Structural properties of linear switched systems: observability, controllability, minimality

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Outline of the course

- Reminder: structural properties of linear systems.
- Observability of linear switched systems.
- Reachability/controllability of linear switched systems.
- Minimality of linear switched systems.
- Kalman-Ho realization algorithm

Linear Time Invariant (LTI) state-space representation

$$\Sigma : \begin{cases} \sigma x(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

$A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$.

$$\sigma x(t) = \begin{cases} \dot{x}(t) & \text{continuous time} \\ x(t+1) & \text{discrete time} \end{cases}$$

(A, B, C): shorthand notation.

Observability: general definition

General non-linear system

$$\sigma x(t) = f(x(t), u(t)), \;\; y = h(x(t), u(t)).$$

u – input, y – output.

$y(z, u)$ – output signal from initial state z under input u.

called observable (in the sense of indistinguishability), if

$$\forall z_1 \neq z_2 : \exists u : y(z_1, u) \neq y(z_2, u)$$

i.e. for any two initial states $z, z^{'}$ **there exists** an input $u(.)$ such that the corresponding outputs $y, y^{'}$ are different.

called observable (in the sense of state reconstruction), if

$$\forall z_1 \neq z_2 : \forall u : y(z_1, u) \neq y(z_2, u)$$

i.e. for any two initial states $z, z^{'}$ **for all** inputs $u(.)$ such that the corresponding outputs $y, y^{'}$ are different.

Observability: general definition

Observability in the sense of state reconstruction $\implies$ observability in the sense indistinguishability.

Observability in the sense of state reconstruction is necessary for observer design.

Observability in the sense of indistinguishability is necessary for minimal dimesional state-space representations.

Observability: linear case

$$f(x,u) = Ax + Bu, \;\; h(x,u) = Cx$$

Observability in the sense of state reconstruction $\iff$ observability in the sense indistinguishability.

For linear systems

$$y(z,u) = y(z,0) + y(0,u)$$
$$y(z^{'},u) = y(z^{'},0) + y(0,u)$$
$$y(z,u) \neq y(z^{'},u) \iff y(z,0) + y(0,u) \neq y(z^{'},0) + y(0,u)$$
$$\iff y(z,0) \neq y(z^{'},0)$$
$$\exists u : y(z,u) \neq y(z^{'},u) \iff y(z,0) \neq y(z^{'},0) \iff$$
$$\forall u : y(z,u) \neq y(z^{'},u)$$

Observability rank condition

(A, B, C) observable,

$\iff$

$$\mathrm{rank}\ \begin{bmatrix} C^T & A^T C^T & \cdots & (A^{n-1})^T C^T \end{bmatrix}^T = n$$

$\iff$

$$\bigcap_{k=0}^{\infty} \ker CA^k = \{0\}$$

$\iff$

$$\forall z \neq 0, \exists k \geq 0 : 0 \neq CA^k z = \begin{cases} \frac{d^k}{dt^k} y(z,0)(t)|_{t=0} & \text{cont. time} \\ y(z,0)(t) & \text{disc. time} \end{cases}$$

$\iff$

$\forall z \neq 0 : y(z,0) \neq 0 \iff \forall z_1 \neq z_2 : y(z_1 - z_2, 0) \neq 0 \iff$
$\forall z_1 \neq z_2 : y(z_1, 0) \neq y(z_2, 0)$.

Observability: application

Observability $\implies$ existence of a Luenberger-observer

Observability reduction: replace a system (A, B, C) with another one with the same input-output behavior.
Basis $b_1, \ldots, b_n$ such that $b_{o+1}, \ldots, b_n$ spans

$$\ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

In the new basis

$$A = \begin{bmatrix} A_o & 0 \\ \star & A_{uo} \end{bmatrix}, \; B = \begin{bmatrix} B_o \\ \star \end{bmatrix}, \; C = \begin{bmatrix} C_o, & 0 \end{bmatrix}$$

(A_o, B_o, C_o) is observable, has the same input-output behavior as (A, B, C).

Detour: input-output behaviors

Two different ways to view a system:

- System of equations: $\sigma x(t) = f(x(t), u(t)),\ \ y = h(x(t))$
- Set of observed input-output pairs (y, u) (see 'Behavioral approach' by Jan C. Willems).

Input-output behavior of $\sigma x(t) = f(x(t), u(t)),\ \ y = h(x(t))$

$$\mathcal{B}_{f,h} = \{(u, y) \mid \exists x : \sigma x(t) = f(x(t), u(t)),\ \ y = h(x(t))\}$$

Input-output behavior is what we want to control, state-space representation is a tool for control synthesis.

Input-output behavior, input-output function, observability

Input-output function from initial state x_0:

$$I_{f,h,x_0} : u \mapsto y \text{ s.t.} \sigma x(t) = f(x(t), u(t)),\ \ y = h(x(t)), x(0) = x_0.$$

Relationship between the two:

$$\mathcal{B}_{f,h} = \bigcup_{x_0, u} \{(I_{x_0,f,h}(u), u)\}$$

Observability (in th sense of indistinguishability) $\iff$ the function $x_0 \mapsto I_{f,h,x_0}$ is one-to-one

Observability (in the sense of state reconstruction) $\iff$ for every (u, y) there exists unique x_0 s.t. $I_{f,h,x_0}(u) = y$.

Input-output behavior of linear systems

I_{A,B,C,x_0}, $\mathcal{B}_{A,B,C}$ - input-output function I_{f,h,x_0}/input-output behavior $\mathcal{B}_{f,h}$, $f(x,u) = Ax + Bu$, $h(x,u) = Cx$.

Nice properties:

$$I_{A,B,C,x_0}(u) = I_{A,B,C,x_0}(0) + I_{A,B,C,0}(u)$$

- $I_{A,B,C,x_0}(0) = \begin{cases} Ce^{At}x_0 & \text{cont. time} \\ CA^t x_0 & \text{disc. time} \end{cases}$ – depends (linearly) on the initial state, independent of input
- $I_{A,B,C,0}(u) = \begin{cases} \int_0^t Ce^{A(t-s)}Bu(s)ds \\ \sum_{s=0}^{t-1} CA^{(t-s-1)}Bu(s) \end{cases}$ – depends (linearly) on the input, not on initial state.

