# Approximation of infinite dimensional linear dynamical models

## ... and its applications

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ONERA
THE FRENCH AEROSPACE LAB

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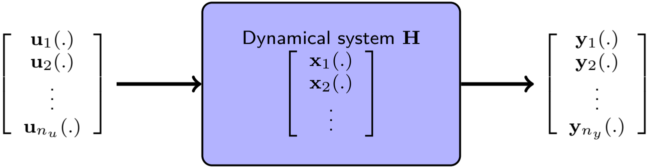
mor Digital Systems

# Introduction and motivating examples

## Finite and infinite dimensional linear dynamical models...




$$\mathbf{u}(s) \quad \mathbf{H}(s) \quad = \quad \mathbf{y}(s)$$

**Finite dynamical** models have a **finite** number of singularities, *e.g.*

$$\mathbf{H}(s) = \frac{1}{1+s} \in \mathcal{RL}_2$$

**Infinite dynamical models** have an **infinite** number of singularities, *e.g.*

$$\mathbf{H}(s) = \frac{1}{1+s+e^{-s}} \in \mathcal{L}_2$$

$$\mathcal{L}_p : \{\mathbf{H} : \mathbb{C} \rightarrow \mathbb{C}^{n_y \times n_u}, ||\mathbf{H}||_p < \infty\}$$

$$\mathcal{RL}_p : \{ \text{ rational } \mathcal{L}_p \text{ functions } \}$$

# Introduction and motivating examples

**... and where do they come from?**

**Finite representations** are largely used in industry and academic research.
**Infinite representation** are less explored (or in specific cases, *e.g.* delay literature).

Finite dynamical models come from

- spatial meshing of **PDE**

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$

- standard mechanical equations

$$M\ddot{\mathbf{x}}(t) = C\dot{\mathbf{x}}(t) + K\mathbf{x}(t) + B\mathbf{u}(t)$$

- structured....

$$(J - H)\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$

Infinite dynamical models may come

- exact solution of linear **PDE**

$$\mathbf{y}(s) = e^{-\sqrt{s}}\mathbf{u}(s)$$

- delays in the loop

$$\mathbf{y}(s) = \frac{1}{1 + s + e^{-s}}\mathbf{u}(s)$$

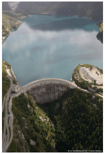
- discretisation of control laws

$$\mathbf{y}(s) = C(e^{sh}I - A)^{-1}B\mathbf{u}(s)$$

**Infinite dimensional dynamical models describe a larger class of systems**

## Introduction and motivating examples

### Today's talk



Approximation of infinite dimensional models:

Part 1 ... some generalities and tools

Part 2 ... for linear **PDE** modelling and analysis
applied on a hydro-electrical open channel

Part 3 ... for stability approximation of $\mathcal{L}_2$ functions
applied on a bundle of **TDS** models

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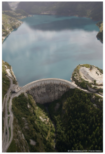

Approximation of infinite dimensional models:

Part 1 ... some generalities and tools

Part 2 ... for linear **PDE** modelling and analysis applied on a hydro-electrical open channel

Part 3 ... for stability approximation of $\mathcal{L}_2$ functions applied on a bundle of **TDS** models

Main message:

- Approximation is a **pivotal tool**
- ... and **(locally optimal) solutions** exist
- ... as well as numerical tools: **MOR toolbox**

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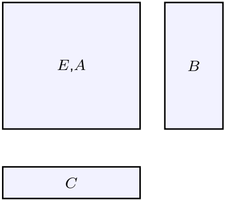
http://mordigitalsystems.fr/

# Preliminaries in model approximation

## A (rather general Petrov-Galerkin finite) linear rational model approximation problem

Let $\mathbf{H} : \mathbb{C} \to \mathbb{C}^{n_y \times n_u}$ be a $n_u$ inputs $n_y$ outputs, complex-valued function describing a **LTI** dynamical system as a **DAE** of order $n$, with realisation $\mathcal{S}$:

$$\mathcal{S} : \begin{cases} E\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) &= C\mathbf{x}(t) \end{cases}$$



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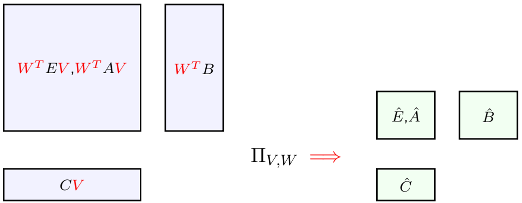
the approximation problem consists in finding $V, W \in \mathbb{R}^{n \times r}$ (with $r \ll n$) **spanning $\mathcal{V}$ and $\mathcal{W}$ subspaces** and forming a projector $\Pi_{V,W} = VW^T$, such that

$$\hat{\mathcal{S}} : \begin{cases} W^T E V \dot{\hat{\mathbf{x}}}(t) &= W^T A V \hat{\mathbf{x}}(t) + W^T B \mathbf{u}(t) \\ \hat{\mathbf{y}}(t) &= C V \hat{\mathbf{x}}(t) \end{cases}$$

well approximates $\mathbf{H}$.



# Preliminaries in model approximation

**A (rather general Petrov-Galerkin finite) linear rational model approximation problem**

## Truncation (mostly dense) *e.g.*

- Modal, $\{V, W\}$ are eigenvectors subspaces
- Balanced, $\{V, W\}$ come from Lyapunov and SVD subspaces
- Singular perturbation, $\{V, W\}$ come from Lyapunov and SVD subspaces

## Interpolation (mostly sparse) *e.g.*

- Moment matching (quite general formulation)
- Rational (Padé, Markov, generalised), $\{V, W\}$ are Krylov subspaces
- Multi-point ($\mathcal{H}_2$ optimal or not), $\{V, W\}$ are generalised Krylov subspaces

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This framework mainly works for finite order (structured) models.
What about realisation-free models?

