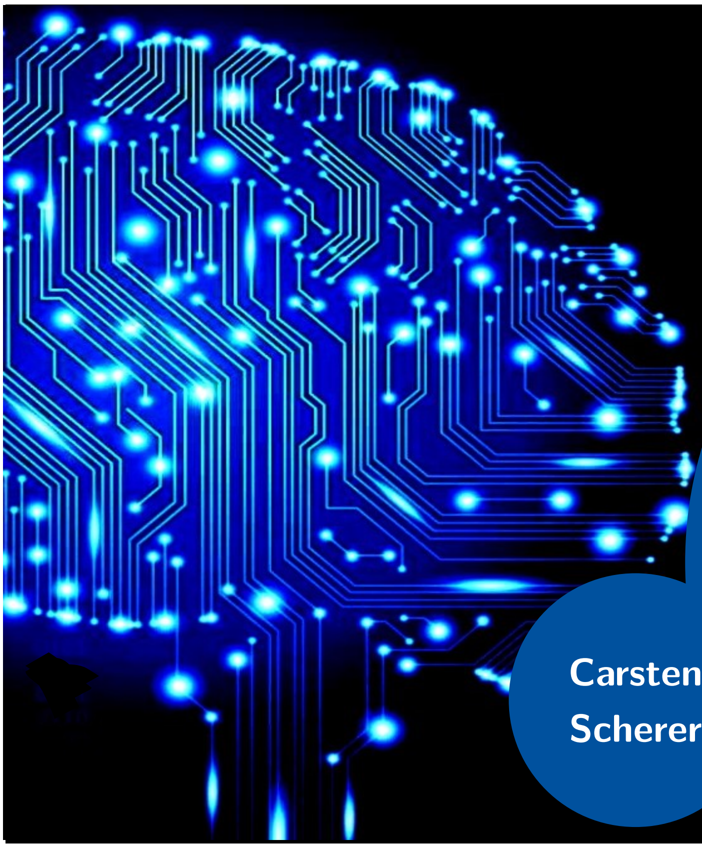
University of Stuttgart Germany

SimTech

mst



A Tutorial about IQCs for Robust Analysis

Carsten Scherer

Supported by
the German Research Foundation (DFG)
within the Cluster of Excellence
Data-Integrated Simulation Science (EXC 2075)

Outline

The standard robustness framework

Hard IQCs: Circle criterion as paradigm example

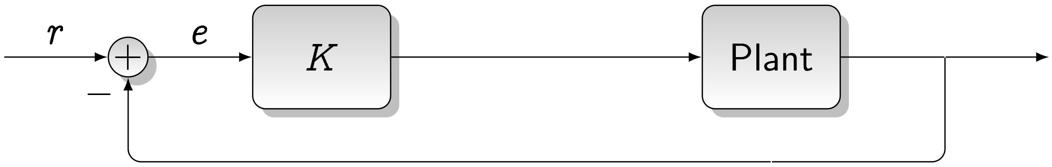
General IQC theorem: Dynamic multipliers

Ramifications

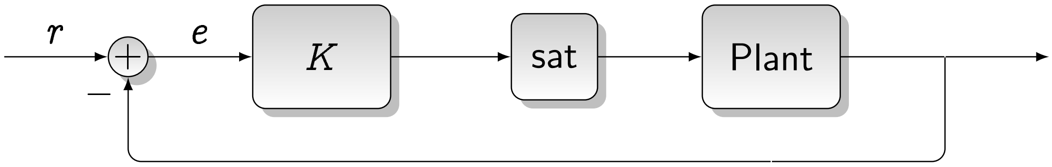
Conclusions and outlook

Classical Control Loop

Classical multi-input multi-output feedback loop:



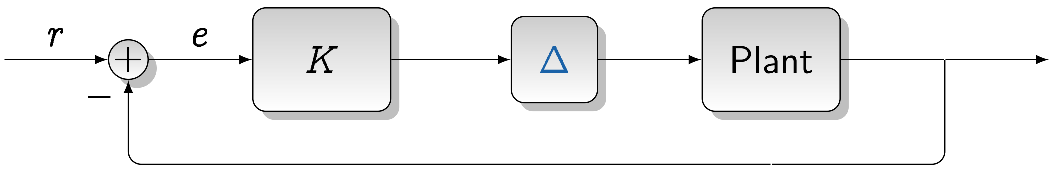
Saturation at plant input:



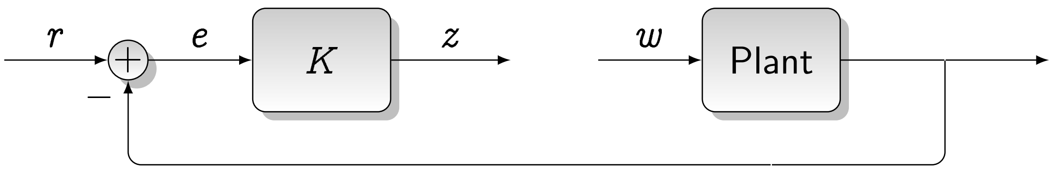
Are stability and performance preserved?

Classical Control Loop

View complicating block as uncertainty Δ:

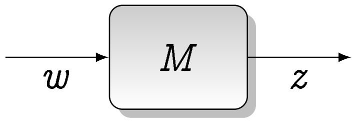


Give names to input and output of Δ and disconnect:

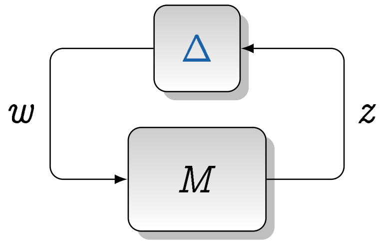


Standard Configuration: Analysis

System with input w and output z compactly written as $z = Mw$:



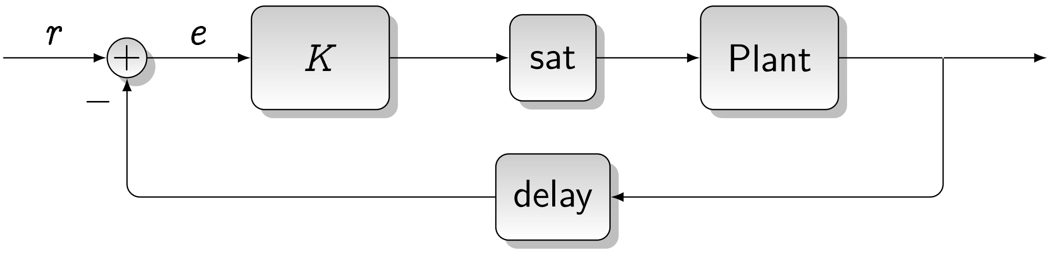
Original system obtained by reconnecting Δ as $w = \Delta(z)$:



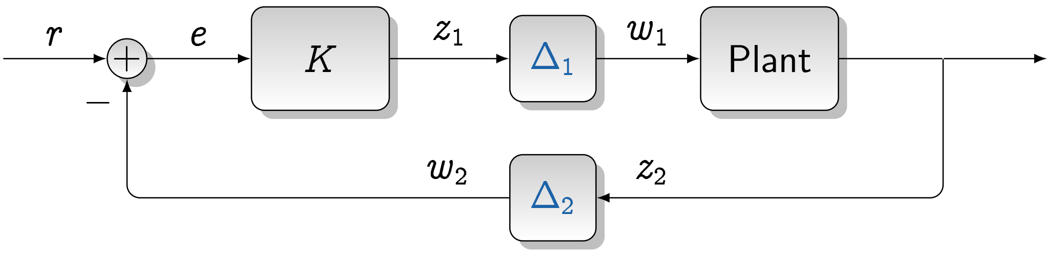
Classical configuration of absolute stability and robust control!

