{k}})^{\{0:p_{\bar{k}}\}} \end{bmatrix}$$

Quadratic functions of monomials

- Problem reformulated as proving $F(\Delta) = \Delta^{\{0:p\}*}(F_0 + F_1^* F_1)\Delta^{\{0:p\}} \succeq 0$
  - under constraints

$$F_{ik}(\Delta_k) = {\Delta_k}^{\{0:p_k\}*}\Phi_{ik}{\Delta_k}^{\{0:p_k\}} \succeq 0, \qquad \Delta_k \in CO\{\Delta_k^{[1]}, \ldots, \Delta_k^{[\bar{v}_k]}\}$$

$$F_{ek}(\Delta_k) = {\Delta_k}^{\{0:p_k\}*}\Phi_{ek}{\Delta_k}^{\{0:p_k\}} = 0,$$

  - This is a subclass of the original problem (work in progress to extended the result)

Positivstellensatz

- Case of scalar constraints : Putinar's Positivstellensatz

$$F(\delta) \succeq 0 \quad \forall \delta : f_i(\delta) \geq 0$$

$$\Uparrow$$

$$\exists d_0(\delta), D_i(\delta) \quad SOS \quad : \quad F_D(\delta) = d_0(\delta)F(\delta) - \sum D_i(\delta) f_i(\delta) \quad SOS$$

▲ Lossless, under some assumptions & polynomials $D_i(\delta)$ with sufficiently high order

▼ Formulation for matrix indeterminates and PMI constraints ?

Positivstellensatz

- Problem reformulated as proving $F(\Delta) = \Delta^{\{0:p\}*}(F_0 + F_1^* F_1)\Delta^{\{0:p\}} \succeq 0$
  - under constraints

$$F_{ik}(\Delta_k) = {\Delta_k}^{\{0:p_k\}*}\Phi_{ik}{\Delta_k}^{\{0:p_k\}} \succeq 0, \qquad \Delta_k \in CO\{\Delta_k^{[1]}, \dots, \Delta_k^{[\bar{v}_k]}\}$$
$$F_{ek}(\Delta_k) = {\Delta_k}^{\{0:p_k\}*}\Phi_{ek}{\Delta_k}^{\{0:p_k\}} = 0,$$

- Sufficient condition : $\exists D_k(\Delta) \quad SOS, \quad \exists G_k(\Delta) = G_k^*(\Delta)$ such that

$$\Delta^{\{0:p\}*}\left(F_0 + F_1^* F_1 - \text{diag}\left(\cdots D_k(\Delta) \boxtimes \Phi_{ik} + G_k(\Delta) \boxtimes \Phi_{ek} \cdots\right)\right)\Delta^{\{0:p\}} \quad SOS$$

  - under polytopic constraints $\Delta_k \in CO\{\Delta_k^{[1]}, \dots, \Delta_k^{[\bar{v}_k]}\}$ $\quad$ ($\bar{v}_k = 0$ if no constraint)

- $\boxtimes$ is a Kronecker-like product
- Manipulation of Δ variables is difficult
- We restrict the search to $D(\Delta)$, $G(\Delta)$ affine in the Δ_k that are in polytopes.

SDP relaxation

- Semidefinite relaxation to prove that a polynomial is SOS
- Exploit degrees of freedom in writing the polynomial in a basis of monomials

$$\Delta^{\{0:p\}*} F \Delta^{\{0:p\}} = \Delta^{\{0:\hat{p}\}*} (\hat{F} + V) \Delta^{\{0:\hat{p}\}}$$

- V is structured such that $\Delta^{\{0:\hat{p}\}*} V \Delta^{\{0:\hat{p}\}} = 0$
- $\hat{p} \geq p$: larger basis of monomials
- SDP test of SOS $\quad \exists V \ : \ \hat{F} + V \succeq 0$

- Example of a scalar polynomial

$$\begin{pmatrix} 1 \\ \delta \\ \delta^2 \end{pmatrix}^T \begin{bmatrix} 1 & 0 & v_1 \\ 0 & 2 - 2v_1 & 0 \\ v_1 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ \delta \\ \delta^2 \end{pmatrix} = 1 + 2\delta^2$$

- Formula to build systematically V (not by inspection of all monomials) ?