  $I_{A,B,C,0} \Longleftrightarrow$ transfer function $H(s) = C(sI - A)^{-1}B$.

Input-output behavior of linear systems

- Transfer function are often identified with input-output behavior.
- But: Transfer functions do not capture all the input-output behavior.
- Transfer functions capture the input-output behavior which we can control and observe.

Motivating example

Consider two linear systems

$$\sigma x = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 2 & 1 \end{bmatrix} x$$

$$\sigma x = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} -2 & 1 \end{bmatrix} x$$

They have the same input-output behavior from zero initial state (transfer functions are the same).

Yet, $u = -2y$ stabilizes the first system, and not the second.

What is the problem ? Which model is the wrong one ?

Observability: exercise

Is (A, B, C) below observable ?

$$A = \begin{bmatrix} -3 & 0 & -1 \\ 0 & -3 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}^T$$

Use the definition and the rank condition to motivate your answer.

If it is not observable, find two states z, z' s.t. $I_{A,B,C,z} = I_{A,B,C,z'}$.

Perform observability reduction.

Input-output behavior: exercise

Consider two linear systems

$$\sigma x = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 2 & 1 \end{bmatrix} x$$

$$\sigma x = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} -2 & 1 \end{bmatrix} x$$

Do they have the same input-output function from the zero initial sate ?

Do they have the same input-output behavior ?

Observability reduction (A_o, B_o, C_o) revisited

Correspondence between input-output functions

$$I_{A_o,B_o,C_o,Px_0} = I_{A,B,C,x_0}, \; P = \begin{bmatrix} I_o \\ 0 \end{bmatrix}.$$

Transfer functions of (A_o, B_o, C_o) and (A, B, C) are equal:

$$I_{A_o,B_o,C_o,0} = I_{A,B,C,0}$$

The set of input-output functions (hence the input-output behavior) are preserved by observability reduction:

$$\bigcup_{x_o} I_{A_o,B_o,C_o} = \bigcup_{x_0} I_{A,B,C,x_0}, \; \mathcal{B}_{A,B,C} = \mathcal{B}_{A_o,B_o,C_o}.$$

Control synthesis can be done on (A_o, B_o, C_o) instead of (A, B, C) (attention, unstable unobserved modes !).

Reachability & controllability

$x(z, u)(t)$ – state of (A, B, C) at time t, under input u, initial state z.

A state z, is reachable from x_0, if $z = x(x_0, u)(T)$ for some u and T.

(A, B, C) is reachable, if all states are reachable from 0.

(A, B, C) is controllable, if for any $z, z^{'}$, there exists u and T s.t. $x(z, u)(T) = z^{'}$.

Conditions for reachabilit

(A, B, C) reachable, $\iff$

$$\text{rank} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n$$

$\iff$
$\text{Span}\{A^k Bu \mid k \geq 0, u \in \mathbb{R}^m\} = n$
$\iff$
(A^T, C^T, B^T) is observable.

Controllability (in cont. time or in disc. time if A is invertible) $\iff$ reachability.

$\text{Span}\{A^k Bu \mid k \geq 0, u \in \mathbb{R}^m\}$ set of reachable states $x(0, u)(t)$ from zero.

Conditions for reachability

Main idea:

- $\mathrm{Span}\{A^k Bu \mid k \geq 0, u \in \mathbb{R}^m\}$ is the smallest vector space which contains states reachable from zero.
- The set of states reachable from zero is a vector space.

The proof of the equivalence of controllability and reachability is difficult: if a state can be reached from zero, then zero can be reached from that state.

Reachability reduction

Basis $b_1, \ldots, b_n$ such that $b_1, \ldots, b_r$ spans

$$\mathrm{Im}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

In the new basis

$$A = \begin{bmatrix} A_r & \star \\ 0 & A_{uc} \end{bmatrix}, \; B = \begin{bmatrix} B_r \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} C_r & \star \end{bmatrix}$$

(A_r, B_r, C_r) is reachable, and has the same input-output function from the zero initial state as (A, B, C)

$$I_{A_r, B_r, C_r, 0} = I_{A,B,C,0}$$

A state which is not reachable from zero cannot be influenced by inputs.

Reachability reduction

Reachability reduction preserves the input-output function generated by initial state 0.

It is not true that (A, B, C) has the same input-output behavior as (A_r, B_r, C_r).

When replacing (A, B, C) by (A_r, B_r, C_r), we lose behavior which cannot be controlled.

Reachability & controllability

Perform reachability reduction on

$$\dot{x} = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix} x + \begin{bmatrix} -2 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

and

$$\dot{x} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

Calculate the input-output functions of the reduced systems from 0

Calculate the inpt-output functions of the original systems from $[0, 1]^T$.

Transforming an LTI to a minimal one

Minimization procedure

1. Transform (A, B, C) to a reachable (A_r, B_r, C_r) with the same input-output function from the initial state zero.
2. Transform $(A_r, B_{,r}, C_r)$ to an observable (A_m, B_m, C_m) with the same input-output function from the initial state zero.

- (A_m, B_m, C_m) is reachable and observable, its input-output function from zero is the same as (A, B, C).

$$I_{A_m,B_m,C_m,0} = I_{A,B,C,0}$$

- Dimension of (A_m, B_m, C_m) is the smallest among all $(A^{'}, B^{'}, C^{'})$ s.t.

$$I_{A',B',C',0} = I_{A,B,C,0}$$

Minimality

Let I be an input-output function.