Classical Control Loop

Saturation at plant input and delay at plant output:

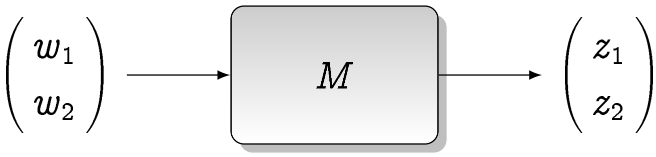


Now rewrite with **two** uncertainties as

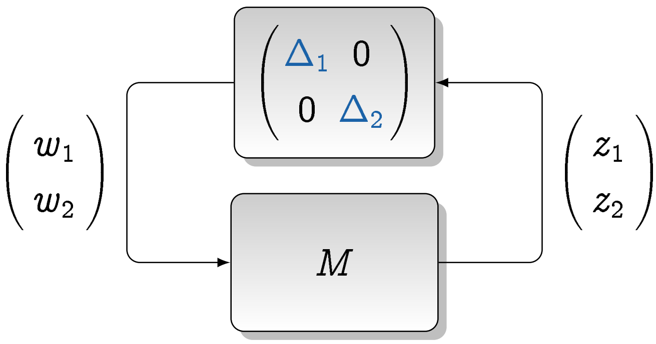


Standard Configuration: Analysis

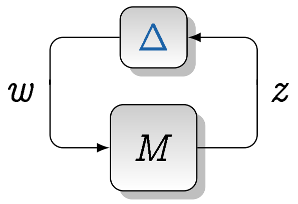
After disconnecting uncertainties we get



Original system obtained with $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \Delta_1(z_1) \\ \Delta_2(z_2) \end{pmatrix} := \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$:



Motivation



This configuration is extremely **flexible**:

- M comprises information about specific control configuration
- Δ represents complicating elements or uncertainties
- Is MIMO loop: Can capture structured systems/uncertainties

Provides **unified framework** for developing theory/algorithms:

- Just one configuration for multitude of interconnections
- M typically is linear time-invariant system
- Δ captured by input-output properties (abstraction)
- Highly modular

Example I

Time-varying uncertain system:

$$\dot{x}(t) = \begin{pmatrix} -1 & 2\delta_1(t) \\ -\frac{1}{2+\delta_1(t)} & -0.1 + 3\delta_2(t) \end{pmatrix} x(t) \ \text{ with } \ |\delta_1(t)| \leq r, \ \ |\delta_2(t)| \leq r.$$

Can be written as nominal system

$$\dot{x}(t) = \begin{pmatrix} -1 & 0 \\ -.5 & -0.1 \end{pmatrix} x(t) + \left(\begin{array}{cc|c} 0 & 2 & 0 \\ -.5 & -2 & 1.5 \end{array}\right) \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}$$

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \left(\begin{array}{cc} -.5 & -4 \\ 0 & 1 \\ \hline 0 & 2 \end{array}\right) x(t) + \left(\begin{array}{cc|c} -.5 & -2 & 1.5 \\ 0 & 0 & 0 \\ \hline 0 & 1 & 0 \end{array}\right) \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}$$

with the time-varying feedback gains

$$w_1(t) = \delta_1(t) z_1(t) \ \text{ and } \ w_2(t) = \delta_2(t) z_2(t).$$

Example I

Time-varying uncertain system:

$$\dot{x}(t) = \begin{pmatrix} -1 & 2\delta_1(t) \\ -\frac{1}{2+\delta_1(t)} & -0.1 + 3\delta_2(t) \end{pmatrix} x(t) \ \text{ with } \ |\delta_1(t)| \leq r, \ \ |\delta_2(t)| \leq r.$$

Compactly expressed as a **nominal** linear system

$$\dot{x} = Ax + Bw, \quad z = Cx + Dw$$

in feedback with the **uncertainty**

$$w = \Delta(z).$$

The uncertainty Δ is a system which takes the input signal $z(.)$ into the output signal $w(.)$ according to the law

$$w(t) = \left(\begin{array}{cc|c} \delta_1(t) & 0 & 0 \\ 0 & \delta_1(t) & 0 \\ \hline 0 & 0 & \delta_2(t) \end{array}\right) z(t).$$

Example II

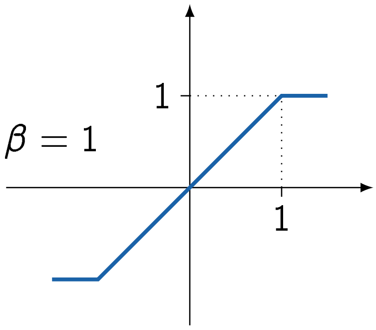
Nonlinear system

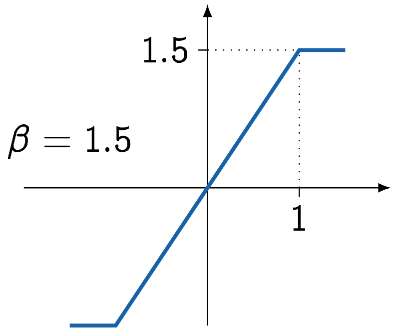
$$\dot{x}(t) = Ax(t) + B\,\mathrm{sat}_\beta(Cx(t))$$

with saturation function

$$\mathrm{sat}_\beta(z) = \begin{cases} \beta z & \text{for} \quad |z| \leq 1 \\ \beta\,\mathrm{sign}(z) & \text{for} \quad |z| > 1 \end{cases}$$

Graphs of saturation functions:





Example II

Compactly described as feedback interconnection

$$\left.\begin{aligned} \dot{x} &= Ax + Bw \\ z &= Cx \end{aligned}\right\} \quad \text{and} \quad w = \Delta(z)$$

with Δ taking the input $z(.)$ into the output $w(.)$ as

$$w(t) = \mathrm{sat}_\beta(z(t)).$$

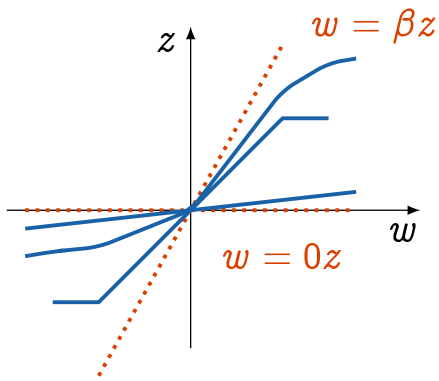
Question of **absolute stability theory:**

Is loop stable for all

$$w(t) = \varphi(z(t))$$

with a static nonlinearity φ which satisfies the **sector condition**

$$\varphi(z)(\beta z - \varphi(z)) \geq 0 \quad \text{for} \quad z \in \mathbb{R}.$$

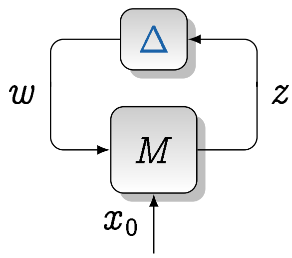


Outline

Setup

$$w = \Delta(z)$$

$$\begin{aligned} \dot{x} &= Ax + Bw, \quad x(0) = x_0 \\ z &= Cx + Dw \end{aligned}$$



Classical feedback interconnection with **linear system** in forward path and **uncertainty** in feedback path.

- Linear system **nominally stable**:

  All eigenvalues of A are in open left-half plane.

  Transfer matrix denoted as $M(s) = C(sI - A)^{-1}B + D$.

- Uncertainty Δ very general. Diagonal combination of systems:

  Linear, nonlinear, dynamic, infinite-dimensional, ...

