Converse Lyapunov–Krasovskii theorems for uncertain retarded differential equations [1]

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[1] Ihab Haidar, Paolo Mason and Mario Sigalotti, *Converse Lyapunov–Krasovskii theorems for uncertain retarded differential equations*, Provisionally accepted as regular paper, *Automatica*, 2014.

Outline

- Retarded Functional Differential Equation (RFDE)
- Switching system approach
- Results
- Conclusion

Problem

Consider the following Retarded Functional Differential Equation (RFDE)

$$(\Sigma) \qquad \dot{x}(t) = L(t)x_t \qquad t \geq 0,$$

where

- $x(t) \in \mathbb{R}^n$: the system state at time t
- $x_t : \theta \mapsto x(t+\theta), \quad \theta \in [-r, 0]$: the history function
- $x_0 = \varphi \in X$: an initial condition
- $L : [0, +\infty) \to \mathcal{L}(X, \mathbb{R}^n)$: a bounded linear operator

Typical examples

1.

$$\left| \begin{array}{lll} \dot{x}(t) & = & A_0 x(t) + A_1 x(t-\tau(t)) \qquad t \geq 0 \\ x(\theta) & = & \varphi(\theta), \quad \theta \in [-r, 0] \end{array} \right.$$

for some $n \times n$ matrices A_0 and A_1 and $\tau : [0, +\infty) \to [-r, 0]$.

2.

$$\left| \begin{array}{lll} \dot{x}(t) & = & \int_0^r A(t,\theta)\, x(t-\theta) d\theta, \qquad t \geq 0 \ , \\ x(\theta) & = & \varphi(\theta), \end{array} \right.$$

$A(t,\theta)$ is a $n \times n$ matrix uniformly bounded with respect to t and $\theta \in [0, r]$ and measurable with respect to θ.

Existence and uniqueness of a solution

- $X = C([-r,0],\mathbb{R}^n)$ or $X = H^1([-r,0],\mathbb{R}^n)$
- $L(\cdot)\varphi : t \mapsto L(t)\varphi$ is a measurable function $\quad \forall \varphi \in C([-r,0],\mathbb{R}^n)$
- there exists a positive constant m such that

$$(K): \qquad |L(t)\varphi| \leq m\|\varphi\|_C \qquad \forall \varphi \in C([-r,0],\mathbb{R}^n)$$

Lemma

Consider the linear RFDE given by system (Σ). Let X be the Banach spaces $C([-r,0],\mathbb{R}^n)$ or $H^1([-r,0],\mathbb{R}^n)$. Assume that condition (K) holds. For every $\varphi \in X$ there exists a unique solution of (Σ) with initial condition φ.

Three principal approaches

- Lyapunov–Krasovskii: consists of finding a positive functional that decays along the trajectories of the considered systems
- Lyapunov–Razumikhin: enables to employ Lyapunov function instead of Lyapunov functional
- Barnea: consists in reducing the stability problem to an optimization problem

Lyapunov–Krasovskii Theorem

Theorem (Lyapunov–Krasovskii)

Let $u, v, w : [0, +\infty) \to [0, +\infty)$ *are continuous nondecreasing functions,* $u(s)$ *and* $v(s)$ *are positive for* $s > 0$*, and* $u(0) = v(0) = 0$*. If there exists a continuous function* $V : C([-r, 0], \mathbb{R}^n) \to \mathbb{R}$ *such that*

$$u(|\varphi(0)|) \leq V(\varphi) \leq v(\|\varphi\|_C)$$
$$\overline{D}V(\varphi) \leq -w(|\varphi(0)|)$$

then the solution $x = 0$ *of equation (2) is uniformly stable. If* $w(s) > 0$ *for* $s > 0$*, then the solution* $x = 0$ *is exponentially stable.*

$$\overline{D}V(\varphi) = \lim_{t\to 0}\sup \frac{V(x_t(\varphi)) - V(\varphi)}{t}$$
$$\underline{D}V(\varphi) = \lim_{t\to 0}\inf \frac{V(x_t(\varphi)) - V(\varphi)}{t}$$

$$x(k+1) = A_0 x(k) + A_1 x(k-\tau(k)), \qquad 0 < \tau(k) \leq m$$

Let

$$z(k) = [x^T(k), \ldots, x^T(k-m)]^T \quad \text{and} \quad \sigma : \mathbb{Z}^+ \to \mathbb{S} = \{1, \ldots, m\}$$

$$z(k+1) = \bar{A}_{\sigma(k)} z(k) \qquad \text{with} \qquad \sigma(k) = \tau(k)$$

where the matrix $\bar{A}_{\sigma(k)}$ switches in the set of possible matrices $\{\bar{A}_1, \cdots, \bar{A}_m\}$

$$\bar{A}_i = \begin{pmatrix} A_0 & 0 & \cdots & 0 & A_1 & 0 & \cdots & 0 \\ I & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & I & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & I & 0 \end{pmatrix}.$$

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[2] L. Hetel, J. Daafouz, and C. Iung. Equivalence between the Lyapunov–Krasovskii functional approach for discrete delay systems and the stability conditions for switched systems. Nonlinear Analysis: Hybrid Systems, 2(3):697–705, 2008.