1. (A, B, C) is a minimal dimensional system such that $I_{A,B,C,0} = I \iff (A, B, C)$ is reachable from zero, and (A, B, C) is observable.
2. If (A, B, C) and $(\hat{A}, \hat{B}, \hat{C})$ are minimal dimensional s.t. $I_{A,B,C,0} = I_{\hat{A},\hat{B},\hat{C},0} = I$ then they are isomorphic: there exists a nonsingular matrix T s.t.:

$$TAT^{-1} = \hat{A}, \; TB = \hat{B}, \; CT^{-1} = \hat{C}.$$

Consequences of minimality for control

- If two reachable and observable LTI systems have the same transfer function, then they are isomorphic and have the same input-output behavior.

  Transfer functions capture the input-output behavior of reachable and observable systems.
- Minimal LTI system which with the same transfer function isomorphic $\implies$
  control design does not depend on the choice of the LTI state-space representation.
- Minimal LTI representations are observable & controllable: observer design and stabilization is always possible.
- Unobservable/uncontrollable eigenvalues are the only potential source of problems.
- Try to use minimal systems for control.
- Further applications: system identification, model reduction.

Definition of linear switched systems

$$
\begin{aligned}
\sigma x(t) &= f(x(t), u(t)),\ \ y(t) = h(x(t), u(t)) \\
f(x, u) &= A_q x + B_q v,\ \ u = (q, v) \\
h(x, u) &= C_q x,\ \ u = (q, v)
\end{aligned}
$$

Inputs $u = (q, v)$
$q \in Q = \{1, 2, \ldots, d\}$ – discrete mode, v – continuous input

Outputs
y – continuous output

Dimension – n, the dimension of the state $x(t)$.

Linear switched systems: simplest class of hybrid systems.

$\{A_q, B_q, C_q\}_{q \in Q}$ – shorthand notation.

Expressions for the state and output

Discrete-time

$$x(x_0,(q,v))(t) = A_{q(t-1)}\cdots A_{q(0)}x_0 + \sum_{k=0}^{t-1} A_{q(t-1)}\cdots A_{q(k+1)}B_{q(k)}v(k)$$

$$y(x_0,(q,v))(t) = C_{q(t)}A_{q(t-1)}\cdots A_{q(0)}x_0 +$$

$$\sum_{k=0}^{t-1} C_{q(t)}A_{q(t-1)}\cdots A_{q(k+1)}B_{q(k)}v(k)$$

Expressions for the state and output

Continuous-time: $q(s) = q_i$ for $s \in [\sum_{j=1}^{i-1} t_j, \sum_{j=1}^{i} t_j)$, $t_j \geq 0$, $t = \sum_{j=1}^{k} t_j$.

$$x(x_0,(q,v))(t) = e^{A_{q_k} t_k} \cdots e^{A_{q_1} t_1} x_0 +$$

$$\sum_{i=1}^{k} \int_0^{t_i} e^{A_{q_k} t_k} \cdots e^{A_{q_i}(t_i - s)} B_{q_i} u(s + \sum_{j=1}^{i-1} t_j)$$

$$y(x_0,(q,v))(t) = C_{q_k} e^{A_{q_k} t_k} \cdots e^{A_{q_1} t_1} x_0 +$$

$$\sum_{i=1}^{k} \int_0^{t_i} C_{q_k} e^{A_{q_k} t_k} \cdots e^{A_{q_i}(t_i - s)} B_{q_i} u(s + \sum_{j=1}^{i-1} t_j)$$

Expressions for state and output trajectories

$Q = \{1, 2, 3\}$

Discrete-time: $q(0) = 1$, $q(1) = 2$, $q(2) = 1$, $q(3) = 2$. Write $x(t), y(t)$ for $t = 0, 1, 2, 3$.

Continuous-time: $k = 4$, $q_1 = 1, q_2 = 2, q_3 = 1, q_4 = 2$. Write $x(t), y(t)$.

Why structure theory of linear switched systems difficult

Local structure of LTI models does not determine the structure of the switched system.
Two modes: $Q = \{1, 2\}$

$$A_1 = \begin{bmatrix} -3 & 0 & -1 \\ 0 & -3 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; C_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}^T$$

$$A_2 = \begin{bmatrix} -4 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \; C_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^T$$

The local subsystems are not observable, but the switched system is (we will see it later).

Observability of linear switched systems

$I_{\{A_q,B_q,C_q\}_{q\in Q},x_0}$ - input-output function I_{f,h,x_0},
$f(x,(q,v)) = A_q x + B_q u$, $h(x,(q,v)) = C_q x$.

$\{A_q, B_q, C_q\}_{q\in Q}$ is observable, if the function $x_0 \mapsto I_{\{A_q,B_q,C_q,\}_{q\in Q},x_0}$ is one-to-one.

Decomposition into autonomus and continuous input-dependent part:

$$I_{\{A_q,B_q,C_q\}_{q\in Q},x_0}((q,v)) =$$
$$I_{\{A_q,B_q,C_q\}_{q\in Q},x_0}((q,0)) + I_{\{A_q,B_q,C_q\}_{q\in Q},0}((q,v))$$

Exercise: Write down the analytic expressions for $I_{\{A_q,B_q,C_q\}_{q\in Q},x_0}((q,0))$ and $I_{\{A_q,B_q,C_q\}_{q\in Q},x_0}((q,v))$ (discrete or cont. time)

Condition for observability

Theorem (Sun & Ge & Lee)

$\{A_q, B_q, C_q\}_{q\in Q}$ *is observable,* $\iff$

$$n = \mathrm{rank}\,[(C_q A_{q_k} A_{q_{k-1}} \cdots A_{q_1})^T \mid q, q_1, \ldots, q_k \in Q, 0 \le k < n]$$

$\iff$

$$\bigcap_{k=0}^{\infty} \bigcap_{q,q_1,\ldots,q_k\in Q} \ker C_q A_{q_k} A_{q_{k-1}} \cdots A_{q_1} = \{0\}$$

Condition for observability

A non-obvious fact from [Sun & Ge & Lee]:

$$\bigcap_{k=0}^{\infty} \bigcap_{q,q_1,\ldots,q_k \in Q} \ker C_q A_{q_k} A_{q_{k-1}} \cdots A_{q_1} =$$
$$\bigcap_{k=0}^{n-1} \bigcap_{q,q_1,\ldots,q_k \in Q} \ker C_q A_{q_k} A_{q_{k-1}} \cdots A_{q_1}.$$

Corollary

If for some q, (C_q, A_q) is an observable pair, then $\{A_q, B_q, C_q\}_{q \in Q}$ is observable.