Notion of Stability

Interconnection is **stable** if the trajectory of

$$\left.\begin{aligned} \dot{x} &= Ax + Bw, \quad x(0) = x_0 \\ z &= Cx + Dw \end{aligned}\right\} \text{ and } \; w = \Delta(z)$$

for any initial condition x_0 generates signal $w(.)$ of finite energy:

$$\|w\|^2 := \int_0^\infty w(t)^T w(t)\, dt < \infty.$$

Can then infer that $x(.)$ and $z(.)$ are of finite energy and

$$\lim_{t\to\infty} x(t) = 0.$$

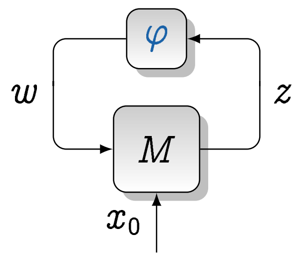
Remark. With $d(t) = Ce^{At}x_0$ the interconnection is equivalent to

$$\left.\begin{aligned} \dot{x} &= Ax + Bw, \qquad x(0) = 0 \\ z &= Cx + Dw + d \end{aligned}\right\} \text{ and } \; w = \Delta(z).$$

Example: Sector Bounded Nonlinearity

Sector bounded Lipschitz nonlinearity:

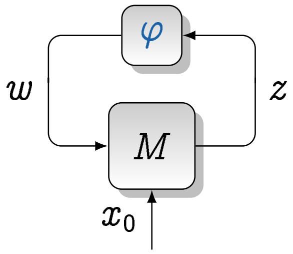
$$\varphi(z)(\beta z - \varphi(z)) \geq 0 \quad \text{for} \quad z \in \mathbb{R}.$$



Example: Sector Bounded Nonlinearity

Sector bounded Lipschitz nonlinearity:

$$\varphi(z)(\beta z - \varphi(z)) \geq 0 \quad \text{for} \quad z \in \mathbb{R}.$$



This can be also expressed as

$$\begin{pmatrix} z \\ \varphi(z) \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 1 & -\frac{2}{\beta} \end{pmatrix} \begin{pmatrix} z \\ \varphi(z) \end{pmatrix} \geq 0$$

for all $z \in \mathbb{R}$.

Circle Criterion

Loop stability guaranteed by frequency domain inequality **(FDI)**

$$\begin{pmatrix} M(i\omega) \\ 1 \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ 1 & -\frac{2}{\beta}1 \end{pmatrix} \begin{pmatrix} M(i\omega) \\ 1 \end{pmatrix} \prec 0 \quad \text{for all} \quad \omega \in [0, \infty].$$

Often expressed as $\text{Re}(M(i\omega)) < \frac{1}{\beta}$ or $\text{Re}(1 - \beta M(i\omega)) > 0$ for all ω.

Intermezzo: KYP Lemma

Let T be a real rational transfer matrix with realization

$$T(s) = C(sI - A)^{-1}B + D \quad \text{where} \quad \text{eig}(A) \cap i\mathbb{R} = \emptyset.$$

Strict Kalman-Yakubovich-Popov Lemma

For any real matrix $P = P^T$ the following statements are equivalent:

- The following frequency domain inequality holds:

$$T(i\omega)^* P T(i\omega) \prec 0 \quad \text{for all} \quad \omega \in [0, \infty].$$

- There exists some $X = X^T$ that satisfies the LMI

$$\begin{pmatrix} A^T X + XA & XB \\ B^T X & 0 \end{pmatrix} + \begin{pmatrix} C & D \end{pmatrix}^T P \begin{pmatrix} C & D \end{pmatrix} \prec 0.$$

Many different formulations exist in literature. This seems the cleanest.

Dissipation Proof: Circle Criterion

Recall FDI

$$\begin{pmatrix} M(i\omega) \\ 1 \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ 1 & -\frac{2}{\beta}1 \end{pmatrix} \begin{pmatrix} M(i\omega) \\ 1 \end{pmatrix} \prec 0 \quad \text{for all} \quad \omega \in [0, \infty].$$

Obtain realization

$$\begin{pmatrix} M(s) \\ 1 \end{pmatrix} = \begin{pmatrix} C \\ 0 \end{pmatrix} (sI - A)^{-1} B + \begin{pmatrix} D \\ 1 \end{pmatrix} \quad \text{with} \quad \text{eig}(A) \subset \mathbb{C}^-.$$

KYP Lemma: FDI implies existence of $X = X^T$ with

$$\begin{pmatrix} A^T X + XA & XB \\ B^T X & 0 \end{pmatrix} + \begin{pmatrix} C & D \\ 0 & 1 \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 1 & -\frac{2}{\beta} \end{pmatrix} \begin{pmatrix} C & D \\ 0 & 1 \end{pmatrix} \prec 0.$$

Left-upper block reads as $A^T X + XA \prec 0$ and hence $X \succ 0$.

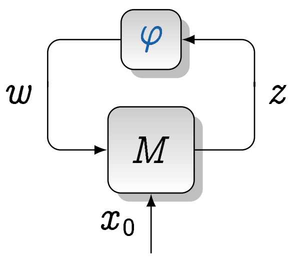
Dissipation Proof: Circle Criterion

Trajectories of interconnection satisfy

$$w(t) = \varphi(z(t))$$

$$\dot{x}(t) = Ax(t) + Bw(t), \; x(0) = x_0$$

$$\begin{pmatrix} z(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} C & D \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ w(t) \end{pmatrix}$$



Perturb LMI: There exists some $\varepsilon > 0$ with

$$\begin{pmatrix} A^T X + XA & XB \\ B^T X & 0 \end{pmatrix} + \begin{pmatrix} C & D \\ 0 & 1 \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 1 & -\frac{2}{\beta} + \varepsilon \end{pmatrix} \begin{pmatrix} C & D \\ 0 & 1 \end{pmatrix} \prec 0.$$

Along trajectory get for all $t \geq 0$:

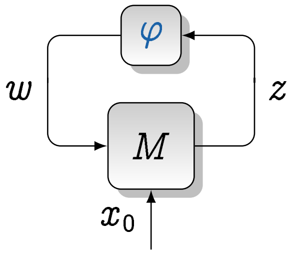
$$\dot{x}(t)^T X x(t) + x(t)^T X \dot{x}(t) + \begin{pmatrix} z(t) \\ w(t) \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 1 & -\frac{2}{\beta} + \varepsilon \end{pmatrix} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} \leq 0.$$

Dissipation Proof: Circle Criterion

Trajectories of interconnection satisfy

$$w(t) = \varphi(z(t))$$

$$\dot{x}(t) = Ax(t) + Bw(t), \; x(0) = x_0$$
$$\begin{pmatrix} z(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} C & D \\ 0 & I \end{pmatrix} \begin{pmatrix} x(t) \\ w(t) \end{pmatrix}$$



Perturb LMI: There exists some $\varepsilon > 0$ with

$$\begin{pmatrix} A^T X + XA & XB \\ B^T X & 0 \end{pmatrix} + \begin{pmatrix} C & D \\ 0 & 1 \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 1 & -\frac{2}{\beta} + \varepsilon \end{pmatrix} \begin{pmatrix} C & D \\ 0 & 1 \end{pmatrix} \prec 0.$$

Along trajectory get for all $t \geq 0$:

$$\frac{d}{dt} x(t)^T X x(t) + \varepsilon\, w(t)^T w(t) + \begin{pmatrix} z(t) \\ w(t) \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 1 & -\frac{2}{\beta} \end{pmatrix} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} \leq 0.$$

Dissipation Proof: Circle Criterion

Integration on $[0, T]$ implies for all $T > 0$:

$$x(T)^T X x(T) - x_0^T X x_0 + \int_0^T \varepsilon\, w(t)^T w(t)\, dt +$$
$$+ \int_0^T \begin{pmatrix} z(t) \\ w(t) \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 1 & -\frac{2}{\beta} \end{pmatrix} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} dt \leq 0.$$

Exploit $w(t) = \varphi(z(t))$ and sector condition to infer for all $T > 0$:

$$\int_0^T \begin{pmatrix} z(t) \\ w(t) \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 1 & -\frac{2}{\beta} \end{pmatrix} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} dt \geq 0.$$

We hence infer for all $T > 0$:

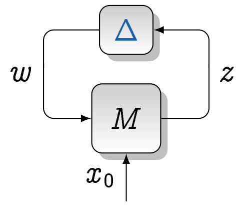
$$x(T)^T X x(T) + \int_0^T \varepsilon\, w(t)^T w(t)\, dt \leq x_0^T X x_0.$$

Since $X \succ 0$ we get stability: $\int_0^\infty w(t)^T w(t)\, dt \leq \frac{1}{\varepsilon} x_0^T X x_0 < \infty.$

Hard IQC-Theorem

$$w = \Delta(z)$$

$$\begin{aligned} \dot{x} &= Ax + Bw, \quad x(0) = x_0 \\ z &= Cx + Dw \end{aligned}$$



Let Δ satisfy the hard **I**ntegral **Q**uadratic **C**onstraint **(IQC)**

$$\int_0^T \begin{pmatrix} z(t) \\ w(t) \end{pmatrix}^T P \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} dt \geq 0 \quad \text{for all} \quad T > 0$$

for any input $z(.)$ and uncertainty output $w = \Delta(z)$.