S-variables for SDP relaxation

- Consider building $H_k(\Delta_k) = J_0 + J_1(I \otimes \Delta_k)J_2 + J_3(I \otimes {\Delta_k}^*)J_4$ such that ${\Delta_k}^{\{0:p_k\}}$ spans the null space of $H_k(\Delta_k)$

$$\underbrace{\begin{bmatrix} \Delta_k & -I & 0 & 0 \\ 0 & {\Delta_k}^* & -I & 0 \\ 0 & 0 & \Delta_k & -I \end{bmatrix}}_{H_k(\Delta_k)} \underbrace{\begin{bmatrix} I \\ \Delta_k \\ {\Delta_k}^*\Delta_k \\ \Delta_k {\Delta_k}^* \Delta_k \end{bmatrix}}_{{\Delta_k}^{\{0:p_k\}}} = 0$$

S-variables for SDP relaxation

- SDP relaxation for one indeterminate Δ_k in polytope, assuming $\Psi(X)$ is affine in X

$$\exists S_k, X^{[v_k]} \; : \; \Psi(X^{[v_k]}) + S_k H_k(\Delta_k^{[v_k]}) + (S_k H_k(\Delta_k^{[v_k]}))^* \succeq 0 \quad \forall v_k = 1 \dots \bar{v}_k$$

$$\Downarrow$$

$$\exists X(\Delta) \; : \; \Delta_k{}^{\{0:p_k\}*} \Psi(X(\Delta_k)) \Delta_k{}^{\{0:p_k\}} \succeq 0 \quad \forall \Delta_k \in CO\{\Delta_k^{[1]}, \dots, \Delta_k^{[\bar{v}_k]}\}$$

- ▲ LMI result for proving PMI constraints over polytopes
- ▲ Can be applied to PMIs such as those build using Positivstellensatz
- ▼ Very large # of decision variables (and $\bar{v}_k$ constraints)
- ▲ We have tools to reduce the size of these LMIs exploiting structure of the data.

- Proof

- Affine Ψ and M_k and $\Delta_k = \sum_{v_k=1}^{\bar{v}_k} \xi_v \Delta^{[v_k]}$

$$\exists S_k, X^{[v_k]} \ : \ \Psi(X^{[v_k]}) + S_k H_k(\Delta_k^{[v_k]}) + (S_k H_k(\Delta_k^{[v_k]}))^* \succeq 0 \quad \forall v_k = 1 \dots \bar{v}_k$$

$$\Updownarrow$$

$$\exists S_k, \ : \ \Psi(X(\Delta)) + S_k H_k(\Delta_k) + (S_k H_k(\Delta_k))^* \succeq 0 \quad \forall \xi_v \geq 0, \quad \sum_{v=1}^{\bar{v}} \xi_v = 1$$

$$X(\Delta) = \sum_{v_k=1}^{\bar{v}_k} \xi_v X^{[v_k]}$$

- By congruence with the fact that $H_k(\Delta_k){\Delta_k}^{\{0:p_k\}} = 0$

$$\Downarrow$$

$$\exists X(\Delta) \ : \ {\Delta_k}^{\{0:p_k\}*} \Psi(X(\Delta)) {\Delta_k}^{\{0:p_k\}} \succeq 0 \quad \forall \Delta_k \in CO\{\Delta_k^{[1]}, \dots, \Delta_k^{[\bar{v}_k]}\}$$

- Conservatism comes for the choice of indeterminate-independent S_k

S-variables and SDPs for SOS

- SDP relaxation for one unbounded Δ_k

$$\exists \hat{S}_k, X \ : \ \Psi(X) + \hat{S}_k H_k(0) + (\hat{S}_k H_k(0))^* \succeq 0 \quad \forall v_k = 1 \dots \bar{v}_k$$

$$\hat{S}_k = \begin{bmatrix} J_4^T & J_2^T \end{bmatrix} \begin{bmatrix} T_k \otimes I & 0 \\ 0 & T_k \otimes I \end{bmatrix} \begin{bmatrix} J_1^T \\ J_3^T \end{bmatrix}, \quad T_k = -T_k^*$$

$$\Downarrow$$

$$\exists X \ : \ {\Delta_k}^{\{0:p_k\}*} \Psi(X) {\Delta_k}^{\{0:p_k\}} \succeq 0 \quad \forall \Delta_k$$

- Structured S-variables provide parameterization of V s.t. $\Delta^{\{0:\hat{p}\}*} V \Delta^{\{0:\hat{p}\}} = 0$

$$V = \hat{S}_k H_k(0) + (\hat{S}_k H_k(0))^*$$

- We conjecture it is a complete parameterization (true for scalar indeterminates)
- SOS-Moment SDP relaxations are a special case of S-variable relaxation