Proof: Exercise

Observability of linear switched systems

$\{A_q, B_q, C_q\}_{q \in Q}$ is observable, if $\forall x_0, x_0^{'}$:

$$\forall q : I_{\{A_q,B_q,C_q\}_{q\in Q},x_0}((q,0)) = I_{\{A_q,B_q,C_q\}_{q\in Q},x_0^{'}}((q,0)) \implies x_0 = x_0^{'},$$

i.e., different initial states can be distinguished by the outputs for zero continuous input and some switching signal.

$I_{\{A_q,B_q,C_q\}_{q\in Q},x_0}((q,0))$ linear in $x_0 \implies$

$$I_{\{A_q,B_q,C_q\}_{q\in Q},x_0}((q,0)) = I_{\{A_q,B_q,C_q\}_{q\in Q},x_0^{'}}((q,0)) \iff I_{\{A_q,B_q,C_q\}_{q\in Q},x_0-x_0^{'}}((q,0)) = 0$$

$\{A_q, B_q, C_q\}_{q \in Q}$ is observable, if

$$(\forall q : I_{\{A_q,B_q,C_q\}_{q\in Q},x_0}((q,0)) = 0) \implies x_0 = 0.$$

Observability of linear switched systems

$$(\forall q \in Q : I_{\{A_q,B_q,C_q\}_{q\in Q},x_0}((q,0)) = 0) \iff$$
$$C_q A_{q_k} A_{q_{k-1}} \cdots A_{q_1} x_0 = 0,\ \forall k \geq 0, q, q_1, \ldots, q_k \in Q$$

Observability: exercise

Two modes: $Q = \{1, 2\}$

$$A_1 = \begin{bmatrix} -3 & 0 & -1 \\ 0 & -3 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; C_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}^T$$

$$A_2 = \begin{bmatrix} -4 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \; C_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^T$$

Check observability

Observability: exercise

$Q = \{1, 2\}$

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$
$$C_1 = C_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

Check observability.

Observability reduction

$$\mathcal{W}^* = \bigcap_{k=0}^{n-1} \bigcap_{q,q_1,\dots,q_k \in Q} \ker C_q A_{q_k} A_{q_{k-1}} \cdots A_{q_1} =$$

$$\bigcap_{k=0}^{\infty} \bigcap_{q,q_1,\dots,q_k \in Q} \ker C_q A_{q_k} A_{q_{k-1}} \cdots A_{q_1}.$$

$b_1, \dots, b_n$ basis s.t. $b_{o+1}, \dots, b_n$ span $\mathcal{W}^*$.
In this new basis,

$$A_q = \begin{bmatrix} A_q^{\mathrm{O}} & 0 \\ A_q^{'} & A_q^{''} \end{bmatrix}, C_q = \begin{bmatrix} C_q^{\mathrm{O}}, & 0 \end{bmatrix}, B_q = \begin{bmatrix} B_q^{\mathrm{O}} \\ B_q^{'} \end{bmatrix},$$

Observability reduction

$\{A_q^{\mathrm{O}}, B_q^{\mathrm{O}}, C_q^{\mathrm{O}}\}_{q \in Q}$ is observable.

The input-output behavior of $\{A_q^{\mathrm{O}}, B_q^{\mathrm{O}}, C_q^{\mathrm{O}}\}_{q \in Q}$ and $\{A_q, B_q, C_q\}_{q \in Q}$ are the same.

$$I_{\{A_q^{\mathrm{O}}, B_q^{\mathrm{O}}, C_q^{\mathrm{O}}\}_{q \in Q}, Px_0} = I_{\{A_q, B_q, C_q\}_{q \in Q}, x_0}.$$

$$P = \begin{bmatrix} I_o & 0 \\ 0 & 0 \end{bmatrix}$$

Last $n - o$ coordinates: unobservable part, does not influence the output, cannot be estimated from the output.

Observability: exercise

$$A_{q_1} = \begin{bmatrix} -3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}, B_{q_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \; C_{q_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T$$

$$A_{q_2} = \begin{bmatrix} -4 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \; B_{q_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \; C_{q_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}^T$$

Perform observability reduction.

Reachability of linear switched systems

$x(z, q, v)(t)$ – state of $\{A_q, B_q, C_q\}_{q \in Q}$ at time t, under input v, switching signal q, and initial state z.

A state z, is called reachable from x_0, if $z = x(x_0, u)(T)$ for some u and T.

$\{A_q, B_q, C_q\}_{q \in Q}$ is called reachable from x_0, if all states are reachable from x_0.

$\{A_q, B_q, C_q\}_{q \in Q}$ is called span-reachable from x_0, if the linear span of all states reachable from zero is the whole state-space.

$\{A_q, B_q, C_q\}_{q \in Q}$ is called controllable, if for any z, z', there exists u and T s.t. $x(z, u)(T) = z'$.

Reachability of linear switched systems

Theorem (Sun & Ge & Lee)

$\{A_q, B_q, C_q\}_{q \in Q}$ *is span-reachable from* 0, $\iff$

$$n = \mathrm{rank}\, [A_{q_k} A_{q_{k-1}} \cdots A_{q_1} B_{q_0} \mid q_0, q_1, \ldots, q_k \in Q, k < n]$$

$\iff$

$$n = \dim \mathrm{Span}\{A_{q_k} A_{q_{k-1}} \cdots A_{q_1} B_{q_0} v \mid q_0, q_1, \ldots, q_k \in Q, k \geq 0, v\}$$

In continuous time or in discrete-time if A_q are invertible, then

- *span reachability from* 0 *is equivalent to reachability from* 0,
- *reachability from* 0 *is equivalent to controllability.*

Reachability of linear switched systems

A non-obvious fact from [Sun & Ge & Lee]:

$$\mathrm{Span}\{A_{q_k}A_{q_{k-1}}\cdots A_{q_1}B_{q_0}v \mid q_0,q_1,\ldots,q_k\in Q, k\geq 0, v\in\mathbb{R}^m\} =$$
$$\mathrm{Span}\{A_{q_k}A_{q_{k-1}}\cdots A_{q_1}B_{q_0}v \mid q_0,q_1,\ldots,q_k\in Q, n>k\geq 0, v\in\mathbb{R}^m\}$$

Corollary

If for some q, (A_q, B_q) is a controllable pair, then $\{A_q, B_q, C_q\}_{q\in Q}$ is span-reachable from 0.