Hard IQC Theorem. Stability is guaranteed if the LMI

$$\begin{pmatrix} A^T X + XA & XB \\ B^T X & 0 \end{pmatrix} + \begin{pmatrix} C & D \\ 0 & I \end{pmatrix}^T P \begin{pmatrix} C & D \\ 0 & I \end{pmatrix} \prec 0.$$

does have a solution $X \succ 0$.

Example I+II

Time-varying uncertain system saturated system:

$$\dot{x}(t) = \begin{pmatrix} -1 & 2\delta_1(t) \\ -\frac{1}{2+\delta_1(t)} & -0.1 + 3\delta_2(t) \end{pmatrix} x(t) + \begin{pmatrix} \text{sat}_\beta(x_1(t)) \\ 0 \end{pmatrix}.$$

Rewrite as linear system

$$\dot{x}(t) = \underbrace{\begin{pmatrix} -1 & 0 \\ -.5 & -0.1 \end{pmatrix}}_{A} x(t) + \underbrace{\left(\begin{array}{cc|c|c} 0 & 2 & 0 & 1 \\ -.5 & -2 & 1.5 & 0 \end{array}\right)}_{B} \begin{pmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{pmatrix}$$

$$\begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{pmatrix} = \underbrace{\left(\begin{array}{cc} -.5 & -4 \\ 0 & 1 \\ \hline 0 & 2 \\ \hline 1 & 0 \end{array}\right)}_{C} x(t) + \underbrace{\left(\begin{array}{cc|c|c} -.5 & -2 & 1.5 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)}_{D} \begin{pmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{pmatrix}$$

in feedback with

$$w_1(t) = \delta_1(t) z_1(t), \quad w_2(t) = \delta_2(t) z_2(t) \quad \text{and} \quad w_3(t) = \text{sat}_\beta(z_3(t)).$$

Example I+II

If $|\delta_1(t)| \leq r$ and $|\delta_2(t)| \leq r$ the trajectories with

$$w_1(t) = \delta_1(t) z_1(t), \quad w_2(t) = \delta_2(t) z_2(t) \quad \text{and} \quad w_3(t) = \mathrm{sat}_\beta(z_3(t))$$

satisfy the hard IQC

$$\int_0^T \begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ \hline w_1(t) \\ w_2(t) \\ w_3(t) \end{pmatrix}^T \underbrace{\left(\begin{array}{ccc|ccc} D & 0 & 0 & \frac{1}{r}G & 0 & 0 \\ 0 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & h \\ \hline \frac{1}{r}G^T & 0 & 0 & -\frac{1}{r^2}D & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{r^2}d & 0 \\ 0 & 0 & h & 0 & 0 & -\frac{2}{\beta}h \end{array}\right)}_{P} \begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ \hline w_1(t) \\ w_2(t) \\ w_3(t) \end{pmatrix} dt \geq 0$$

in case that

$$D \succ 0, \quad G + G^T = 0 \quad \text{and} \quad d > 0 \quad \text{and} \quad h > 0.$$

Is a **routine** combination of static D/G-scalings and sector condition.

Example I+II

Test whether there exists a **multiplier** P in above class such that

$$X \succ 0, \quad \begin{pmatrix} A^TX + XA & XB \\ B^TX & 0 \end{pmatrix} + \begin{pmatrix} C & D \\ 0 & I \end{pmatrix}^T P \begin{pmatrix} C & D \\ 0 & I \end{pmatrix} \prec 0.$$

- Is standard LMI problem. Very easy to implement e.g. via Yalmip.
- If answer is yes we have guaranteed stability.
- If answer is no the test might be conservative.

  Can use full block multipliers for (often drastic) improvements!

Old and recently emerging again: Can work with **time-varying** $P(t)$ structured as above and find solution $X(t) \gg 0$ of the **differential LMI**

$$\begin{pmatrix} \dot{X}(t) + A^TX(t) + X(t)A & X(t)B \\ B^TX(t) & 0 \end{pmatrix} + \begin{pmatrix} C & D \\ 0 & I \end{pmatrix}^T P(t) \begin{pmatrix} C & D \\ 0 & I \end{pmatrix} \ll 0.$$

Large Variety of Techniques

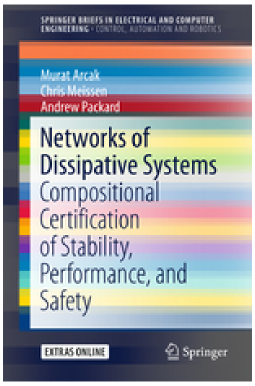
- Input-Output Approach
  - Small-gain, passivity, conic separation (Zames)
  - Topological separation (Safonov)
  - Stability multipliers (Desoer, Vidyasagar)
  - Integral quadratic constraints (Megretski, Rantzer)
- Dissipativity Approach
  - Absolute stability (Popov, Yakubovich, Brockett, J.L. Willems)
  - Theory of dissipative dynamical systems (J.C. Willems)
  - Abundance of LMI results in literature

Linked through Kalman-Yakubovich-Popov Lemma.

A long-standing gap was closed only recently!

General Framework

- Exhibits **general principle** behind huge variety of stability tests:
  - to handle structured uncertainties in **robust control**
  - allowing general **operator** uncertainties
  - for **networked** interconnected systems



- Extends seamlessly to **performance**
- Basis for robust and LPV **synthesis**
  Controller transformation or elimination
- Various extensions **tricky**:
  Popov, Yakubovich, delays

Trouble: Does not work for much more powerful **dynamic IQCs**!

Outline

Example III

Time-invariant uncertain system saturated system:

$$\dot{x}(t) = \begin{pmatrix} -1 & 2\delta \\ -\frac{1}{2+\delta} & -0.1 \end{pmatrix} x(t) + \begin{pmatrix} \mathrm{sat}_\beta(x_1(t)) \\ 0 \end{pmatrix}, \quad \delta \in [0, r].$$

Reduce conservatism with **frequency-dependent** D/G scalings for δ.
Reduce conservatism with **Zames-Falb multiplier** for sat_β.

Recall that L_2 denotes the set of **finite energy** signals on $[0, \infty)$:

$$||x||^2 := \int_0^\infty x(t)^T x(t)\, dt < \infty.$$

Such signals have a **Fourier transform** denoted as $\hat{x}$.

Uncertainties are assumed to be stable in the following sense:

$$z \in L_2 \;\text{ implies }\; \Delta(z) \in L_2.$$

For $z \in L_2$ it makes sense to consider $\hat{w}$ for $w = \Delta(z)$.

Example III

Time-invariant uncertain system saturated system:

$$\dot{x}(t) = \begin{pmatrix} -1 & 2\delta \\ -\frac{1}{2+\delta} & -0.1 \end{pmatrix} x(t) + \begin{pmatrix} \text{sat}_\beta(x_1(t)) \\ 0 \end{pmatrix}, \quad \delta \in [0, r].$$

Reduce conservatism with **frequency-dependent** D/G scalings for δ.