# Preliminaries in model approximation

## A (rather general) linear model approximation problem

Let us consider $\mathbf{H}$, a $n_u$ inputs, $n_y$ outputs linear dynamical system described by the **complex-valued function from $\mathbf{u}$ to $\mathbf{y}$, of order** $n$ ($n$ large or $\infty$)

$$\mathbf{H} : \mathbb{C} \to \mathbb{C}^{n_y \times n_u},$$

the model approximation problem consists in finding $\hat{\mathbf{H}}$ of order $r \ll n$

$$\hat{\mathbf{H}} : \mathbb{C} \to \mathbb{C}^{n_y \times n_u},$$

that well reproduces the input-output behaviour of $\mathbf{H}$.

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that well reproduces the input-output behaviour of $\mathbf{H}$ and equipped with a given realisation, *e.g.*

$$\hat{\mathcal{S}} : \begin{cases} \hat{E}\dot{\hat{\mathbf{x}}}(t) &= \hat{A}\hat{\mathbf{x}}(t) + \hat{B}\mathbf{u}(t) \\ \hat{\mathbf{y}}(t) &= \hat{C}\hat{\mathbf{x}}(t) \end{cases} \quad \text{or } \hat{\mathcal{S}}_d : \begin{cases} \hat{E}\dot{\hat{\mathbf{x}}}(t) &= \hat{A}\hat{\mathbf{x}}(t) + \hat{B}\hat{\boldsymbol{\Delta}}_i(\mathbf{u}(t)) \\ \hat{\mathbf{y}}(t) &= \hat{\boldsymbol{\Delta}}_o(\hat{C}\hat{\mathbf{x}}(t)) \end{cases}$$

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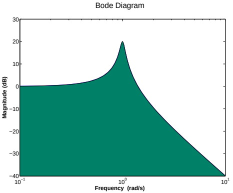
"Well reproduce..."?
$\hat{\mathbf{H}}$ is a "good" approximation of $\mathbf{H}$ if
for the same driving $\mathbf{u}(t)$, $\mathcal{E}(t) = \mathbf{y}(t) - \hat{\mathbf{y}}(t)$ is "small"

# Preliminaries in model approximation

$\mathcal{H}_2$ **optimality conditions**[1] [2]

## $\mathcal{H}_2$ model approximation

$$\hat{\mathbf{H}} := \arg \min_{\substack{\mathbf{G} \in \mathcal{H}_2 \\ \mathbf{rank}(\mathbf{G}) = r \ll n}} ||\mathbf{H} - \mathbf{G}||_{\mathcal{H}_2}$$




Energy to an impulse input

$$||\mathbf{H}||^2_{\mathcal{H}_2} := \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{tr}\left(\overline{\mathbf{H}(\imath\nu)}\mathbf{H}^T(\imath\nu)\right) d\nu$$

Note that: $||\mathbf{y}(t) - \hat{\mathbf{y}}(t)||_{L_\infty} \leq ||\mathbf{H} - \hat{\mathbf{H}}||_{\mathcal{H}_2} ||\mathbf{u}(t)||_{L_2}$

[1] S. Gugercin and A C. Antoulas and C A. Beattie, *"$\mathcal{H}_2$ Model Reduction for Large Scale Linear Dynamical Systems"*, SIAM Journal on Matrix Analysis and Applications, vol. 30(2), June 2008, pp. 609-638.

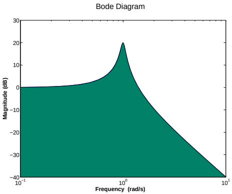
[2] K. A. Gallivan, A. Vanderope, and P. Van-Dooren, *"Model reduction of MIMO systems via tangential interpolation"*, SIAM Journal of Matrix Analysis and Application, vol. 26(2), February 2004, pp. 328-349.

# Preliminaries in model approximation

### $\mathcal{H}_2$ optimality conditions [1]

**Input / output delays structured $\mathcal{H}_2$ model approximation**

$$\hat{\mathbf{H}}_d := \arg \min_{\substack{\mathbf{G} \in \mathcal{H}_\infty \\ \mathbf{rank}(\mathbf{G}) = r \ll n}} ||\mathbf{H} - \mathbf{G}||_{\mathcal{H}_2}$$

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[1] I. Pontes Duff, C. P-V and C. Seren, *"$\mathcal{H}_2$-optimal model approximation by input / output-delay structured reduced order models"*, in Systems & Control Letters, vol. 117, July 2018, pp. 60-67.

# Preliminaries in model approximation

## $\mathcal{H}_2$ optimality conditions[2]

$$\hat{\mathbf{H}}(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B} = \sum_{l=1}^{r} \frac{\hat{\phi}_l}{s - \hat{\lambda}_l}$$

Let $\hat{\mathbf{H}}$ be a $r$-th order asymptotically stable model with semi-simple poles only, equipped with $\hat{\mathcal{S}} : (\hat{E}, \hat{A}, \hat{B}, \hat{C}, \hat{D})$. If $\hat{\mathbf{H}}$ is solution of the $\mathcal{H}_2$ approximation problem, then

$$\begin{array}{rcl} \mathbf{H}(-\hat{\lambda}_l) & = & \hat{\mathbf{H}}(-\hat{\lambda}_l) \\ \mathbf{H}'(-\hat{\lambda}_l) & = & \hat{\mathbf{H}}'(-\hat{\lambda}_l) \end{array}$$

where $\hat{\lambda}_l$ are the eigenvalues of $(\hat{E}, \hat{A})$.