Proof: Exercise

Reachability of linear switched systems

Main idea:

- $\mathrm{Span}\{A_{q_k} A_{q_{k-1}} \cdots A_{q_1} B_{q_0} v \mid q_0, q_1, \ldots, q_k \in Q, k \geq 0, v \in \mathbb{R}^m\}$
  is the smallest vector space which contains states reachable from zero.

  In continuous time or in discrete-time if A_q are invertible, then there exists a switching signal q and an interval $[0, T]$ s.t.
- - The linear span of
    $$\{x(0, (q, v))(t) \mid v \text{ continuous input}, t \in [0, T]\}$$
    contains the set of all states which are reachable from zero.
  - The set
    $$\{x(0, (q, v))(t) \mid v \text{ continuous input}, t \in [0, T]\}$$
    is a vector space.

The proof of the equivalence of controllability and reachability is difficult.

Reachability: exercise

Two modes: $Q = \{1, 2\}$

$$A_1 = \begin{bmatrix} -3 & 0 & -1 \\ 0 & -3 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; C_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}^T$$

$$A_2 = \begin{bmatrix} -4 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \; C_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^T$$

Check reachability

Reachability: exercise

$Q = \{1, 2\}$

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$C_1 = C_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

Check reachability.

Reachability reduction

$$\mathcal{V}^* = \mathrm{Span}\{A_{q_k}A_{q_{k-1}}\cdots A_{q_1}B_{q_0}v \mid q_0, q_1, \ldots, q_k \in Q, k \geq 0, v \in \mathbb{R}^m\}$$

Choose a basis $b_1, \ldots, b_n$ s.t. $b_1, \ldots, b_r$ span $\mathcal{V}^*$.

In this new basis,

$$A_q = \begin{bmatrix} A_q^{\mathrm{R}} & A_q^{'} \\ 0 & A_q^{''} \end{bmatrix}, C_q = \begin{bmatrix} C_q^{\mathrm{R}}, & C_q^{'} \end{bmatrix}, B_q = \begin{bmatrix} B_q^{\mathrm{R}} \\ 0 \end{bmatrix}, \quad (1)$$

$\{A_q^{\mathrm{R}}, B_q^{\mathrm{R}}, C_q^{\mathrm{R}}\}_{q \in Q}$ is span-reachable from 0.

The input-output function from zero of $\{A_q^{\mathrm{R}}, B_q^{\mathrm{R}}, C_q^{\mathrm{R}}\}_{q \in Q}$ and $\{A_q, B_q, C_q\}_{q \in Q}$ are the same.

$$I_{\{A_q^{\mathrm{R}}, B_q^{\mathrm{R}}, C_q^{\mathrm{R}}\}_{q \in Q}, 0} = I_{\{A_q, B_q, C_q\}_{q \in Q}, 0}.$$

Last $n - r$ coordinates: uncontrollable part, cannot be influenced by continuous inputs.

Reachability: exercise

$$A_{q_1} = \begin{bmatrix} -3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}, B_{q_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} C_{q_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T$$

$$A_{q_2} = \begin{bmatrix} -4 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} B_{q_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} C_{q_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}^T$$

Apply reachability reduction.

Minimization

- Apply reachability reduction to $\{A_q, B_q, C_q\}_{q\in Q}$ to get $\{A_q^{\mathrm{R}}, B_q^{\mathrm{R}}, C_q^{\mathrm{R}}\}_{q\in Q}$.
- Apply observability reduction to $\{A_q^{\mathrm{R}}, B_q^{\mathrm{R}}, C_q^{\mathrm{R}}\}_{q\in Q}$ to get $\{A_q^m, B_q^m, C_q^m\}_{q\in Q}$.

$\{A_q^m, B_q^m, C_q^m\}_{q\in Q}$ is span-reachable from 0, observable, and its input-output function from 0 is the same as that of $\{A_q, B_q, C_q\}_{q\in Q}$, i.e.,

$$I_{\{A_q^m, B_q^m, C_q^m\}_{q\in Q},0} = I_{\{A_q, B_q, C_q\}_{q\in Q},0}.$$

State-space dimension of $\{A_q^m, B_q^m, C_q^m\}_{q\in Q}$ is $\leq$ state-space dimension of $\{A_q, B_q, C_q\}_{q\in Q}$.

Minimality

Let I be an input-output function.

Theorem (Pet06,Pet07,Pet11a,Pet13)

- $\{A_q, B_q, C_q\}_{q\in Q}$ *is a minimal dimensional among all linear switched systems whose input-output function from* 0 *is* I,
  $\iff$
  $\{A_q, B_q, C_q\}_{q\in Q}$ *is observable and span-reachable from* 0.
- $\{A_q^m, B_q^m, C_q^m\}_{q\in Q}$ *is minimal dimensional among all linear switched systems with the same input-output function from* 0.
- *If* $\{A_q, B_q, C_q\}_{q\in Q}$ *and* $\{\hat{A}_q, \hat{B}_q, \hat{C}_q\}_{q\in Q}$ *are minimal dimensional s.t.* $I_{\{A_q,B_q,C_q\}_{q\in Q},0} = I_{\{\hat{A}_q,\hat{B}_q,\hat{C}_q\}_{q\in Q},0} = I$ *then they are isomorphic:*
  there exists a nonsingular matrix T *s.t.:*

$$\forall q : TA_q T^{-1} = \hat{A}_q,\ TB_q = \hat{B}_q,\ CT_q^{-1} = \hat{C}_q.$$

Counter-examples

- If at least one of the continuous subsystems are minimal, then the switched system is minimal.
- A switched system can be minimal (resp. observable, reachable), without any of the subsystems being minimal (resp. observable, reachable).
- Certain linear switched systems can never be brought to a form where all the continuous subsystems are minimal.

