Conditions asymptotiquement nécessaires et suffisantes pour des inégalités polynomiales matricielles du second ordre

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Summary

- Introduction and previous results
- Main Result
- Numerical Examples
- Conclusion and Perspectives

Introduction

The negativeness of second order matrix-valued polynomials:

$$P(\tau) = \tau^2\Phi_2 + \tau\Phi_1 + \Phi_0 < 0, \quad (1)$$

where $\Phi_i \in \mathbb{R}^{n\times n}$ $(i=0,1,2)$ and $\tau \in [\underline{\tau}, \bar{\tau}]$.

- Often occurs for stability analysis or synthesis in the Time-varying delay systems framework. For instance when considering Looped Lyapunov-Krasovskii Functionals (LKF) in Sampled-Data controller design (see e.g. [Gao et al., 2020]).
- Extensive recent studies are made to provide relaxed LMI conditions satisfying (1) (see e.g. the recent survey in [Zhang et al., 2022] or the recent results in [Liu et al., 2023]).

Goal of this paper:

- to provide further relaxed LMI-based conditions (or at least an efficient alternative),
- to show that such approach may also be useful for some standard robust control problem, going beyond the traditional context of Time-Varying delay systems.

Overview of usual and recent approaches

The negativeness of second order matrix-valued polynomials:

$$P(\tau) = \tau^2\Phi_2 + \tau\Phi_1 + \Phi_0 < 0,$$

where $\Phi_i \in \mathbb{R}^{n\times n}$ $(i = 0, 1, 2)$ and $\tau \in [\underline{\tau}, \bar{\tau}]$.

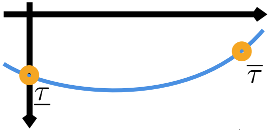
How to get LMI-based conditions satisfying (1)?

Geometrical based methods

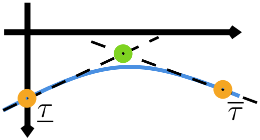
NS conditions inspired by robust control techniques

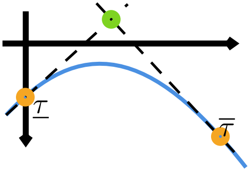
Overview of usual and recent approaches: Geometric methods

[Kim, 2011]: $\forall\tau\in[\underline{\tau},\bar{\tau}]$, the matrix-valued polynomial inequality (1) holds if $P(\underline{\tau})<0$, $P(\bar{\tau})<0$ and $\Phi_2\geq 0$.

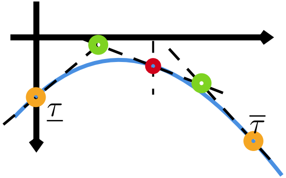


[Park and Park, 2020]: $\forall\tau\in[\underline{\tau},\bar{\tau}]$, the matrix-valued polynomial inequality (1) holds if $P(\underline{\tau})<0$, $P(\bar{\tau})<0$ and $P(\underline{\tau})+P(\bar{\tau})-\Delta\tau^2<0$.





[Liu et al., 2023]: $\forall\tau\in[\underline{\tau},\bar{\tau}]$ and a given integer $N\in\mathbb{N}^*$, the quadratic polynomial inequality (1) holds if $P(\underline{\tau})<0$, $P(\bar{\tau})<0$, and $P\left(\underline{\tau}+\frac{i-1}{N}\Delta\tau\right)+P\left(\underline{\tau}+\frac{i}{N}\Delta\tau\right)-\frac{1}{N^2}\Delta\tau^2\Phi_2<0$, $\forall i\in\mathbb{I}_N^*$.