Suppose the transfer matrix H has no poles in $i\mathbb{R}$ and satisfies

$$H(i\omega)^* + H(i\omega) \succ 0 \;\text{ for }\; \omega \in [0, \infty].$$

For $\delta \in [0, r]$ and $w(t) = \delta z(t)$ with $z \in L_2$ we have

$$\int_{-\infty}^{\infty} \begin{pmatrix} \hat{z}(i\omega) \\ \hat{w}(i\omega) \end{pmatrix}^* \begin{pmatrix} 0 & H(i\omega)^* \\ H(i\omega) & -\frac{1}{r}[H(i\omega)^* + H(i\omega)] \end{pmatrix} \begin{pmatrix} \hat{z}(i\omega) \\ \hat{w}(i\omega) \end{pmatrix} d\omega \geq 0.$$

This is a **frequency-domain** IQC. Looks bombastic but is a triviality!

Why?

Clearly $w(t) = \delta z(t)$ implies $\hat{w}(i\omega) = \delta \hat{z}(i\omega)$. Then observe

$$\begin{pmatrix} \hat{z}(i\omega) \\ \hat{w}(i\omega) \end{pmatrix}^* \begin{pmatrix} 0 & H(i\omega)^* \\ H(i\omega) & -\frac{1}{r}[H(i\omega)^* + H(i\omega)] \end{pmatrix} \begin{pmatrix} \hat{z}(i\omega) \\ \hat{w}(i\omega) \end{pmatrix} =$$

$$= \hat{z}(i\omega)^* \begin{pmatrix} I \\ \delta I \end{pmatrix}^T \begin{pmatrix} 0 & H(i\omega)^* \\ H(i\omega) & -\frac{1}{r}[H(i\omega)^* + H(i\omega)] \end{pmatrix} \begin{pmatrix} I \\ \delta I \end{pmatrix} \hat{z}(i\omega) =$$

$$= \hat{z}(i\omega)^* \left[[H(i\omega)^* + H(i\omega)]\, \delta \left(1 - \frac{1}{r}\delta\right) \right] \hat{z}(i\omega) \geq 0.$$

Integration over frequency gives IQC.

Example III

Time-invariant uncertain system saturated system:

$$\dot{x}(t) = \begin{pmatrix} -1 & 2\delta \\ -\frac{1}{2+\delta} & -0.1 \end{pmatrix} x(t) + \begin{pmatrix} \text{sat}_\beta(x_1(t)) \\ 0 \end{pmatrix}, \quad \delta \in [0, r].$$

Reduce conservatism with **Zames-Falb multiplier** for sat_β.

Let $h(i\omega) := g - \hat{f}(i\omega)$ and the inverse Fourier transform of f satisfy

$$f(t) \geq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} f(t)\, dt < g.$$

Then $w(t) = \text{sat}_\beta(z(t))$ with $z \in L_2$ satisfies

$$\int_{-\infty}^{\infty} \begin{pmatrix} \hat{z}(i\omega) \\ \hat{w}(i\omega) \end{pmatrix}^* \begin{pmatrix} 0 & h(i\omega)^* \\ h(i\omega) & -\frac{1}{\beta}[h(i\omega)^* + h(i\omega)] \end{pmatrix} \begin{pmatrix} \hat{z}(i\omega) \\ \hat{w}(i\omega) \end{pmatrix} d\omega \geq 0.$$

Classical but not obvious. Not so well-known that it's due to **convexity**!

Key Ideas of Proof

The potential $V(x) := \int_0^x \text{sat}_\beta(z)\, dz$ is **convex** since sat_β is monotone.

The standard subgradient inequality from convex analysis implies

$$V'(x)\,(x - y) \geq V(x) - V(y) \quad \text{for all} \quad x, y \in \mathbb{R}.$$

Since $w(t) = \text{sat}_\beta(z(t))$ we hence get for all $t, \tau \in \mathbb{R}$:

$$w(t)\,(z(t) - z(t-\tau)) \geq V(z(t)) - V(z(t-\tau)).$$

Since z has finite energy we infer

$$\int_{-\infty}^{\infty} w(t)\,(z(t) - z(t-\tau))\, dt \geq 0.$$

Since $f(\tau) \geq 0$ and $g > \int_{-\infty}^{\infty} f(\tau)\, d\tau$ we get

$$\int_{-\infty}^{\infty} w(t) \left(g z(t) - \int_{-\infty}^{\infty} f(\tau) z(t-\tau)\, d\tau \right) dt \geq 0.$$

Parseval gives the IQC for $\beta = \infty$:

$$\int_{-\infty}^{\infty} \hat{w}(i\omega)^*\, h(i\omega) \hat{z}(i\omega)\, d\omega \geq 0.$$

Parametrization of Multipliers

With any stable transfer matrix ψ one parameterizes H as

$$H = \psi^* Q \psi \quad \text{with a real matrix } \; Q.$$

Example. If H is SISO and

$$\psi(s) = \begin{pmatrix} 1 \\ \frac{1}{s+1} \\ \frac{1}{(s+1)^2} \end{pmatrix}, \quad Q = \begin{pmatrix} q_{11} & q_{12} & q_{12} \\ q_{21} & 0 & 0 \\ q_{31} & 0 & 0 \end{pmatrix}$$

we get

$$\psi^* Q \psi = q_{31} \frac{1}{(1-s)^2} + q_{21} \frac{1}{1-s} + q_{11} + q_{12} \frac{1}{1+s} + q_{31} \frac{1}{(1+s)}.$$

Parametrization of Multipliers

With any stable transfer matrix ψ one parameterizes H as

$$H = \psi^* Q \psi \quad \text{with a real matrix } \; Q.$$

Observe that we directly obtain

$$\begin{pmatrix} 0 & H^* \\ H & -\frac{1}{r}[H^* + H] \end{pmatrix} = \begin{pmatrix} \psi & -\frac{1}{r}\psi \\ 0 & \psi \end{pmatrix}^* \underbrace{\begin{pmatrix} 0 & Q^T \\ Q & 0 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} \psi & -\frac{1}{r}\psi \\ 0 & \psi \end{pmatrix}}_{\Psi}.$$

This motivates that multipliers in IQC theory are often described as

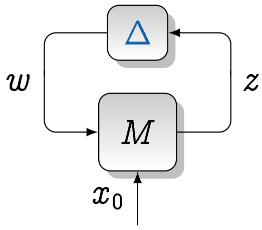
$$\Psi^* P \Psi$$

with a **fixed stable dynamic filter** Ψ and a **real symmetric structured matrix variable** P that is contained in a convex set described by LMIs.

IQC-Theorem

$$w = \Delta(z)$$

$$\begin{aligned} x &= Ax + Bw, \quad x(0) = x_0 \\ z &= Cx + Dw \end{aligned}$$



For any $z \in L_2$ let $w = \Delta(z) \in L_2$ depend **causally** on z and satisfy

$$\int_{-\infty}^{\infty} \begin{pmatrix} \hat{z}(i\omega) \\ \hat{w}(i\omega) \end{pmatrix}^* \Psi(i\omega)^* P \Psi(i\omega) \begin{pmatrix} \hat{z}(i\omega) \\ \hat{w}(i\omega) \end{pmatrix} d\omega \geq 0.$$

IQC Theorem. Stability is guaranteed if

$$\begin{pmatrix} M(i\omega) \\ I \end{pmatrix}^* \Psi(i\omega)^* P \Psi(i\omega) \begin{pmatrix} M(i\omega) \\ I \end{pmatrix} \prec 0 \text{ for all } \omega \in [0, \infty]$$

and the multiplier is positive/negative.

