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Explicit robustness margins for discrete-time linear systems with PWA control

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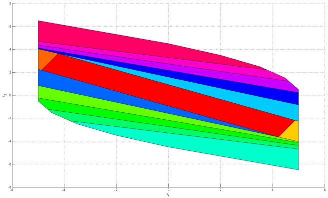
Outline

PWA dynamical systems: the state partition

A *polyhedral partition* of a compact set $\mathcal{X} \subset \mathbb{R}^n$ is defined as follows

1) $\mathcal{X} = \bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i, \; N \in \mathbb{N}_+$
2) $\mathcal{X}_i$ is polyhedral for $\forall i \in \mathcal{I}_N$
3) $int(\mathcal{X}_i) \bigcap int(\mathcal{X}_j) = \emptyset$ with $i \neq j, \; (i,j) \in \mathcal{I}_N^2$
4) $(\mathcal{X}_i, \mathcal{X}_j)$ are neighbours if $(i,j) \in \mathcal{I}_N^2$, $i \neq j$ and $dim(\mathcal{X}_i \cap \mathcal{X}_j) = n - 1$



PWA dynamical: the PWA control for a LTI system

- A nominal discrete time linear system

$$x(k+1) = A_0 x(k) + B_0 u(k) \tag{1}$$

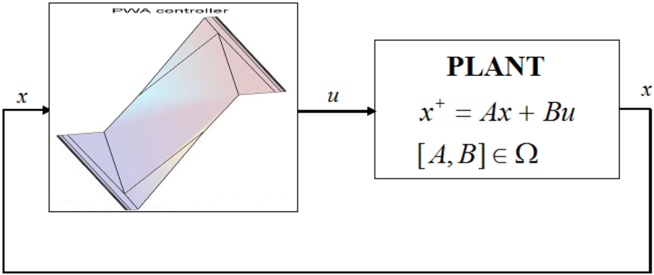
- A function $u_{pwa} : \mathcal{X} \to \mathbb{R}^p$ defined over a polyhedral partition $\mathcal{X} = \bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i \subset \mathbb{R}^n$ by

$$\begin{gathered} u_{pwa}(x) = G_i x + g_i \text{ for } x \in \mathcal{X}_i \\ G_i \in \mathbb{R}^{p \times n} \; g_i \in \mathbb{R}^{p \times 1} \end{gathered} \tag{2}$$

is a *piecewise affine control law* over the partition $\mathcal{X}$.

Motivation

- The PWA structure of the control law is imposed here without any further discussion on the design method.
- Several popular techniqueslead to such a problem formulation, among which we can mention for example the Model Predictive Control in its explicit form (Bemporad et al, Seron et al, Tondel et al., Olaru and Dumur).



PWA systems with uncertainties

- The nominal closed loop dynamics represent a PWA system:

$$x(k+1) = f_{pwa}(x(k)) = (A_0 + B_0 G_i)x(k) + B_0 g_i \quad \text{for } x(k) \in \mathcal{X}_i \tag{3}$$

- But functioning in the presence of:
  - (Time-varying) Polytopic uncertainty

$$\Omega = conv\{(A_1, B_1), ..., (A_L, B_L)\}. \tag{4}$$

If $(A, B) \in \Omega$ then $\exists\ \lambda_1, ..., \lambda_L \geq 0$, $\sum_{i=1}^{L} \lambda_i = 1$ and

$$(A, B) = \sum_{i=1}^{L} \lambda_i (A_i, B_i)$$

  - Additive distrubances:

$$x(k+1) = Ax + B(G_i x + g_i) + w. \tag{5}$$

  - Restricted resolution for controller implementation.

We aim to answer several questions:

- How to determine the robustness margin with respect to a given PWA controller?
- How to determine the fragility margin of a given PWA controller?
- How to compute the set of tolerable additive disturbance with respect to a given PWA controller?
- How to compute the tolerable errors for the representation of the regions in the state space partition associated with a given PWA controller?

Positive invariance

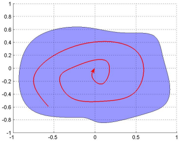
Definition

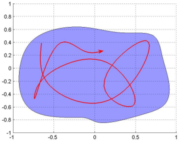
A set $\mathcal{X} \in \mathbb{R}^n$ is positively invariant with respect to the system $x(k+1) = f(x(k))$ if for any $x_0 \in \mathcal{X}$, the solution $x(k, x_0)$ satisfies $x(k, x_0) \in \mathcal{X}$ for $k \in \mathbb{N}$.