Overview of usual and recent approaches: NS Conditions

[Chen et al., 2022, de Oliveira and Souza, 2020]: $\forall \tau \in [\underline{\tau}, \bar{\tau}]$, the quadratic polynomial inequality (1) holds if and only if there exist $0 < D = D^\top \in \mathbb{R}^{p \times p}$ and a skew-symmetric matrix $G \in \mathbb{R}^{p \times p}$ such that:

$$\begin{bmatrix} P(\underline{\tau}) & \frac{1}{2}\Phi_1 + \underline{\tau}\Phi_2 \\ \star & \Phi_2 \end{bmatrix} < \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}^\top \begin{bmatrix} -D & G \\ \star & D \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$$

where $H_1 = \begin{bmatrix} \Delta\tau I & 0 \end{bmatrix}$ and $H_2 = \begin{bmatrix} \Delta\tau I & -2I \end{bmatrix}$

[Park and Park, 2020]: $\forall \tau \in [\underline{\tau}, \bar{\tau}]$, the quadratic polynomial inequality (1) holds if and only if if there exists $0 \leq M + M^\top \in \mathbb{R}^{p \times p}$ such that:

$$\begin{bmatrix} P(\underline{\tau}) & \frac{1}{2}\Phi_1 + \underline{\tau}\Phi_2 + \Delta\tau M \\ \star & \Phi_2 - M - M^\top \end{bmatrix} < 0 \qquad (2)$$

.

- Shown to be equivalent in [Zhang et al., 2022];
- Usual concerns about these NS in the literature: introduction of additional decision variables, problems for large-sized matrix inequalities which already involves a huge number of decision variables;
- There is still extensive research efforts to provide less conservative sufficient conditions!

In the sequel, we proposed a new alternative...

Main Result

Summarized by the following Theorem, we provides new LMI conditions based on:

- partitioning the polynomial parameter range,
- rewriting (1) as an homogeneous polynomial constraint,
- applying Young's inequality for more relaxed conditions.

Theorem

For a pre-fixed number of partitioning intervals $N \in \mathbb{N}^*$, the quadratic polynomial inequality (1) holds $\forall \tau \in [\underline{\tau}, \bar{\tau}]$ such that the inequalities:

$$\begin{array}{ll} \text{I) } P(\underline{\tau})<0, & \text{II) } P(\bar{\tau}_i)<0, \\ \text{III) } 2P(\underline{\tau}_i)+T(\underline{\tau}_i,\bar{\tau}_i)<0, & \text{IV) } 2P(\bar{\tau}_i)+T(\underline{\tau}_i,\bar{\tau}_i)<0, \end{array} \quad (3)$$

are satisfied with $T(\underline{\tau}_i,\bar{\tau}_i) = 2\underline{\tau}_i\bar{\tau}_i\Phi_2 + (\bar{\tau}_i - \underline{\tau}_i)\Phi_1 + 2\Phi_0$, $\underline{\tau}_i = \underline{\tau} + \frac{(i-1)(\bar{\tau}-\underline{\tau})}{N}$ and $\bar{\tau}_i = \underline{\tau} + \frac{i(\bar{\tau}-\underline{\tau})}{N}$.

Main Result - Proof

- For any given $N \in \mathbb{N}^*$, consider the partition of the interval range of the parameter τ as $[\underline{\tau}, \bar{\tau}] = \cup_{i=1}^{N} [\underline{\tau}_i, \bar{\tau}_i]$.

- $\forall i \in \mathbb{I}_N^*$ and $\forall \tau \in [\underline{\tau}_i, \bar{\tau}_i]$, we define:

$$\alpha_{1i} = \frac{(\tau - \underline{\tau}_i)N}{\Delta\tau} \text{ and } \alpha_{2i} = \frac{(\bar{\tau}_i - \tau)N}{\Delta\tau} \tag{4}$$

where $\alpha_{1i} \in [0,1]$, $\alpha_{2i} \in [0,1]$ and $\alpha_{1i} + \alpha_{2i} = 1$

- We have that $\boxed{\tau = \alpha_{1i}\bar{\tau}_i + \alpha_{2i}\underline{\tau}_i}$, therefore the matrix-valued polynomial (1) can be rewritten as:

$$(\alpha_{1i}\bar{\tau}_i + \alpha_{2i}\underline{\tau}_i)^2 \Phi_2 + (\alpha_{1i}\bar{\tau}_i + \alpha_{2i}\underline{\tau}_i)\Phi_1 + \Phi_0 < 0 \tag{5}$$

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Main Result - Proof

- That is to say, by homogenization, since $(\alpha_{1i}+\alpha_{2i})^2=\alpha_{1i}+\alpha_{2i}=1$:

$$\alpha_{1i}^2 P(\bar{\tau}_i) + \alpha_{1i}\alpha_{2i} T(\underline{\tau}_i, \bar{\tau}_i) + \alpha_{2i}^2 P(\underline{\tau}_i) < 0 \tag{6}$$

which is now an second-order homogeneous polynomial in α_{1i} and α_{2i}.