Example

$$A_{q_1} = \begin{bmatrix} -3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}, B_{q_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} C_{q_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T$$

$$A_{q_2} = \begin{bmatrix} -4 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} B_{q_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} C_{q_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}^T$$

This system is neither observable nor reachable, hence it is not minimal.

Example: cont

After minimization, we obtain

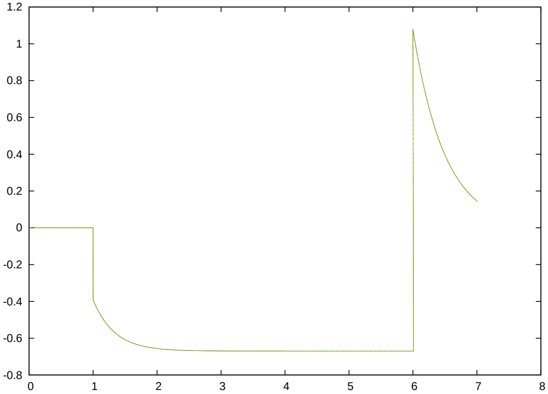
$$A_{q_1} = \begin{bmatrix} -3 & 0 & -0.02 \\ 0 & -3 & 0 \\ 0.98 & 0 & 0.006 \end{bmatrix}, B_{q_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \; C_{q_1} = \begin{bmatrix} 0.95 \\ 0 \\ -0.31 \end{bmatrix}^T$$

$$A_{q_2} = \begin{bmatrix} -4 & 0 & -0.02 \\ 0 & -2 & 0 \\ 0.98 & 0 & -0.99 \end{bmatrix} B_{q_2} = \begin{bmatrix} 0.31 \\ 0 \\ 0.95 \end{bmatrix} \; C_{q_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^T$$

The system above is minimal, but none of the subsystems is minimal

Example: cont

If we simulate the two systems for white noise input and switching sequence $(q_2, 1)(q_1, 2)(q_1, 3)(q_2, 1)$.



Further consequences

- For linear switched systems which are observable and span-reachable from zero, the input-output function from 0 captures all the input-output behavior.
- It is impossible to estimate the state for non-observable linear switched systems. The converse need not be true.
- It is impossible to control (stabilize) a linear switched system with continuous inputs , if it is not span-reachable from zero. The converse need not be true.
- Minimal switched systems isomorphic $\implies$ control depends only on the input-output behavior not on the choice of the state-space representation.
- Existence of quadratic (control) Lyapunov functions, storage functions is a property of input-output behavior.

Linear Time Invariant (LTI) state-space representation

$\Sigma = (A, B, C)$.

Input-output map $Y_\Sigma = I_{A,B,C,0}$ maps input $u(.)$ to output $y(.)$, initial state $x(0) = 0$.

$$Y_\Sigma(u)(t) = \begin{cases} \int_0^t Ce^{A(t-s)}Bu(s)ds \\ \sum_{s=0}^{t-1} CA^{(t-s)}Bu(s) \end{cases}$$

Σ is a realization of $Y : u(.) \mapsto y(.)$, iff $Y_\Sigma = Y$.

Realization problem

For the specified input-output map Y find a (preferably minimal) linear system Σ such that Σ realizes Y.

Impulse response

A potential input-output map of a linear system is determined by its impulse response:

Impulse response $G(t)$

$$Y(u(.), t) = \begin{cases} \int_0^t G(t-s)u(s)ds & \text{continuous time} \\ \sum_{s=0}^{t-1} G(t-s)u(s) & \text{discrete time} \end{cases}$$

Σ is a realization, iff

$$G(t) = Ce^{At}B \text{ (cont.time)}$$
$$G(t) = CA^{t}B \text{ (disc.-time)}$$

Markov parameters

Markov parameters

$$M_k = \begin{cases} \frac{d^k}{dt^k} G(t)\big|_{t=0} & \text{continuous time, or} \\ G(k+1) & \text{discrete time} \end{cases}$$

Classical step. Σ is a realization of $Y \iff M_k = CA^k B$

Existence of a realization

Recall M_k – Markov parameters*

Hankel matrix of Y $\quad H_Y = \begin{bmatrix} M_0 & M_1 & M_2 & \cdots \\ M_1 & M_2 & M_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$

Theorem

- *Y has a realization by an LTI* $\iff \operatorname{rank} H_Y < +\infty$.
- $\operatorname{rank} H_Y$ *is the dimension of a minimal LTI realization of Y.*

Ho-Kalman algorithm

1. Find a factorization

$$H_{N,N+1} = \begin{bmatrix} M_0 & M_1 & \cdots & M_N \\ M_1 & M_2 & \cdots & M_{N+1} \\ \vdots & \vdots & \vdots & \vdots \\ M_{N-1} & M_N & \cdots & M_{2N-1} \end{bmatrix} = OR$$

   s.t. O full column rank, R full row rank.
   (e.g, SVD: $H_{N,N+1} = U\Sigma V^T$, $O = U\Sigma^{1/2}$, $R = \Sigma^{1/2}V^T$).

2. $R = \begin{bmatrix} R_1 & , R_2, & \cdots, & R_{N+1} \end{bmatrix}$, $O = \begin{bmatrix} O_1 \\ O_2 \\ \vdots \\ O_N \end{bmatrix}$.