Variant of **Megretski, Rantzer (97)**

Dissipation Proof

Stability FDI:

$$\begin{pmatrix} M(i\omega) \\ I \end{pmatrix}^* \Psi(i\omega)^* P \Psi(i\omega) \begin{pmatrix} M(i\omega) \\ I \end{pmatrix} \prec 0 \text{ for all } \omega \in [0, \infty].$$

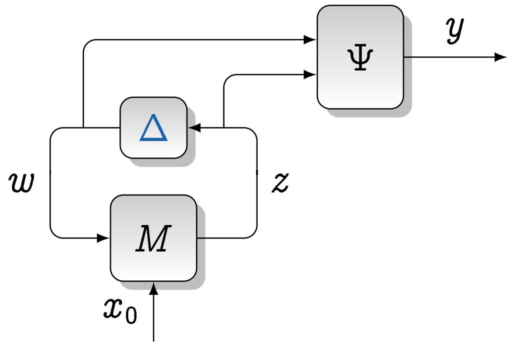
With minimal realization $\Psi = \begin{pmatrix} \Psi_1 & \Psi_2 \end{pmatrix} = \left[\begin{array}{c|cc} A_\Psi & B_{\Psi 1} & B_{\Psi 2} \\ \hline C_\Psi & D_{\Psi 1} & D_{\Psi 2} \end{array}\right]$ get

$$\Psi \begin{pmatrix} M \\ I \end{pmatrix} = \left[\begin{array}{cc|c} A_\Psi & B_{\Psi 1} C & B_{\Psi 1} D + B_{\Psi 2} \\ 0 & A & B \\ \hline C_\Psi & D_{\Psi 1} C & B_{\Psi 1} D + B_{\Psi 2} \end{array}\right] =: \left[\begin{array}{c|c} A_f & B_f \\ \hline C_f & D_f \end{array}\right], \quad \sigma(A_f) \subset \mathbb{C}^-.$$

KYP Lemma: Stability FDI implies existence of $X = X^T$ with

$$\begin{pmatrix} A_f^T X + X A_f & X B_f \\ B_f^T X & 0 \end{pmatrix} + \begin{pmatrix} C_f & D_f \end{pmatrix}^T P \begin{pmatrix} C_f & D_f \end{pmatrix} \prec 0.$$

Dissipation Proof



Literally as in the static case we get dissipation inequality for $T > 0$:

$$\begin{pmatrix} x_\psi(T) \\ x(T) \end{pmatrix}^T X \begin{pmatrix} x_\psi(T) \\ x(T) \end{pmatrix} - \begin{pmatrix} x_\psi(0) \\ x_0 \end{pmatrix}^T X \begin{pmatrix} x_\psi(0) \\ x_0 \end{pmatrix} + \\ + \int_0^T \varepsilon\, w(t)^T w(t)\, dt + \int_0^T y(t)^T P y(t)\, dt \leq 0.$$

Trouble: Neither $X \succ 0$ nor $\int_0^T y(t)^T P y(t)\, dt \geq 0$ are true any more!

Technical Result

Theorem. The FDIs

$$\begin{pmatrix} M \\ I \end{pmatrix}^* \Psi^* P \Psi \begin{pmatrix} M \\ I \end{pmatrix} \overset{i\mathbb{R}}{\prec} 0 \quad \text{and} \quad \Psi_1^* P \Psi_1 \overset{i\mathbb{R}}{\succ} 0$$

guarantee the existence of stabilizing solution Z of ARE

$$A_\Psi^T Z + Z A_\Psi + C_\Psi^T P C_\Psi - \\ - (Z B_\Psi + C_\Psi^T P D_\Psi)(D_\Psi^T P D_\Psi)^{-1}(B_\Psi^T Z + D_\Psi^T P C_\Psi) = 0.$$

Moreover, all solutions $X = X^T$ of

$$\begin{pmatrix} A_f^T X + X A_f & X B_f \\ B_f^T X & 0 \end{pmatrix} + \begin{pmatrix} C_f & D_f \end{pmatrix}^T P \begin{pmatrix} C_f & D_f \end{pmatrix} \prec 0$$

satisfy the coupling condition $X - \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix} \succ 0.$

Special case: **Seiler (15)**, **Veenman, S (13)** General case: **S, Veenman (18)**

Consequences for Uncertainty

It is **routine** (Parseval) that the IQC

$$\int_{-\infty}^{\infty} \begin{pmatrix} \hat{z}(i\omega) \\ \hat{w}(i\omega) \end{pmatrix}^{*} \Psi(i\omega)^{*} P \Psi(i\omega) \begin{pmatrix} \hat{z}(i\omega) \\ \hat{w}(i\omega) \end{pmatrix} d\omega \geq 0$$

translates for $y = \Psi \begin{pmatrix} z \\ w \end{pmatrix}$ into the **infinite horizon** time-domain IQC

$$\int_0^T y(t)^T P y(t)\, dt + \int_T^\infty y(t)^T P y(t)\, dt \geq 0.$$

Theorem. Suppose the multiplier also satisfies

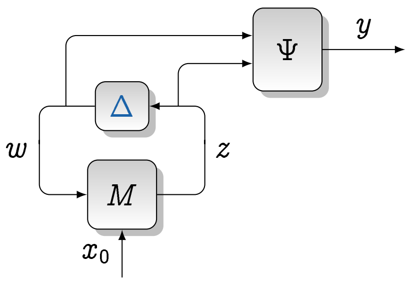
$$\Psi_2^* P \Psi_2 \overset{i\mathbb{R}}{\preccurlyeq} 0.$$

Then the following finite horizon IQC with terminal cost holds:

$$\int_0^T y(t)^T P y(t)\, dt + x_\Psi(T)^T Z x_\Psi(T) \geq 0 \quad \text{for all} \quad T \geq 0.$$

Special case: **Pfifer, Seiler (16)** General case: **S, Veenman (18)**

Dissipation Proof



Recall dissipation inequality for $T > 0$:

$$\begin{pmatrix} x_\psi(T) \\ x(T) \end{pmatrix}^T X \begin{pmatrix} x_\psi(T) \\ x(T) \end{pmatrix} - \begin{pmatrix} x_\psi(0) \\ x_0 \end{pmatrix}^T X \begin{pmatrix} x_\psi(0) \\ x_0 \end{pmatrix} + \\ + \int_0^T \varepsilon\, w(t)^T w(t)\, dt + \int_0^T y(t)^T P y(t)\, dt \leq 0.$$

Combined with finite horizon IQC we get for $T > 0$:

$$\begin{pmatrix} x_\psi(T) \\ x(T) \end{pmatrix}^T \left[X - \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix} \right] \begin{pmatrix} x_\psi(T) \\ x(T) \end{pmatrix} - \begin{pmatrix} x_\psi(0) \\ x_0 \end{pmatrix}^T X \begin{pmatrix} x_\psi(0) \\ x_0 \end{pmatrix} + \\ + \int_0^T \varepsilon\, w(t)^T w(t)\, dt \leq 0.$$

Dissipation Proof

Conclusions: For all x_0 infer that $w \in L_2$ (stability) and $T > 0$:

$$\begin{pmatrix} x_\psi(T) \\ x(T) \end{pmatrix}^T \left[X - \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix} \right] \begin{pmatrix} x_\psi(T) \\ x(T) \end{pmatrix} \leq \begin{pmatrix} x_\psi(0) \\ x_0 \end{pmatrix}^T X \begin{pmatrix} x_\psi(0) \\ x_0 \end{pmatrix}$$

- Dissipation proof of IQC theorem for **positive/negative** multipliers:

$$\Psi^* P \Psi = \begin{pmatrix} \succcurlyeq 0 & * \\ * & \preccurlyeq 0 \end{pmatrix}.$$

  Recently handled general case and multi-valued uncertainties.

  S, Veenman (18), S, Holicki (18)

- Benefit: Absolute stability criteria imply hard time-domain constraints!

  Fetzer, S, Veenman (18)

- Permits to **routinely** merge IQC stability results with a multitude of existing time-domain dissipation constraints.

Geralization of IQC Theorem

New IQC Theorem

- Let Δ satisfy a finite-horizon IQC with terminal cost Z:

$$\int_0^T y(t)^T P y(t)\, dt + x_\Psi(T)^T Z x_\Psi(T) \geq 0 \quad \text{for all} \quad T \geq 0$$

  holds along all filtered trajectories $y = \Psi \begin{pmatrix} z \\ \Delta(z) \end{pmatrix}$.