**$\mathcal{H}_2$ optimality is recast a bi-tangential Hermite interpolation at the reduced order model eigenvalues** (same comment if input/output delays enter in the game)

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# Preliminaries in model approximation

## Approximation by interpolation[3] [4]

**(Tangential) interpolation is the path to this $\mathcal{H}_2$ problem**

**SISO** model: given $\mathbf{H}$, seek a reduced-order system $\hat{\mathbf{H}}$, such that

$$\begin{array}{rcll} \hat{\mathbf{H}}(\mu_i) & = & \mathbf{H}(\mu_i) & i = 1, \dots, q \\ \hat{\mathbf{H}}(\lambda_j) & = & \mathbf{H}(\lambda_j) & j = 1, \dots, k \end{array}$$

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**MIMO** model (tangential): in a similar way, given $\mathbf{H}$, seek $\hat{\mathbf{H}}$, such that

$$\begin{array}{rcll} \mathbf{l}_i^H \hat{\mathbf{H}}(\mu_i) & = & \mathbf{l}_i^H \mathbf{H}(\mu_i) & i = 1, \dots, q \\ \hat{\mathbf{H}}(\lambda_j)\mathbf{r}_j & = & \mathbf{H}(\lambda_j)\mathbf{r}_j & j = 1, \dots, k \end{array}$$

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# Preliminaries in model approximation

## Rational interpolation in the Loewner framework

Given $\mathbf{H}(s)$ and $\{\mu_1, \ldots, \mu_q\}$, $\{\lambda_1, \ldots, \lambda_k\}$, we seek $\hat{\mathbf{H}}$, s.t.

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$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{H}(\mu_1)-\mathbf{H}(\lambda_1)}{\mu_1-\lambda_1} & \dots & \frac{\mathbf{H}(\mu_1)-\mathbf{H}(\lambda_k)}{\mu_1-\lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{H}(\mu_q)-\mathbf{H}(\lambda_1)}{\mu_q-\lambda_1} & \dots & \frac{\mathbf{H}(\mu_q)-\mathbf{H}(\lambda_k)}{\mu_q-\lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}$$

$$\mathbb{L}_\sigma = \begin{bmatrix} \frac{\mu_1\mathbf{H}(\mu_1)-\mathbf{H}(\lambda_1)\lambda_1}{\mu_1-\lambda_1} & \dots & \frac{\mu_1\mathbf{H}(\mu_1)-\mathbf{H}(\lambda_k)\lambda_k}{\mu_1-\lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mu_q\mathbf{H}(\mu_q)-\mathbf{H}(\lambda_1)\lambda_1}{\mu_q-\lambda_1} & \dots & \frac{\mu_q\mathbf{H}(\mu_q)-\mathbf{H}(\lambda_k)\lambda_k}{\mu_q-\lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}$$

$\mathbf{W} = \begin{bmatrix} \mathbf{H}(\sigma_1) & \dots & \mathbf{H}(\sigma_r) \end{bmatrix}$ and $\mathbf{V}^T = \begin{bmatrix} \mathbf{H}(\sigma_1) & \dots & \mathbf{H}(\sigma_r) \end{bmatrix}$

$\hat{\mathbf{H}}(s) = \mathbf{W}(\mathbb{L}_\sigma - s\mathbb{L})^{-1}\mathbf{V} \quad \Rightarrow$ Rational interpolation

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$$\begin{array}{rcll}\hat{\mathbf{H}}(\sigma_i) & = & \mathbf{H}(\sigma_i) & i=1,\ldots,r \\ \hat{\mathbf{H}}'(\sigma_i) & = & \mathbf{H}'(\sigma_i) & \end{array}$$

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$$\mathbb{L} = \begin{bmatrix} \mathbf{H}'(\sigma_1) & \dots & \frac{\mathbf{H}(\sigma_1)-\mathbf{H}(\sigma_r)}{\sigma_1-\sigma_r} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{H}(\sigma_r)-\mathbf{H}(\sigma_1)}{\sigma_r-\sigma_1} & \dots & \mathbf{H}'(\sigma_r) \end{bmatrix} \in \mathbb{C}^{r\times r}$$

$$\mathbb{L}_\sigma = \begin{bmatrix} (s\mathbf{H}(s))'_{s=\sigma_1} & \dots & \frac{\sigma_1\mathbf{H}(\sigma_1)-\sigma_r\mathbf{H}(\sigma_r)}{\sigma_1-\sigma_r} \\ \vdots & \ddots & \vdots \\ \frac{\sigma_r\mathbf{H}(\sigma_r)-\sigma_1\mathbf{H}(\sigma_1)}{\sigma_r-\sigma_1} & \dots & (s\mathbf{H}(s))'_{s=\sigma_r} \end{bmatrix} \in \mathbb{C}^{r\times r}$$

$$\mathbf{W} = \begin{bmatrix} \mathbf{H}(\sigma_1) & \dots & \mathbf{H}(\sigma_r) \end{bmatrix} \text{ and } \mathbf{V}^T = \begin{bmatrix} \mathbf{H}(\sigma_1) & \dots & \mathbf{H}(\sigma_r) \end{bmatrix}$$

$\hat{\mathbf{H}}(s) = \mathbf{W}(\mathbb{L}_\sigma - \mathbb{L}s)^{-1}\mathbf{V}$ $\Rightarrow$ Hermite interpolation

# Preliminaries in model approximation

## Rational interpolation in the Loewner framework

The rational function $\hat{\mathbf{H}}(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B}$ interpolates $\mathbf{H}(s)$ at points $\sigma_i$ iff.

$$\left[\hat{E}\right]_{ij} = \begin{cases} -\dfrac{\left(\mathbf{H}(\sigma_i) - \mathbf{H}(\sigma_j)\right)}{\sigma_i - \sigma_j} & i \neq j \\ -\mathbf{H}'(\sigma_i) & i = j \end{cases}$$

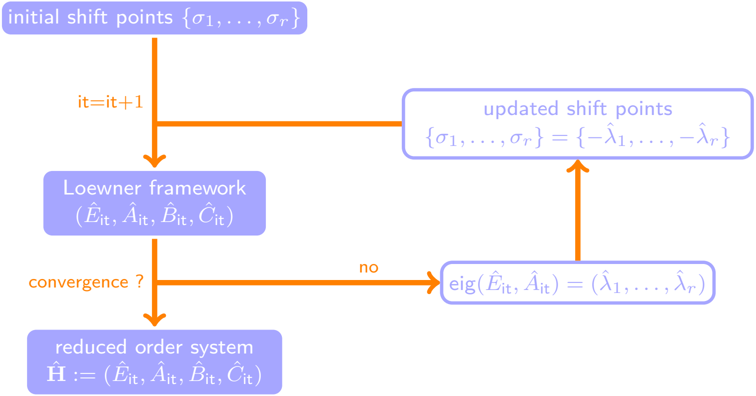
$$\left[\hat{A}\right]_{ij} = \begin{cases} -\dfrac{\left(\sigma_i\mathbf{H}(\sigma_i) - \sigma_j\mathbf{H}(\sigma_j)\right)}{\sigma_i - \sigma_j} & i \neq j \\ -(s\mathbf{H}(s))'|_{s=\sigma_i} & i = j \end{cases}$$

$$\hat{C} = \begin{bmatrix} \mathbf{H}(\sigma_1) & \dots & \mathbf{H}(\sigma_r) \end{bmatrix} \text{ and } \hat{B} = \begin{bmatrix} \mathbf{H}(\sigma_1) \\ \vdots \\ \mathbf{H}(\sigma_r) \end{bmatrix}$$

Use an iterative scheme to find $\sigma_i$.