- If $T(\underline{\tau}_i,\bar{\tau}_i)<0$, (6) is satisfied:

$$\boxed{\text{I) } P(\underline{\tau})<0, \quad \text{II) } P(\bar{\tau}_i)<0}$$

- If $T(\underline{\tau}_i,\bar{\tau}_i)\geq 0$, the Young inequality $\boxed{\alpha_{1i}\alpha_{2i}\leq \frac{1}{2}(\alpha_{1i}^2+\alpha_{2i}^2):}$ applies and (6) is satisfied:

$$\alpha_{1i}^2\left(P(\bar{\tau}_i) + \frac{1}{2} T(\underline{\tau}_i, \bar{\tau}_i)\right) + \alpha_{2i}^2\left(P(\underline{\tau}_i) + \frac{1}{2} T(\underline{\tau}_i, \bar{\tau}_i)\right) < 0$$

$$\Leftrightarrow \boxed{\text{III) } \quad 2P(\underline{\tau}_i)+T(\underline{\tau}_i,\bar{\tau}_i)<0, \ \text{IV) } 2P(\bar{\tau}_i)+T(\underline{\tau}_i,\bar{\tau}_i)<0}$$

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Example 1: Scalar-valued polynomial (particular case)

- For this first example, let us consider the particular case of a scalar-valued polynomial inequality:

$$P(\tau) = \tau^2 10a + \tau 10 + b - a < 0, \quad \tau \in [0,1] \tag{7}$$

where a and b are two real parameters dedicated to compare the feasibility fields of the considered conditions.

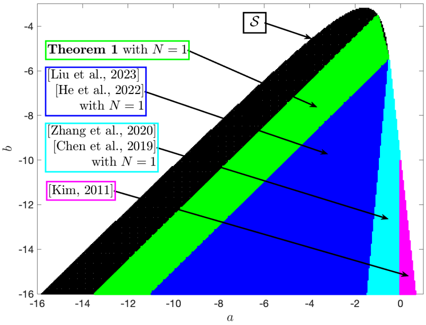
- Since (7) is a scalar quadratic inequality, from the roots of $P(\tau)$, we have $P(\tau) < 0$ for all $(a,b) \in S$ such that:

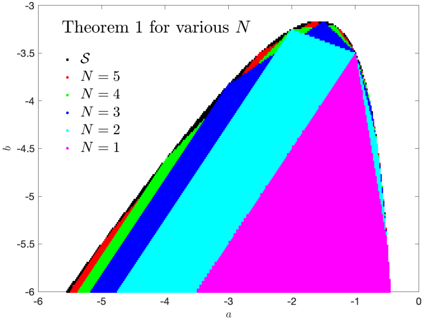
$$S = \left\{ (a,b) \in \mathbb{R}^2 : \left| \begin{array}{l} P(0) = b - a < 0, \\ P(1) = 9a + b + 10 < 0, \\ b - a - \frac{5}{2a} < 0, \text{if } -\frac{1}{2a} \in [0,1]. \end{array} \right. \right\} \tag{8}$$

This exact characterization of S will be used to evaluate the conservatism of the different considered conditions.

Example 1: Scalar-valued polynomial (particular case)

Conservatism comparison w.r.t. feasibility fields





- From this figures, we see that the conditions of Theorem 1 are less conservative than the geometrical approaches from previous literature.
- Theorem 1 provides Asymptotically Necessary and Sufficient Conditions as far as N increases!