- Let there exists a solution X of

$$\begin{pmatrix} A_f^T X + X A_f & X B_f \\ B_f^T X & 0 \end{pmatrix} + \begin{pmatrix} C_f & D_f \end{pmatrix}^T P \begin{pmatrix} C_f & D_f \end{pmatrix} \prec 0$$

  that satisfies $X - \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix} \succ 0.$

Then the above **conclusions** can be drawn.

No additional assumptions whatsoever required! **S, Veenman (18)**

Example III

Time-invariant uncertain system saturated system:

$$\dot{x}(t) = \begin{pmatrix} -1 & 2\delta \\ -\frac{1}{2+\delta} & -0.1 \end{pmatrix} x(t) + \begin{pmatrix} \text{sat}_\beta(x_1(t)) \\ 0 \end{pmatrix}, \quad \delta \in [0, r].$$

Rewrite as linear system

$$\dot{x}(t) = \underbrace{\begin{pmatrix} -1 & 0 \\ -.5 & -0.1 \end{pmatrix}}_{A} x(t) + \underbrace{\left(\begin{array}{cc|c} 0 & 2 & 1 \\ -.5 & -2 & 0 \end{array}\right)}_{B} \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}$$

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \underbrace{\left(\begin{array}{cc} -.5 & -4 \\ 0 & 1 \\ \hline 1 & 0 \end{array}\right)}_{C} x(t) + \underbrace{\left(\begin{array}{cc|c} -.5 & -2 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)}_{D} \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}$$

in feedback with

$$w_1(t) = \delta z_1(t), \quad w_2(t) = \text{sat}_\beta(z_2(t)).$$

Example III

Time-invariant uncertain system saturated system:

$$\dot{x}(t) = \begin{pmatrix} -1 & 2\delta \\ -\frac{1}{2+\delta} & -0.1 \end{pmatrix} x(t) + \begin{pmatrix} \text{sat}_\beta(x_1(t)) \\ 0 \end{pmatrix}, \quad \delta \in [0, r].$$

Combine earlier individual IQCs for causal and stable uncertainties

$$w_1(t) = \delta z_1(t), \quad w_2(t) = \text{sat}_\beta(z_2(t))$$

to infer

$$\int_{-\infty}^{\infty} \left(\begin{array}{c} \hat{z}_1 \\ \hat{z}_2 \\ \hline \hat{w}_1 \\ \hat{w}_2 \end{array}\right)^* \underbrace{\left(\begin{array}{cc|cc} 0 & 0 & H^* & 0 \\ 0 & 0 & 0 & h^* \\ \hline H & 0 & -\frac{1}{r}[H^* + H] & 0 \\ 0 & h^* & 0 & -\frac{1}{\beta}[h^* + h] \end{array}\right)}_{\Pi} \left(\begin{array}{c} \hat{z}_1 \\ \hat{z}_2 \\ \hline \hat{w}_1 \\ \hat{w}_2 \end{array}\right) d\omega \geq 0.$$

This is a positive/negative multiplier by inspection!

Example III

Time-invariant uncertain system saturated system:

$$\dot{x}(t) = \begin{pmatrix} -1 & 2\delta \\ -\frac{1}{2+\delta} & -0.1 \end{pmatrix} x(t) + \begin{pmatrix} \mathrm{sat}_\beta(x_1(t)) \\ 0 \end{pmatrix}, \quad \delta \in [0, r].$$

The parametrizations

$$H = \psi_1^* Q_1 \psi_1 \ \text{ and } \ h = \psi_2^* Q_2 \psi_2$$

routinely lead to

$$\Pi = \underbrace{\left(\begin{array}{cc|cc} \psi_1 & 0 & -\frac{1}{r}\psi_1 & 0 \\ 0 & \psi_2 & 0 & -\frac{1}{\beta}\psi_2 \\ \hline 0 & 0 & \psi_1 & 0 \\ 0 & 0 & 0 & \psi_2 \end{array}\right)^*}_{} \underbrace{\left(\begin{array}{cc|cc} 0 & 0 & Q_1^T & 0 \\ 0 & 0 & 0 & Q_2^T \\ \hline Q_1 & 0 & 0 & 0 \\ 0 & Q_2 & 0 & 0 \end{array}\right)}_{P} \underbrace{\left(\begin{array}{cc|cc} \psi_1 & 0 & -\frac{1}{r}\psi_1 & 0 \\ 0 & \psi_2 & 0 & -\frac{1}{\beta}\psi_2 \\ \hline 0 & 0 & \psi_1 & 0 \\ 0 & 0 & 0 & \psi_2 \end{array}\right)}_{\Psi}.$$

Exactly in right format to implement stability FDI as LMI!

Example III

Time-invariant uncertain system saturated system:

$$\dot{x}(t) = \begin{pmatrix} -1 & 2\delta \\ -\frac{1}{2+\delta} & -0.1 \end{pmatrix} x(t) + \begin{pmatrix} \mathrm{sat}_\beta(x_1(t)) \\ 0 \end{pmatrix}, \quad \delta \in [0, r].$$

Constraints for parametrizations $H = \psi_1^* Q_1 \psi_1$ and $h = \psi_2^* Q_2 \psi_2$:

For dynamic D/G multiplier observe that

$$H^* + H = \psi_1^* \left[Q_1^T + Q_1 \right] \psi_1 \overset{i\mathbb{R}}{\succ} 0$$

is equivalent to existence of solution $Y_1 = Y_1^T$ of the LMI

$$\begin{pmatrix} A_{\psi_1}^T Y_1 + Y_1 A_{\psi_1} & Y_1 B_{\psi_1} \\ B_{\psi_1}^T Y_1 & 0 \end{pmatrix} + \begin{pmatrix} C_{\psi_1} & D_{\psi_1} \end{pmatrix}^T \left[Q_1^T + Q_1 \right] \begin{pmatrix} C_{\psi_1} & D_{\psi_1} \end{pmatrix} \succ 0.$$

Straightforward to implement as LMI constraint!

Example III

Time-invariant uncertain system saturated system:

$$\dot{x}(t) = \begin{pmatrix} -1 & 2\delta \\ -\frac{1}{2+\delta} & -0.1 \end{pmatrix} x(t) + \begin{pmatrix} \text{sat}_\beta(x_1(t)) \\ 0 \end{pmatrix}, \quad \delta \in [0, r].$$

Constraints for parametrizations $H = \psi_1^* Q_1 \psi_1$ and $h = \psi_2^* Q_2 \psi_2$:
For Zames-Falb multiplier choose

$$\psi_2(s) = \begin{pmatrix} 1 \\ \frac{1}{s+1} \\ \frac{1}{(s+1)^2} \end{pmatrix}, \quad Q_2 = \begin{pmatrix} q_{11} & q_{12} & q_{12} \\ q_{21} & 0 & 0 \\ q_{31} & 0 & 0 \end{pmatrix}$$

and recall

$$\psi^* Q \psi = q_{31}\frac{1}{(1-s)^2} + q_{21}\frac{1}{1-s} + q_{11} + q_{12}\frac{1}{1+s} + q_{31}\frac{1}{(1+s)}.$$