# Preliminaries in model approximation

**TF-IRKA**[5]




[5] C. Beattie and S. Gugercin, *"Realization-independent $\mathcal{H}_2$-approximation"*, Proceedings of the 51st IEEE Conference on Decision and Control, 2012.

# Preliminaries in model approximation

## Rational approximation on a TDS example

$\mathbf{y}(s) = \frac{1}{1+s+e^{-s}}\mathbf{u}(s)$

```
% xdt(t) = -x(t) - x(t-1) + u
%    y(t) =  x(t)
delayT(1) = struct('delay',1,'a',-1,'b',0,'c',0,'d',0);
Hss = delayss(-1,1,1,0,delayT);
```

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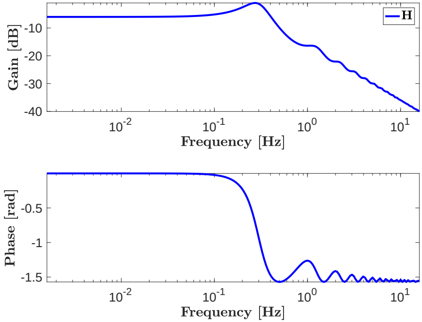
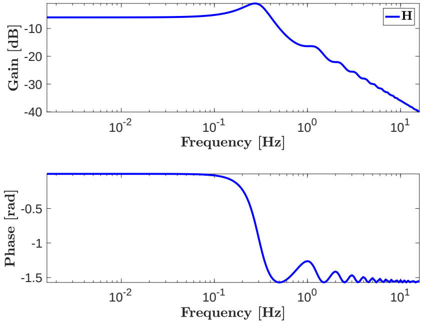


# Preliminaries in model approximation

## Rational approximation on a TDS example

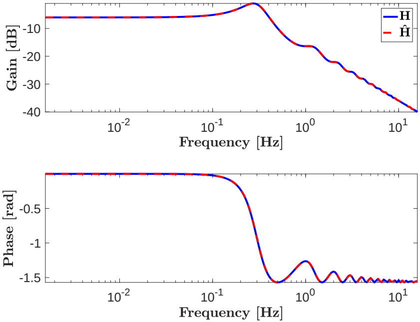
```
% MOR Toolbox
W  = logspace(-2,2,200);
H  = @(s) 1/(1+s+exp(-s));
FR = mor.bode(H,W);
Hr = mor.lti({W,FR},[]); % Rational approximation r=37
```



# Preliminaries in model approximation

## Rational approximation on a TDS example

```
% MOR Toolbox
W  = logspace(-2,2,200);
H  = @(s) 1/(1+s+exp(-s));
FR = mor.bode(H,W);
Hr = mor.lti({W,FR},[]); % Rational approximation r=37
```

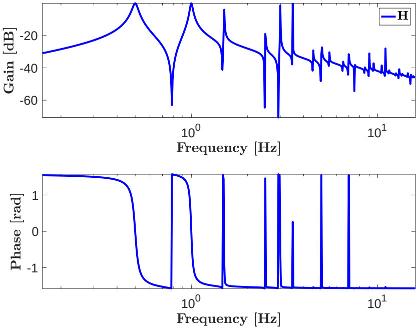


# Preliminaries in model approximation

## Rational approximation on a PDE example[6]

Vibrating string (ends fixed & control and observation distributed along the string)

$$\mathbf{H}(s) = \frac{\frac{s}{2}\sinh(s) + 2\cosh(\frac{s}{2}) - 3\cosh^2(\frac{s}{2}) + 1}{s(s+\frac{1}{2})\sinh(s) + 2\cosh(\frac{s}{2}) - 3\cosh^2(\frac{s}{2}) + 1}$$

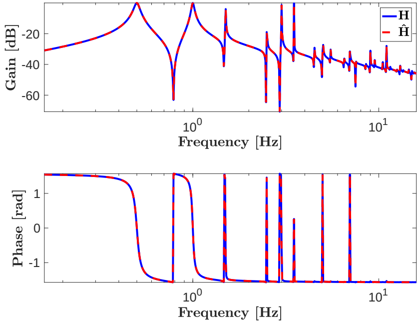



6 R. Curtain and K. Morris, *"Transfer functions of distributed parameter systems: A tutorial"*, Automatica, 45(5), 2009, pp. 1101-1116.