3. $B = R_1$, $C = O_1$, and A solves

$$A \begin{bmatrix} R_1, & R_2, & \cdots, & R_N \end{bmatrix} = \begin{bmatrix} R_2, & R_3, & \cdots, & R_{N+1} \end{bmatrix}$$

Correctness of Ho-Kalman algorithm and partial realization

$$H_{N,N} = \begin{bmatrix} M_0 & M_1 & \cdots & M_{N-1} \\ M_1 & M_2 & \cdots & M_N \\ \vdots & \vdots & \vdots & \vdots \\ M_{N-1} & M_N & \cdots & M_{2N-2} \end{bmatrix},$$

$$H_{N+1,N} = \begin{bmatrix} M_0 & M_1 & \cdots & M_{N-1} \\ M_1 & M_2 & \cdots & M_N \\ \vdots & \vdots & \vdots & \vdots \\ M_N & M_{N+1} & \cdots & M_{2N-1} \end{bmatrix}$$

Correctness of Ho-Kalman algorithm and partial realization

Theorem (Ho-Kalman algorithm & partial realization)

- $\text{rank } H_{N,N} = \text{rank } H_Y \implies (A, B, C)$ *is a minimal realization of* Y
- *If* Y *has a realization of dimension less than* N*, then* $\text{rank } H_{N,N} = \text{rank } H_Y$.
- $\text{rank } H_{N,N} = \text{rank } H_{N+1,N} = \text{rank } H_{N,N+1} \implies (A, B, C)$ *is a so called* $2N$ *realization of* Y*, i.e.*

$$M_k = CA^kB, \; k = 0, 1, \ldots, 2N-1$$

Impulse response of linear switched systems

- Potential input-output map Y of a linear switched system
  1. Maps switching signal $q(.)$ and input $u(.)$ to output $y(.)$.
  2. Linear in continuous input $u()$.

- Y is completely described by its impulse response

  Impulse response for switching $q(.)$

  Switching $q()$: stay in discrete mode $q_1, \ldots, q_k$ for times $t_1, \ldots, t_k$.

$$G_{q_1 \ldots q_k}(t_1, \ldots, t_k) = Y(q(.), \sigma_0)$$

  - σ_0 is the Dirac-delta for continuous-time
  - $\sigma_0(0) = 1$, $\sigma_0(t) = 0, t > 0$ for discrete-time

Markov parameters for linear switched systems

Markov parameters, $q_0, q \in Q$ – discrete modes, $j = 1, 2, \ldots, m$

$$S_{q,q_0}(q_1 q_2 \cdots q_k) = \left\{ \begin{array}{l} G_{q_0 q_1 \cdots q_k q}(1, 1, \ldots, 1) \\ \frac{d}{dt_1} \cdots \frac{d}{dt_k} G_{q_0 q_1 \cdots q_k q}(0, t_1, \ldots, t_k, 0)|_{t_1 = \cdots = t_k = 0} \end{array} \right\}$$

Markov parameters are indexed by sequences of discrete modes Q^*

Σ is a realization of Y $\iff$

$$S_{q,q_0}(q_1 q_2 \cdots q_k) = C_q A_{q_k} \cdots A_{q_1} B_{q_0}$$

Hankel matrix for linear switched systems

$Q = \{1, 2, \ldots, D\}$

$v_1 \prec \ldots \prec v_k, \ldots$ lexicographic ordering of all sequences.

$$M(v) = \begin{bmatrix} S_{1,1}(v) & \ldots & S_{1,D}(v) \\ \vdots & \ldots & \vdots \\ S_{D,1}(v) & \ldots & S_{D,D}(v) \end{bmatrix}$$

Hankel matrix: H_Y

$$H_Y = \begin{bmatrix} M(v_1 v_1) & M(v_2 v_1) & \cdots & M(v_k v_1) & \cdots \\ M(v_1 v_2) & M(v_2 v_2) & \cdots & M(v_k v_2) & \cdots \\ M(v_1 v_3) & M(v_2 v_3) & \cdots & M(v_k v_3) & \cdots \\ \vdots & \vdots & \ldots & \vdots & \ldots \end{bmatrix},$$

Realization theorem for linear switched systems

Theorem (Pet06,Pet07,Pet11a,Pet13)

- *Y has a realization* $\iff \mathrm{rank} \;\; H_Y < +\infty$,

Realization algorithm [Pet06,Pet11,Pet13]

$$H_{Y,N+1,N} = \begin{bmatrix} M(v_1 v_1) & \cdots & M(v_{\mathbf{M}(N)} v_1) \\ \vdots & \cdots & \vdots \\ M(v_1 v_{\mathbf{M}(N)}) & \cdots & M(v_{\mathbf{M}(N)} v_{\mathbf{M}(N)}) \\ M(v_1 v_{\mathbf{M}(N+1)}) & \cdots & M(v_{\mathbf{M}(N)} v_{\mathbf{M}(N+1)}) \end{bmatrix}$$

$\mathbf{M}(N)$ – number of sequences over Q of length at most N

1: $H_{f,N+1,N} = OR$
2: $B_q = m(q-1)+1, \ldots, mq$th columns of R.
3: $C_q = p(q-1)+1, \ldots, pq$th rows of O.
4: $A_q = \bar{O}^{+} O_q$

- $\bar{O}$ – the block rows of O which are indexed by $v_1, \ldots, v_N$.
- $\bar{O}^{+}$–pseudo-inverse of $\bar{O}$.
- O_q – shifted $\bar{O}$: the row of O_q indexed by sequence v is the row of O indexed by sequence qv.