Main result

- $F(\Delta) = \Delta^{\{0:p\}*}(F_0 + F_1^* F_1)\Delta^{\{0:p\}} \succeq 0$
- under constraints

$$F_{ik}(\Delta_k) = {\Delta_k}^{\{0:p_k\}*}\Phi_{ik}{\Delta_k}^{\{0:p_k\}} \succeq 0, \qquad \Delta_k \in CO\{\Delta_k^{[1]}, \dots, \Delta_k^{[\bar{v}_k]}\}$$
$$F_{ek}(\Delta_k) = {\Delta_k}^{\{0:p_k\}*}\Phi_{ek}{\Delta_k}^{\{0:p_k\}} = 0,$$

$$\Leftarrow \exists D_k^{[v\mathcal{K}_2]} \succeq 0, G_k^{[v\mathcal{K}_2]} = G_k^{[v\mathcal{K}_2*]}, \hat{S}_k^{[v\mathcal{K}_2]}, S_k^{[v\mathcal{K}_2\setminus k]}$$

$$\hat{F}_1^{\perp *}\left(\hat{F}_0 - \text{diag}\left(\begin{matrix} \vdots \\ D_k^{[v\mathcal{K}_2]} \boxtimes \Phi_{ik} + G_k^{[v\mathcal{K}_2]} \boxtimes \Psi_{ek} \\ \vdots \end{matrix}\right)\right)\hat{F}_1^{\perp}$$
$$+ \sum_{k\in\mathcal{K}_1}\left\{\hat{F}_1^{\perp *}\hat{S}_k^{[v\mathcal{K}_2]}\hat{H}_k(0)\hat{F}_1^{\perp}\right\}^{\mathcal{H}} + \sum_{k\in\mathcal{K}_2}\left\{S_k^{[v\mathcal{K}_2\setminus k]}\hat{H}_k(\Delta_k^{[v_k]})\hat{F}_1^{\perp}\right\}^{\mathcal{H}} \succeq 0$$

- $\mathcal{K}_1, \mathcal{K}_2$ indices of uncertainties without and with polytopic constraints respectively

Main result

■ $F(\Delta) = \Delta^{\{0:p\}*}(F_0 + F_1^* F_1)\Delta^{\{0:p\}} \succeq 0$ under PMI, PME & Polytopic constraints

$$\Leftarrow \exists D_k^{[v\mathcal{K}_2]} \succeq 0, G_k^{[v\mathcal{K}_2]} = G_k^{[v\mathcal{K}_2 *]}, \hat{S}_k^{[v\mathcal{K}_2]}, S_k^{[v\mathcal{K}_2 \backslash k]}$$

$$\hat{F}_1^{\perp *}\left(\hat{F}_0 - \text{diag}\left(D_k^{[v\mathcal{K}_2]} \boxtimes \Phi_{ik} + G_k^{[v\mathcal{K}_2]} \boxtimes \Psi_{ek} \right)\right)\hat{F}_1^{\perp}$$
$$+ \sum_{k\in\mathcal{K}_1} \left\{ \hat{F}_1^{\perp *} \hat{S}_k^{[v\mathcal{K}_2]} H_k(0) \hat{F}_1^{\perp} \right\}^{\mathcal{H}} + \sum_{k\in\mathcal{K}_2} \left\{ S_k^{[v\mathcal{K}_2 \backslash k]} H_k(\Delta_k^{[v_k]}) \hat{F}_1^{\perp} \right\}^{\mathcal{H}} \succeq 0$$

▲ Hierarchy of SDP relaxations as order p is increased

▼ As all LMI formulations numerical burden is rapidly prohibitive

▲ Complicated formula, but not difficult to code (no need for symbolic manipulation)

▲ Size of LMIs is not so huge when exploiting the structure $F_0 + F_1^* F_1$

▲ When applied to special cases we get exactly the existing LMIs of the litterature

▼ Can we provide new results ?

▼ Assuming all indeterminates are matrices. Exploit commutativity of scalars ?

▼ Is the hierarchy complete (proof of losslessness at some order of relaxation) ?

Conclusion

- ▲ Ongoing work to explore links between many existing results in Robust Control
- ▲ Method inspired by SOS-Moments relaxations and our S-variable approach
- ▲ Motivation for dealing with polynomial matrix inequalities of matrix indeterminates
- ▼ Sub-case of all possible polynomial matrix inequalities of matrix indeterminates
- ▼ Numerical experiments to be done