# Preliminaries in model approximation

## Rational approximation on a PDE example

```
% MOR Toolbox
H    = @(s) (s/2*sinh(s)+2*cosh(s/2)-3*cosh(s/2)^2+1 ) / ...
FR   = mor.bode(H,W);
Hr   = mor.lti({W,FR},[]); % Rational approximation r=60
Hred = mor.lti(Hr,r);      % H2 approximation r=40,30,20,10
```



## Preliminaries in model approximation

### Rational approximation on a PDE example

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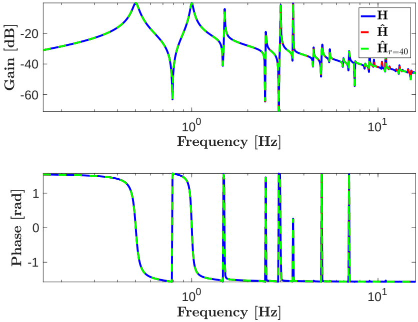

# Preliminaries in model approximation

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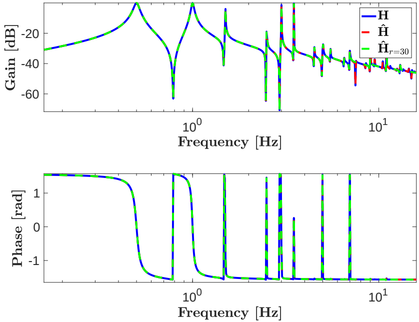

# Preliminaries in model approximation

## Rational approximation on a PDE example

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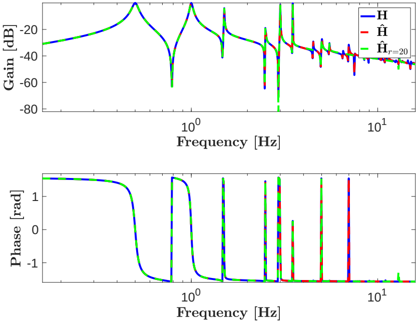

# Preliminaries in model approximation

## Rational approximation on a PDE example

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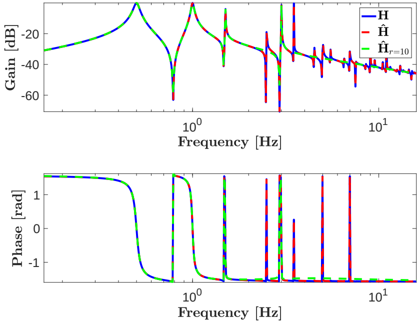


# Linear parametrised PDE model approximation

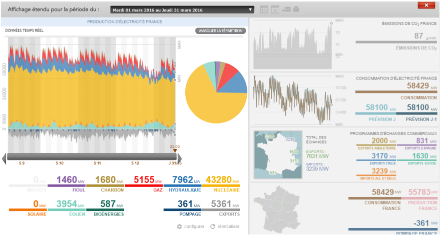
## An industrial example[7]

### Hydraulics green electricity ($\approx 10\%$)

- Dams
- Run-of-the-river

### Run-of-the-river ($\approx 5\%$)

- In France, provides 3.6GW
- Rely open-channel hydraulic systems
- Need for analysis and control




[7] http://alsace.edf.com/actions/fonctionnement-des-centrales-hydroelectriques-sur-le-rhin/

# Linear parametrised PDE model approximation

**An industrial example**

## Modelling assumptions

- No discharge, no infiltration, one dimensional flow, small bed slope, small stream line, negligible vertical acceleration
- Input $\mathbf{u}$: boundary conditions $q_e(t)$ and $q_s(t)$
- Output $\mathbf{y}$: water depth
- $t, x$ are the time and spatial variables

$$\begin{aligned} \frac{\partial S}{\partial t} + \frac{\partial Q}{\partial x} &= 0 \\ \frac{\partial Q}{\partial t} + \frac{\partial (Q^2/S)}{\partial x} + gS\frac{\partial H}{\partial x} &= gS(I-J), \end{aligned}$$

$x \in [0 \; ; \; L]$ is the spatial variable, $H(x,t)$ the water depth, $S(x,t)$ the wetted area, $Q(x,t)$ the discharge...

**Under some assumptions, can be rewritten as a nominal flow parametrised irrational function**

# Linear parametrised PDE model approximation

## An industrial example[8]

$$h(s, x_m, \delta) = \mathbf{G}_e(s, x_m, \delta) q_e(s) - \mathbf{G}_s(s, x_m, \delta) q_s(s)$$

$$\begin{aligned}
\mathbf{G}_e(s, x_m, \delta) &= \frac{\lambda_1(s) e^{\lambda_2(s)L + \lambda_1(s)x_m} - \lambda_2(s) e^{\lambda_1(s)L + \lambda_2(s)x_m}}{B_0 s (e^{\lambda_1(s)L} - e^{\lambda_2(s)L})} \\
\mathbf{G}_s(s, x_m, \delta) &= \frac{\lambda_1(s) e^{\lambda_1(s)x_m} - \lambda_2(s) e^{\lambda_2(s)x_m}}{B_0 s (e^{\lambda_1(s)L} - e^{\lambda_2(s)L})}
\end{aligned}$$

- Irrational transfer function
- Infinite order equation

8 V. Dalmas, G. Robert, C. Poussot-Vassal, I. Pontes Duff and C. Seren, *"From infinite dimensional modelling to parametric reduced order approximation: Application to open-channel flow for hydroelectricity"*, in Proceedings of the 15th European Control Conference (ECC'16), Aalborg, Denmark, July, 2016.

# Linear parametrised PDE model approximation

## An industrial example[8]

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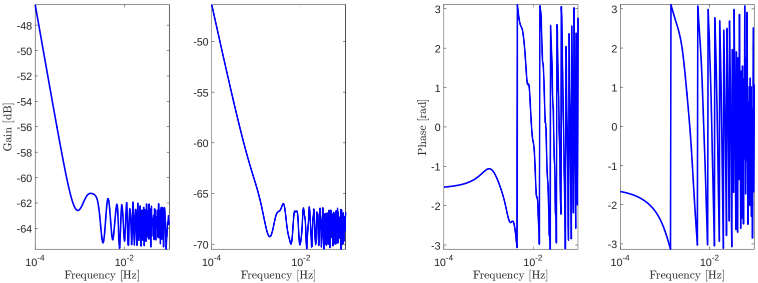



- Delay behaviour is obvious
- Not $\mathcal{H}_2$ function

8 V. Dalmas, G. Robert, C. Poussot-Vassal, I. Pontes Duff and C. Seren, *"From infinite dimensional modelling to parametric reduced order approximation: Application to open-channel flow for hydroelectricity"*, in Proceedings of the 15th European Control Conference (ECC'16), Aalborg, Denmark, July, 2016.