We parametrize the operator $t \mapsto L(t)$

- Let $\mathbb{S}$ be an index set (which can be uncountable).
- Let $\sigma(\cdot) : [0, +\infty) \longrightarrow \mathbb{S}$ be a measurable signal
- $\sigma(\cdot)$ parametrizes (Σ)

$$(\Sigma) : \qquad \dot{x}(t) = L_{\sigma(t)} x_t,$$

- there exists a positive constant m such that

$$(K) : \qquad |L_\sigma \varphi| \leq m \|\varphi\|_C \qquad \forall \varphi \in C([-r, 0], \mathbb{R}^n), \sigma \in \mathbb{S}$$

Semigroup associated to each candidate

With any $\sigma \in \mathbb{S}$

$$\dot{x}(t) = L_\sigma x_t,$$

one can associate a C_0-semigroup

$$T_\sigma(t) : X \to X \quad \text{defined by} \quad T_\sigma(t)(\varphi) = x_t$$

with infinitesimal generator $\mathcal{A}_\sigma$ given by

$$\begin{aligned} D(\mathcal{A}_\sigma) &= \left\{ \varphi \in X : \frac{d\varphi}{d\theta} \in X, \frac{d\varphi}{d\theta}(0) = L_\sigma \varphi \right\}, \\ \mathcal{A}_\sigma \varphi &= \frac{d\varphi}{d\theta}. \end{aligned}$$

Switched system representation: picewise constant case

- The evolution operator corresponding to a piecewise constant signal

$$\sigma(t) = \sum_{k \geq 0} \mathbb{1}_{[t_k, t_{k+1})}(t)\sigma_k$$

with $t_0 = 0$, $t_k < t_{k+1}$ for $k \geq 0$ is given by

$$T_{\sigma(\cdot)}(t) = T_{\sigma_k}(t - t_k)T_{\sigma_{k-1}}(t_k - t_{k-1})...T_{\sigma_0}(t_1 - t_0) \qquad t \in [t_k, t_{k+1}).$$

- The evolution is then given by the following switched system

$$(\Sigma) \longrightarrow (\Sigma_s) : \qquad \begin{array}{rcl} x_t & = & T_{\sigma(\cdot)}(t)x_0, \\ x_0 & = & \varphi \in X. \end{array}$$

Theorem (F.M. Hante and M. Sigalotti[a])

[a] F.M. Hante and M. Sigalotti. Converse Lyapunov theorems for switched systems in Banach and Hilbert spaces. SIAM J. Control Optim., 49(2):752–770, 2011.

The conditions

(i) *there exist* $M \geq 1$ *and* $w > 0$ *such that*

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq Me^{wt}, \quad t \geq 0, \ \sigma(\cdot)\text{-uniformly},$$

(ii) *there exists a function* $V : X \to [0, \infty)$ *such that* $\sqrt{V(\cdot)}$ *is a norm on* X,

$$V(\varphi) \leq c\|\varphi\|_X^2$$

for some constant $c > 0$ *and*

$$\overline{D}_\sigma V(\varphi) \leq -\|\varphi\|_X^2, \quad \sigma \in \mathbb{S}, \varphi \in X.$$

are necessary and sufficient for the existence of constants $K \geq 1$ *and* $\mu > 0$ *such that*

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq Ke^{-\mu t}, \quad t \geq 0, \ \sigma(\cdot)\text{-uniformly}.$$

Uniform exponenetial boundedness

Lemma

Suppose that condition (K) holds. If $X = C([-r, 0], \mathbb{R}^n)$ or $H^1([-r, 0], \mathbb{R}^n)$ then the solutions of (Σ_s) are $\sigma(\cdot)$-uniformly exponentially bounded.

Proof.