Partial realization theorem for linear switched systems

$$H_{Y,N,N} = \begin{bmatrix} M(v_1 v_1) & \cdots & M(v_{\mathbf{M}(N)} v_1) \\ \vdots & \cdots & \vdots \\ M(v_1 v_{\mathbf{M}(N)}) & \cdots & M(v_{\mathbf{M}(N)} v_{\mathbf{M}(N)}) \end{bmatrix}$$

$$H_{Y,N,N+1} = \begin{bmatrix} M(v_1 v_1) & \cdots & M(v_{\mathbf{M}(N)} v_1) & M(v_{\mathbf{M}(N+1)} v_1) \\ \vdots & \cdots & \vdots & \vdots \\ M(v_1 v_{\mathbf{M}(N)}) & \cdots & M(v_{\mathbf{M}(N)} v_{\mathbf{M}(N)}) & M(v_{\mathbf{M}(N+1)} v_{\mathbf{M}(N)}) \end{bmatrix}$$

Theorem (Pet11b,Pet13)

1. *If* $\mathrm{rank}\ H_{Y,N,N} = \mathrm{rank}\ H_{Y,N,N+1} = \mathrm{rank}\ H_{Y,N+1,N}$ *then the result of the algorithm recreates the Markov-parameters* $M(v_1), \ldots M(v_{\mathbf{M}(2N+1)})$.
2. *If* $N \geq$ *the dimension of a realization of* Y*, then the algorithm returns a minimal realization of* Y*.*

Example

Consider the switched system from the previous example and Y the input-output map of that system.

$$H_{Y,2,1} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & -1 \\ 1 & 0 & -3 & 0 & -2 & 0 \\ 0 & -1 & 0 & 3 & 0 & 4 \\ -3 & 0 & 9 & 0 & 6 & 0 \\ 0 & -1 & 0 & 4 & 0 & 5 \\ -2 & 0 & 6 & 0 & 4 & 0 \\ 0 & 3 & 0 & -9 & 0 & -12 \\ 9 & 0 & -27 & 0 & -18 & 0 \\ 0 & 4 & 0 & -12 & 0 & -16 \\ 6 & 0 & -18 & 0 & -12 & 0 \\ 0 & 4 & 0 & -12 & 0 & -16 \\ 6 & 0 & -18 & 0 & -12 & 0 \\ 0 & 5 & 0 & -16 & 0 & -21 \\ 4 & 0 & -12 & 0 & -8 & 0 \end{bmatrix}$$

Example: cont.

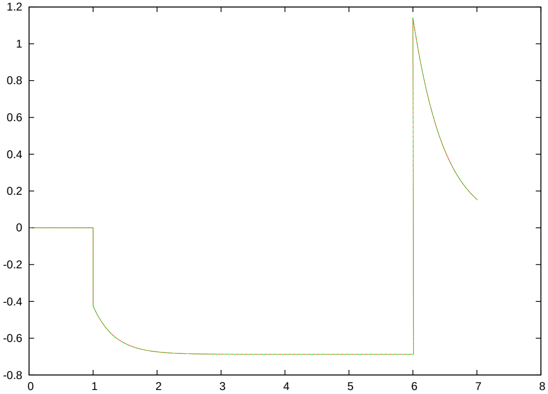
Applying the realization algorithm to $H_{Y,2,1}$ yields.

$$A_{q_1} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3.02 & 0.17 \\ 0 & -0.32 & 0.018 \end{bmatrix}, B_{q_1} = \begin{bmatrix} -1.9 \\ 0 \\ 0 \end{bmatrix}, C_{q_1} = \begin{bmatrix} 0 \\ 0.21 \\ 0.46 \end{bmatrix}^T$$

$$A_{q_2} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -4.02 & 0.17 \\ 0 & -0.32 & -0.98 \end{bmatrix}, B_{q_2} = \begin{bmatrix} 0 \\ 1.25 \\ -0.57 \end{bmatrix}, C_{q_2} = \begin{bmatrix} -0.53 \\ 0 \\ 0 \end{bmatrix}^T$$

Example: cont

If we simulate the two systems for white noise input and switching sequence $(q_2, 1)(q_1, 2)(q_1, 3)(q_2, 1)$.



Further work

- The results above can be extended to linear jumps and bilinear local equations.
- The results can be extended to LPV systems.
- Extension to stochastic jump-Markov linear systems.
- Application to model reduction, system identification.

References

R. E. Kalman.
Advanced theory of linear systems.
In *Topics in Mathematical System Theory*, pages 237–339.
McGraw-Hill, New York, 1969.

T. Kailath, *Linear Systems*. Prentice Hall, 1980.

T. Katayama.
Subspace Methods for System Identification.
Springer-Verlag, 2005.

References

M. Petreczky.
Realization theory of linear and bilinear switched systems: A formal power series approach: Part i.
ESAIM Control, Optimization and Calculus of Variations, 17:410–445, 2011.

M. Petreczky, L. Bako, and van J.H. Schuppen.
Realization theory for discrete-time linear switched systems.
Automatica, 49(11):3337–3344, 2013.

M. Petreczky and J. H. Van Schuppen.
Partial-realization of linear switched systems: A formal power series approach.
Automatica, 47(10):2177–2184, 2011.

References

M. Petreczky and J.H. van Schuppen.
Realization theory for linear hybrid systems.
IEEE Trans. on Automatic Control, 55:2282 – 2297, 2010.

Mihaly Petreczky, Aneel Tanwani and Stephan Trenn.
Observability of Switched Linear Systems.
In *Hybrid Dynamical Systems: Observation and control*, Lecture Notes in Control and Information Sciences, vol. 457, M. Djemai, M. Defoort (eds), Springer Verlag, ISBN: 978-3-319-10795-0, 2015.

Mihaly Petreczky.
Realization theory of linear hybrid systems.
In *Hybrid Dynamical Systems: Observation and control*, Lecture Notes in Control and Information Sciences, vol. 457, M. Djemai, M. Defoort (eds), Springer Verlag, ISBN: 978-3-319-10795-0, 2015.

References

- Zh. Sun and Sh. S. Ge.
  Switched linear systems : control and design.
  Springer, London, 2005.
- S.S. Ge, Zhendong Sun, and T.H. Lee.
  Reachability and controllability of switched linear discrete-time systems.
  IEEE Trans. Automatic Control, 46(9):1437 – 1441, 2001.
- Zhendong Sun, S.S. Ge, and T.H. Lee.
  Controllability and reachability criteria for switched linear systems.
  Automatica, 38:115 – 786, 2002.
- Zhendong Sun and Dazhong Zheng.
  On reachability and stabilization of switched linear systems.
  IEEE Trans. Automatic Control, 46(2):291 – 295, 2001.