# Linear parametrised PDE model approximation

## Delay structured ROM

... from the open-channel example

$$h(s,x,\delta) = \mathbf{G}_e(s,x,\delta)q_e(s) - \mathbf{G}_s(s,x,\delta)q_s(s)$$

one seeks the input delayed $r$-th order rational function

$$\begin{array}{rcl} \hat{h}(s,\delta) & = & \hat{\mathbf{G}}_\mathbf{e}(s,\delta)q_e(s) - \hat{\mathbf{G}}_\mathbf{s}(s,\delta)q_s(s) \\ \hat{\mathbf{G}}_\mathbf{e}(x,s,\delta) & = & \mathbf{R}_\mathbf{e}(s,\delta)e^{-\tau_e(\delta)s} \\ \hat{\mathbf{G}}_\mathbf{s}(x,s,\delta) & = & \mathbf{R}_\mathbf{s}(s,\delta)e^{-\tau_s(\delta)s} \end{array}$$

- $\mathbf{R}_\mathbf{e}(s,\delta)$ and $\mathbf{R}_\mathbf{s}(s,\delta)$ are rational meromophic functions
- which are linearly dependent on $\delta$,
- and $\tau_e(\delta), \tau_s(\delta) \in \mathbb{R}_+^{n_u}$ is an input delay vector.

# Linear parametrised PDE model approximation

## Delay structured ROM (an approach when the delay is known)

If delays are a-priori known functions, approximation can be done on the shifted function

$$\tilde{h}(s,x,\delta) = \mathbf{G}_e(s,x,\delta)e^{+\tau_e(\delta)s}q_e(s) - \mathbf{G}_s(s,x_m,\delta)e^{+\tau_s(\delta)s}q_s(s)$$

- then apply Loewner
- and go back to $h(s,x,\delta)$

or

- apply **TF-IRKA**[9]
- and go back to $h(s,x,\delta)$

The Loewner approach is preferred for practical reasons in[10].

However, is the fixed delays the best idea? What if you don't a priori know them?

[9] C.A. Beattie, and S. Gugercin, *"Realization-independent $\mathcal{H}_2$-approximation"*, in ProceedingsProceedings of the 51st IEEE CDC, USA, December, 2012.

[10] V. Dalmas, G. Robert, C. Poussot-Vassal, I. Pontes Duff and C. Seren, *"From infinite dimensional modelling to parametric reduced order approximation: Application to open-channel flow for hydroelectricity"*, in Proceedings of the 15th ECC, Denmark, July, 2016.

# Linear parametrised PDE model approximation

## I/O delayed $\mathcal{H}_2$ optimality conditions[11]

### Input / output delays structured $\mathcal{H}_2$ model approximation

$$\hat{\mathbf{H}}_d := \arg \min_{\substack{\mathbf{G} \in \mathcal{H}_\infty \\ \mathbf{rank}(\mathbf{G}) = r \ll n}} ||\mathbf{H} - \mathbf{G}||_{\mathcal{H}_2}$$

where $\hat{\mathbf{H}}_d = \hat{\mathbf{\Delta}}_o \hat{\mathbf{H}} \hat{\mathbf{\Delta}}_i$.

$\mathcal{H}_2$ interpolatory conditions in the delay free case no longer apply

- due to the exponential terms in the transfer function...
- dedicated conditions need to be derived

[11] I. Pontes Duff, C. P-V and C. Seren, *"$\mathcal{H}_2$-optimal model approximation by input / output-delay structured reduced order models"*, in Systems & Control Letters, vol. 117, July 2018, pp. 60-67.

## Linear parametrised PDE model approximation

### I/O delayed $\mathcal{H}_2$ optimality conditions (reminder in the delay-free case)

$$\hat{\mathbf{H}}(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B} = \sum_{l=1}^{r} \frac{\hat{\phi}_l}{s - \hat{\lambda}_l}$$

Let $\hat{\mathbf{H}}$ be a $r$-th order asymptotically stable model with semi-simple poles only, equipped with $\hat{\mathcal{S}} : (\hat{E}, \hat{A}, \hat{B}, \hat{C}, \hat{D})$. If $\hat{\mathbf{H}}$ is solution of the $\mathcal{H}_2$ approximation problem, then

$$\begin{array}{rcl} \mathbf{H}(-\hat{\lambda}_l) & = & \hat{\mathbf{H}}(-\hat{\lambda}_l) \\ \mathbf{H}'(-\hat{\lambda}_l) & = & \hat{\mathbf{H}}'(-\hat{\lambda}_l) \end{array}$$

where $\hat{\lambda}_l$ are the eigenvalues of $(\hat{E}, \hat{A})$.

# Linear parametrised PDE model approximation

## I/O delayed $\mathcal{H}_2$ optimality conditions

$$\hat{\mathbf{H}}(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B} = \sum_{k=1}^{r} \frac{\hat{\phi}_k}{s - \hat{\lambda}_k} \text{ and } \hat{\mathbf{H}}_d = \hat{\mathbf{H}}e^{-\hat{\tau}s}$$

Let $\hat{\mathbf{H}}$ be a $r$-th order asymptotically stable model with semi-simple poles only, equipped with $\hat{\mathcal{S}} : (\hat{E}, \hat{A}, \hat{B}, \hat{C}, \hat{D})$. If $\hat{\mathbf{H}}$ is solution of the $\mathcal{H}_2$ approximation problem, then

$$\begin{array}{rcl} \mathbf{T}_d(-\hat{\lambda}_l) & = & \hat{\mathbf{H}}(-\hat{\lambda}_l) \\ \mathbf{T}_d'(-\hat{\lambda}_l) & = & \hat{\mathbf{H}}'(-\hat{\lambda}_l) \end{array}$$

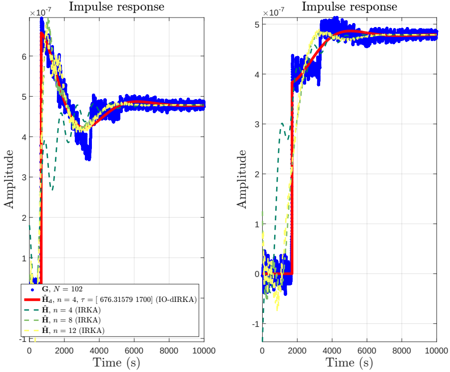
where $\hat{\lambda}_l$ are the eigenvalues of $(\hat{E}, \hat{A})$.