Example 2: Robust control of a discrete-time polytopic system

- Consider a discrete-time convex polytopic system given by [Guerra and Vermeiren, 2004]:

$$x(k+1) = \sum_{i=1}^{2} \rho_i(k)(A_i x(k) + B_i u(k)) \tag{9}$$

where $A_i = \begin{bmatrix} 1 & (-1)^i \beta \\ -1 & -0.5 \end{bmatrix}$, $B_i = \begin{bmatrix} 5+(-1)^{i-1}\beta \\ 2\beta \end{bmatrix}$, $\rho_i(k) \in [0,1]$ and $\rho_1(k) + \rho_2(k) = 1$,

- and the PDC control law given by:

$$u(k) = \sum_{j=1}^{2} \rho_j(k) F_j P^{-1} x(k). \tag{10}$$

where $F_j \in \mathbb{R}^{1\times 2}$ and $P \in \mathbb{R}^{2\times 2}$ are gain matrices to be synthesized.

Example 2: Robust control of a discrete-time polytopic system

- Assuming a quadratic Lyapunov candidate function $V(x(k)) = x^T(k)P^{-1}x(k)$, with $P = P^\top > 0$, the following parameterized LMI provides the design conditions:

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\rho_i(k)\rho_j(k)\Gamma_{ij} < 0, \text{ with } \Gamma_{ij} = \begin{bmatrix} -P & -PA_i^T - F_j^T B_i^T \\ \star & -P \end{bmatrix} \qquad (11)$$

- Usual double-sums relaxation techniques can be found in the literature to solve (11), e.g.:
  - from [Tanaka et al., 1998] solutions hold $\forall \beta \in [0, 1.36]$,
  - from [Tuan et al., 2001] solutions hold $\forall \beta \in [0, 1.71]$.

- Let $\tau = \rho_1(k) \in [0,1]$, since $\rho_2(k) = 1 - \rho_1(k)$, the PLMI (11) can be rewritten as a matrix-valued polynomial inequality:

$$P(\tau) = \tau^2\Phi_2 + \tau\Phi_1 + \Phi_0 < 0$$

with $\Phi_2 = \Gamma_{11} + \Gamma_{22} - \Gamma_{12} - \Gamma_{21}$, $\Phi_1 = \Gamma_{12} + \Gamma_{21} - 2\Gamma_{22}$ and $\Phi_0 = \Gamma_{22}$.

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Example 2: Robust control of a discrete-time polytopic system

Method / N	1	2	3	4	8	13	38
[Kim, 2011]	Unfeas	-	-	-	-	-	-
[Zhang et al., 2020]	1.282	-	-	-	-	-	-
[Liu et al., 2023]	1.360	-	-	-	-	-	-
[Chen et al., 2019]	1.282	1.360	1.618	1.668	1.742	1.757	**1.765**
[He et al., 2022, Liu et al., 2023]	1.360	1.710	1.733	1.742	1.764	**1.765**	**1.765**
Theorem 1	1.710	1.742	1.756	1.764	**1.765**	**1.765**	**1.765**

Table 1: Maximum values of $\beta \in [0, \bar{\beta}]$ obtained according to the number N of partitions considered.

- Theorem 1 provides the less conservative results regarding to the previous geometrical approaches, archiving the optimal value of $\bar{\beta} = 1.765$ with a smaller number of partition N.
- Theorem 1 also overcome some usual relaxation Lemma from the convex polytopic literature (e.g. [Tanaka et al., 1998, Tuan et al., 2001]).

Conclusions and perspectives

- New Asymptotically Necessary and Sufficient conditions have been proposed for matrix-valued quadratic polynomial inequalities,
- Based on homogeneous polynomial constraints, these constitutes an alternative to usual geometrical approaches, which hasn't been investigated before,
- The conservatism reduction brought by our proposal, compared to previous results, has been illustrated through two numerical examples (leaving the usual context of time-varying delay systems).

- Extension of these conditions using Polya's Theorem and other examples for sampled-data control have already been developed but left out here for space reasons (to be submitted in a journal soon),

- We are now focusing on extending these results to higher order polynomials as well as to Multiple Polynomial LPV systems.

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