Impose easy to implement LP constraints

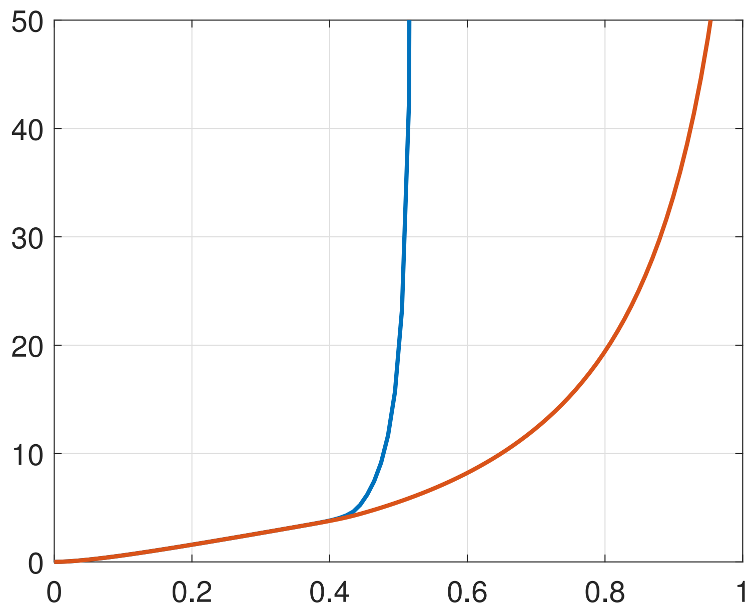
$$q_{31}, q_{21}, q_{12}, q_{13} \geq 0 \quad \text{and} \quad q_{11} \geq q_{31} + q_{21} + q_{12} + q_{13}.$$

Example III: Results

Take $r = 1$, $\beta = 1$. Plot guaranteed L_2-gain bounds of $d \mapsto w$ in

$$z = Mw + d, \quad w = \tau\Delta(z)$$

over $\tau \in [0, 1]$ for **static** and **dynamic** multipliers:

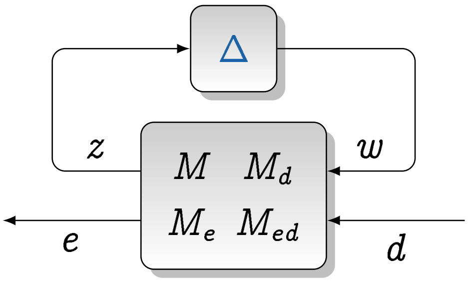


Lessons

- IQC theory is encompassing classical and modern approaches:
  - absolute stability theory
  - μ-theory
  - dissipativity theory
- Is highly flexible and modular:
  - easy to combine uncertainties of diverse nature
  - permits compositional safety verification of complex systems
- Has close links to Lyapunov approach:
  - via dissipativity theory
  - often more powerful/more insightful
- It is not difficult to apply!

Outline

IQC-Theorem: Performance



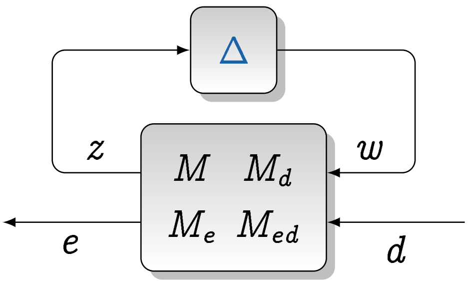
Work with FDI

$$\begin{pmatrix} M & M_d \\ I & 0 \\ \hline M_e & M_{ed} \\ 0 & I \end{pmatrix}^{*} \left(\begin{array}{c|cc} \Psi^* P \Psi & & \\ \hline & I & 0 \\ & 0 & -\gamma^2 I \end{array}\right) \begin{pmatrix} M & M_d \\ I & 0 \\ \hline M_e & M_{ed} \\ 0 & I \end{pmatrix} \overset{i\mathbb{R}}{\prec} 0$$

to guarantee stability and $||e|| \leq \gamma ||d||$ for all loop responses.

Can compute best achievable performance levels γ by LMIs.

IQC-Theorem and Dissipativity Theory

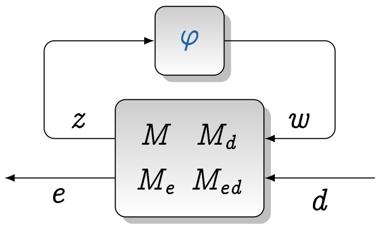


Link to dissipativity theory:

- Straightforward **extension** to time-varying/LPV systems
  - Just replace KYP Lemma by time-varying/LPV versions
  - Can incorporate all classically known multipliers
- Permits **local** stability/performance analysis
  - Guarantee robust ellipsoidal bounds on output
  - Exploit locality to reduce conservatism

Extended IQC-Theorem: New Result

$$w \in \varphi(z)$$
$$z = Mw + M_d d$$
$$e = M_e w + M_{ed} d$$



$\varphi : \mathbb{R} \rightrightarrows \mathbb{R}$ is subdifferential of convex $f : \mathbb{R} \to \mathbb{R}$ with $0 \in \varphi(0)$.

Example: $f(x) = |x|$ leads to $\varphi(x) = \begin{cases} 1 & \text{for } x > 0 \\ [-1, 1] & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$

New: Can use Zames-Falb multipliers in IQC-Theorem.

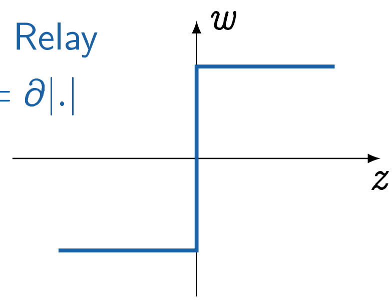
Extended IQC-Theorem: New Result

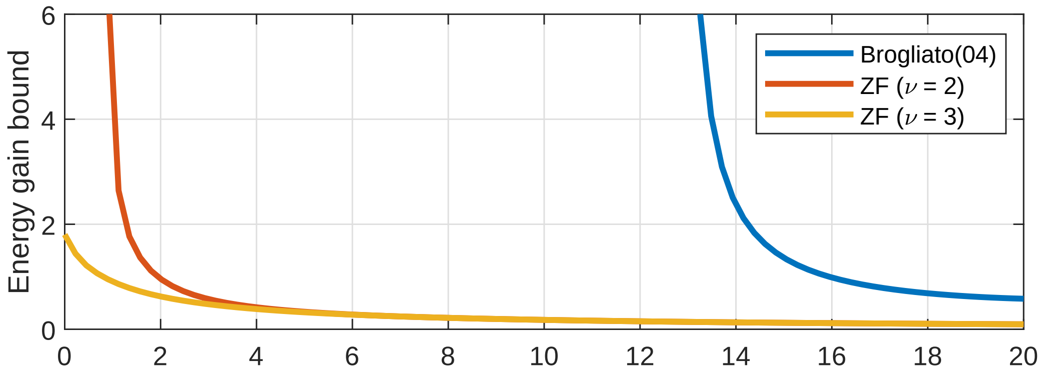
Relay systems:

- Switching control
- Unilateral constraints
- Complementarity systems

Ideal Relay

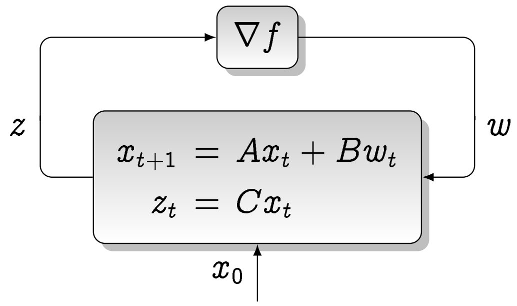
$\varphi = \partial|.|$





Family of systems depending on $\rho \in [0, 20]$

IQC-Theorem in Discrete-Time



Optimization algorithms for strongly convex $f : \mathbb{R}^n \to \mathbb{R}$:

- Gradient descent is a first order linear system.
- Nesterov proposed **accelerated** gradient descent:
  - Much better practical performance
  - Proved fast convergence by estimation sequence
- Better convergence rate show with **first order** Zames-Falb multiplier

Lessard, Recht, Packard (16)

Conclusions and Outlook

- Surveyed classical and more recent IQC-theory
  - Clarified the link to dissipativity theory
  - Illustrated how to apply the framework
  - Discussed the crucial benefits
- Interesting issues
  - Scalability: Exploit interconnection structure
  - Solvers: Dedicated and stable algorithms
  - Controller **synthesis**
- Publications related to this talk:
  https://www.imng.uni-stuttgart.de/mst/publications/