$$\mathbf{T}_d(s) = \sum_{j=1}^{n} \frac{\psi_j}{s - \mu_j} e^{\hat{\tau}\mu_j}.$$

Plus, a delay condition: $\sum_{j=1}^{n} \mu_j \psi_j \left( \sum_{k=1}^{r} \frac{\hat{\phi}_k}{\mu_j + \hat{\lambda}_k} \right) e^{\hat{\tau}\mu_j} = 0.$

# Linear parametrised PDE model approximation

## I/O delayed $\mathcal{H}_2$ optimality conditions



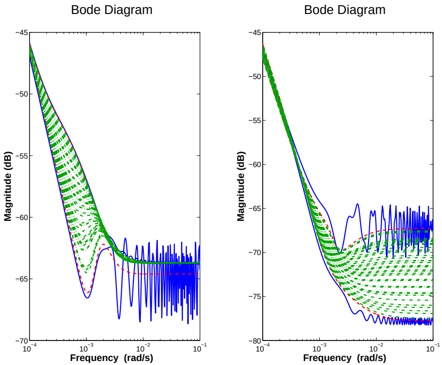

Mismatch errors

- **IO-dIRKA** (r=4) $\mathbf{4.34672 \times 10^{-15}}$
- **IRKA** (r=8): $\approx 6.4 \times 10^{-15}$
- **IRKA** (r=12): $\mathbf{4.06717 \times 10^{-15}}$

- Rational model obtained by Loewner
- Rational ROM, $r = 4$ with and without input delay structure

# Linear parametrised PDE model approximation

## Get the parametrised form




- Irrational model, rational parametrised ROM, $r = 4$
- Parametric model / LFR (using SMAC Toolbox)

# Input-output $\mathcal{L}_2$ stability

## Problem description

Let us now assume that the delayed (hydro-electrical channel) model **is looped with a control law**. The model becomes:

- Time Domain (**TD**)

$$E\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + \sum_{k=1}^{n_d} A_k\mathbf{x}(t-\tau_k) + \mathbf{b}u(t), \quad y(t) = \mathbf{c}^T\mathbf{x}(t),$$

- Frequency Domain (**FD**)

$$\mathbf{H}(s) = \mathbf{c}^T\left(sE - A_0 - \sum_{k=1}^{n_d} A_k e^{-s\tau_k}\right)^{-1}\mathbf{b},$$

**How to evaluate the stability of such a model?**

# Input-output $\mathcal{L}_2$ stability

## Traditional approaches[12] [13] [14]

### Time domain approach

- Lyapunov-Krasovskii functional,
- Lyapunov-Rasumikin functional, or
- Jensen inequalities
- √ applicable to time varying delay
- √ involves LMI, so optimality can be measured
- × ... limited to low $n$ and few delays

### Frequency domain approach

- Bifurcation theory
- Arnoldi like methods
- √ provides the number of unstable modes
- √ applicable to larger $n$
- × ... usually involving single interpolation point

What if $n$ or the number of delays $n_d$ go large?

12 W. Michiels, S.-I. Niculescu, *"Stability, Control, and Computation for Time-Delay Systems"*, SIAM Advances in Design and Control, 2014.

13 C. Briat, *"LPV & Time-Delay Systems Analysis, Observation, Filtering & Control"*, Vol. 3, Springer-Heidelberg, Germany, 2015.

14 R. Siphai, S. Niculescu, C. Abdallah, W. Michels, K. Gu, *"Stability and stabilization of systems with time delay"*, IEEE Control Systems Magazine 2, 2011 38-65.

# Input-output $\mathcal{L}_2$ stability

## Glimpse of stability approximation on a delay example[15]

$$\mathbf{y}(s) = \frac{1}{1+s+e^{-s}}\mathbf{u}(s)$$

```
% xdt(t) = -x(t) - x(t-1) + u
%   y(t) =  x(t)
delayT(1) = struct('delay',1,'a',-1,'b',0,'c',0,'d',0);
Hss = delayss(-1,1,1,0,delayT);
% MOR Toolbox
W = logspace(-2,2,200);
mor.stability(Hss,W)

% Stable = close to machine precision
% Unstable, otherwise
ans =

    6.0729e-26
```

15 A. Seuret & C. P-V., *"SMS exchanges"*, March 12th, 2019.

## Input-output $\mathcal{L}_2$ stability

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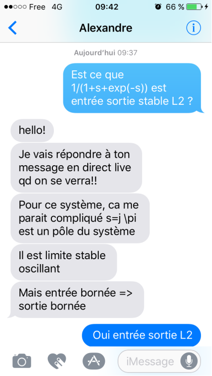


15 A. Seuret & C. P-V., *"SMS exchanges"*, March 12th, 2019.

## Input-output $\mathcal{L}_2$ stability

### $\mathcal{L}_2$ MFSA, Meromorphic Function Stability Approximation[16]

**Require:** $\mathbf{H} \in \mathcal{L}_2$, $\{\omega_i\}_{i=1}^N \in \mathbb{R}_+$, $N \in \mathbb{N}$ and $\epsilon \in \mathbb{R}_+$ (typically twice machine precision)

Sample $\mathbf{H}$ and obtain $\{\omega_i, \mathbf{\Phi}_i\}_{i=1}^N$

Perform an exact Loewner interpolation and obtain $\hat{\mathbf{H}}$ equipped with $\hat{\mathcal{S}} \in \mathbb{S}^0_{n,n_y,n_u}$ which ensures interpolatory conditions

Compute $\hat{\mathbf{H}}_s$ with realisation $\hat{\mathcal{S}}_+ \in \mathbb{S}^+_{n,n_y,n_u}$, the best stable approximation of $\hat{\mathbf{H}}$

Compute the stability index as $S = ||\hat{\mathbf{H}}_s - \hat{\mathbf{H}}||_{\mathcal{L}_2}$

**if** $S < \epsilon$ **then**

  $\mathbf{H}$ is stable

**else**

  $\mathbf{H}$ is unstable

**end if**

**No proof, yet... but some arguments are coming...**

[16] C. P-V. and P. Vuillemin, *"Input-output stability estimation of $\mathcal{L}_2(\imath\mathbb{R})$ functions"*, in maybe somewhere.