1. case $X = C([-r, 0], \mathbb{R}^n)$.
   By integrating system (Σ) and using equation (K), one has for every $t \geq 0$

   $$\|x_t\|_C \leq \|\varphi\|_C + m \int_0^t \|x_s\|_C ds.$$

   Thanks to Gronwall's Lemma, we have

   $$\|x_t\|_C \leq \|\varphi\|_C e^{mt}. \tag{1}$$

2. case $X = H^1([-r, 0], \mathbb{R}^n)$.
   Same reasoning + Poincaré Inequality.

First converse theorem

Theorem

Suppose that condition (K) holds. System (Σ_s) is uniformly exponentially stable in X, if and only if there exists a function $V : X \to [0, \infty)$ such that $\sqrt{V(\cdot)}$ is a norm on X,

$$V(\varphi) \leq c\|\varphi\|_X^2,$$

for some constant $c > 0$ and

$$\underline{D}_\sigma V(\varphi) \leq -\|\varphi\|_X^2, \quad \sigma \in \mathbb{S}, \varphi \in X.$$

Lemma (F.M. Hante and M. Sigalotti[a])

[a]F.M. Hante and M. Sigalotti. Converse Lyapunov theorems for switched systems in Banach and Hilbert spaces. SIAM J. Control Optim., 49(2):752–770, 2011.

Assume that

(i) *there exist* $M \geq 1$ *and* $w > 0$ *such that*

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq Me^{wt}, \quad t \geq 0, \ \sigma(\cdot)\text{-uniformly},$$

(ii) *there exist* $c \geq 0$ *and* $p \in [1, +\infty)$ *such that*

$$\int_0^{+\infty} \|T_{\sigma(\cdot)}(t)x\|_X^p \leq c\|x\|_X^p, \ \sigma(\cdot)\text{-uniformly},$$

for every $x \in X$.

Then there exist $K \geq 1$ *and* $\mu > 0$ *such that*

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq Ke^{-\mu t}, \quad t \geq 0, \ \sigma(\cdot)\text{-uniformly}.$$

Second converse theorem

Theorem

Suppose that condition (K) *holds. Then system* (Σ_s) *is uniformly exponentially stable in* X *if and only if there exists a continuous function* $V : X \to [0, +\infty)$ *such that*

$$V(\varphi) \leq c\|\varphi\|_X^2,$$

for some constant $c > 0$ *and*

$$\underline{D}_\sigma V(\varphi) \leq -|\varphi(0)|^2, \sigma \in \mathbb{S}, \varphi \in X.$$

Proof

1. $$V(x_t) - V(x_0) \leq -\int_0^t |x_s(0)|^2 ds$$

2. $$\int_0^{+\infty} |x_s(0)|^2 ds \leq c\|\varphi\|_X^2$$

3. $$\int_0^t \|x_s\|_{H^1}^2 ds \leq c_1 \int_0^t |x_s(0)|^2 ds + c_2\|\varphi\|_{H^1}^2 ds,$$

4. $$\int_0^{+\infty} \|x_t\|_{H^1}^2 ds \leq c_0\|\varphi\|_{H^1}^2,$$

Extension to measurable cases

$$Q := \{L_\sigma \in \mathcal{L}(X, \mathbb{R}^n) \mid \sigma \in \mathbb{S}\}.$$

Theorem

System (Σ) is uniformly exponentially stable for $L : [0, +\infty) \to Q$ such that $L(\cdot)\varphi$ is measurable for any $\varphi \in X$ if and only if it is uniformly exponentially stable for $L \in PC([0, +\infty), Q)$.

Proof

Lemma

System (Σ) *is uniformly exponentially stable for* $L : [0, +\infty) \to Q$ *such that* $L(\cdot)\varphi$ *is measurable for any* $\varphi \in C([-r, 0], \mathbb{R}^n)$ *if and only if it is uniformly exponentially stable for* $L \in PC\,([0, +\infty), Q)$.

Lemma

Suppose that condition (K) *holds. The following two statements are equivalent:*

(i) *System* (Σ) *is uniformly exponentially stable in* $C([-r, 0], \mathbb{R}^n)$.

(ii) *System* (Σ) *is uniformly exponentially stable in* $H^1([-r, 0], \mathbb{R}^n)$.

Conclusion

- In this work we give a collection of converse Lyapunov–Krasovskii theorems for uncertain retarded functional differential equations.
- The first converse Theorem shows that the existence of a squared norm $V(\cdot)$ on $C([-r, 0], \mathbb{R}^n)$ is a necessary and sufficient condition for the uniform exponential stability of system (Σ).
- By the second converse theorem the assumption that $V(\cdot)$ is a squared norm is dropped.
- One of the novelties of our results is that these functionals may not have a strictly positive norm-dependent lower bound, in contrast with what is known in the literature.

Thank you for your attention