## Input-output $\mathcal{L}_2$ stability

### Arguments to derive an algorithm[17] [18]

The conjecture we claim is in twofolds:

1. One is always able to find a rational model $\hat{\mathbf{H}} \in \mathcal{RL}_2$ that well reproduces $\mathbf{H} \in \mathcal{L}_2$, whatever the complexity of $\mathbf{H}$ is, if we can arbitrarily increase $r$, the dimension of $\hat{\mathbf{H}}$.
   This can be achieved by increasing the Loewner matrix up to a numerical rank loss.
2. If, based on an unstable realisation of $\hat{\mathbf{H}} \in \mathcal{RL}_2$, the optimal stable approximant $\hat{\mathbf{H}}_s \in \mathcal{H}_2$ is close enough to $\hat{\mathbf{H}} \in \mathcal{RL}_2$, in the sense of the $\mathcal{L}_2$-norm, then $\hat{\mathbf{H}}$ is stable and, following previous statement (1.), $\mathbf{H}$ is stable too.
   This step can be achieved by a rational stable approximation followed by a norm computation which threshold is fixed to machine precision.

[17] I. Pontes Duff, P. Vuillemin, C. P-V, C. Briat and C. Seren, *"Approximation of stability regions for large-scale time-delay systems using model reduction techniques"*, in Proceedings of the 14th European Control Conference (ECC'15), Linz, Austria, July, 2015.

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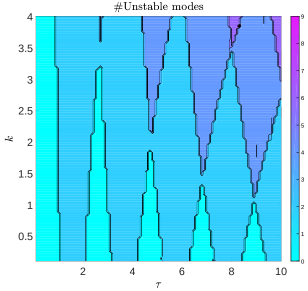
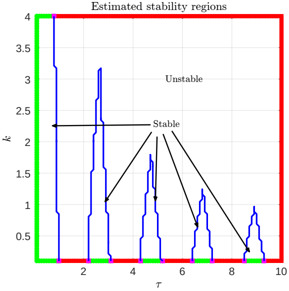
# Input-output $\mathcal{L}_2$ stability

## Time delay examples[19]

$$\mathbf{H}_1(s) = ke^{-\tau s}/(s^2 + w_0^2 - ke^{-\tau s})$$



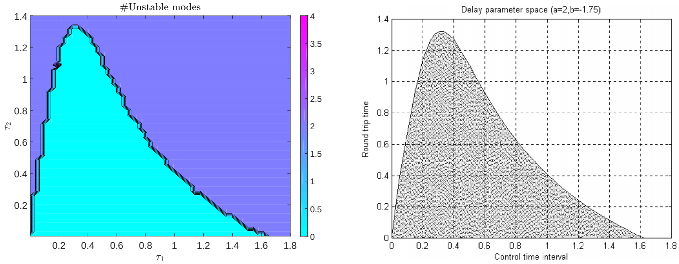

- Left: brute force interpolation-based
- Right: coupled with boundary search algorithm

19 R. Siphai, S. Niculescu, C. Abdallah, W. Michels, K. Gu, *"Stability and stabilization of systems with time delay"*, IEEE Control Systems Magazine 2 (2011) pp. 38-65.

## Input-output $\mathcal{L}_2$ stability

### Time delay examples[20]

$$\mathcal{S}_2 : \dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau_1) + A_2\mathbf{x}(t-\tau_1-\tau_2)$$



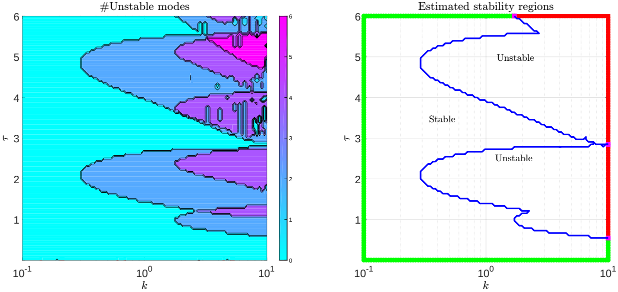

- Left: brute force interpolation-based
- Right: result obtained in S.I. Niculescu's paper

20 S-I. Niculescu, *"On delay robustness analysis of a simple control algorithm in high-speed networks"*, Automatica 38 (2002) pp. 885 - 889.

# Input-output $\mathcal{L}_2$ stability

## Time delay examples[21]

$$\mathcal{S}_3 : \dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau) + \mathbf{b}\mathbf{u}(t),\ \mathbf{y}(t) = \mathbf{c}^T\mathbf{x}(t)$$




- Left: brute force interpolation-based
- Right: coupled with boundary search algorithm

21 A. Seuret, F. Gouaisbaut, *"Hierarchy of LMI conditions for the stability analysis of time-delay systems"*, Systems & Control Letters 81 (2015) pp. 1-7.

## Conclusion

### What to keep in mind?

Model approximation tailored to large-scale models (of course) but also...

- **Infinite dimensional models** (such as)*e.g.* delay, irrational functions)
- **Input - output stability estimation** (and norms)

## Conclusion

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The **MOR Toolbox** is appropriate to treat

- finite order large-scale dynamical models
- infinite order dynamical models
- input-output data dynamical models approximation
- norm and stability estimation of $\mathcal{L}_2$ complex meromorphic functions

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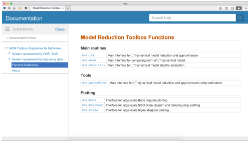

# Approximation of infinite dimensional linear dynamical models

... and its applications

Charles Poussot-Vassal

ONERA
THE FRENCH AEROSPACE LAB

March 2019
COMET and MOSAR Workshop

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