An equivalent definition in a set theoretic framework:

Definition

A set $\mathcal{X} \in \mathbb{R}^n$ is positively invariant with respect to the system $x(k+1) = f(x(k))$ if $f(\mathcal{X}) \subset \mathcal{X}$.





Assumptions

The closed loop nominal PWA systems enjoy the following properties:

1. The set $\mathcal{X}$ is a polytope.
2. The set $\mathcal{X}$ is positively invariant with respect to the nominal model.
3. The nominal model is defined by a pair of matrices (A_0, B_0) contained in the polytopic set of uncertainties, namely $(A_0, B_0) \in \Omega$.
4. The control $u_{pwa} : \mathbb{R}^n \to \mathbb{R}^m$ is continuous.
5. $0 \in int(\mathcal{X})$ and the origin is the only fixed point of the nominal dynamics.
6. The origin is asymptotically stable with $\mathcal{X}$ as basin of attraction.

Outline

The statement of the robustness problem

We consider that the nominal system is used for the design of a PWA control law which in turn will be used in practice for the control of the linear parameter varying dynamics:

$$x(k+1) = Ax(k) + Bu(k) \tag{6}$$

where (A, B) lies in the uncertainty set Ω.

The robustness problem is to identify the subset Ω_{rob}, defined as the largest subset $\Omega_{rob} \subset \Omega$ such that $\mathcal{X}$ is positive invariant with respect to the parameter varying dynamical system:

$$\begin{array}{c} x(k+1) = (A + BG_i)x(k) + Bg_i \\ \text{for } i \text{ such that } x(k) \in X_i \end{array} \tag{7}$$

for all $(A, B) \in \Omega_{rob}$.

The set Ω_{rob} will be denoted *robustness margin*.

Preliminary results

Despite the nonlinearity of the closed loop dynamics, the convexity of the state space partition $\mathcal{X}$ is inherited by the set of uncertainties guaranteeing the positive invariance.

Theorem

The set Ω_{rob} is convex.

Corollary

Let the set

$$\Omega^{\alpha}_{rob} = \Big\{ \alpha \in \mathbb{R}^L_+ | \sum_{j=1}^{L} \alpha_j (A_j + B_j G_i) \mathcal{X}_i \oplus \alpha_j B_j g_i \subset \mathcal{X}, \quad \forall i \in \mathcal{I}_N, \mathbf{1}^T \alpha = 1 \Big\} \tag{8}$$

The sets Ω_{rob} and Ω^{α}_{rob} are isomorphic.

Some notation

The vertices $\mathcal{V}(X)$ define a finite subset of $\mathbb{R}^n$. Using an arbitrary ordering, its elements can be stored as columns of a matrix $V \in \mathbb{R}^{n \times q}$:

$$V = [\mathcal{V}(X)] \tag{9}$$

V

All the vertices in of the partition will form:

$$\mathcal{W} = \{x \in \mathbb{R}^n | \exists i \in \mathcal{I}_N \text{ s.t. } x \in \mathcal{V}(X_i)] \tag{10}$$

WO

and after fixing an arbitrary ordering of the elements, this set will generate a matrix $W \in \mathbb{R}^{n \times p}$:

$$W = [\mathcal{W}] \tag{11}$$

W

which gathers as columns the nonredundent vertices of the collection of regions in the partition $\mathcal{X}_i$. At the same time, the image of the set $\mathcal{W}$ via the affine mapping $f_{pwa}(.)$ leads to

$$U = [f_{pwa}(\mathcal{W})] \tag{12}$$

U

Main result – the vertex representation

Theorem

Consider a nominal LTI system subject to a time-varying, bounded, polytopic uncertainty. For a given piecewise affine control law, the robustness margin is obtained as the projection

$$\Omega^{\alpha}_{rob} = Proj_{\mathcal{S}_L} \mathcal{R} \qquad (13)$$

where $\mathcal{R}$ represents the polyhedral set:

$$\mathcal{R} = \left\{ (\alpha, \Gamma) \in \mathbb{R}^L \times \mathbb{R}^{q \times p}_{+} \middle| \mathbf{1}^T \Gamma = \mathbf{1}^T \right. \\ \left. \sum_{j=1}^{L} \alpha_j (A_j W + B_j U) = V\Gamma \right\} \qquad (14)$$

$\mathcal{S}_L$ denotes the simplex in the positive orthant of $\mathbb{R}^L$.

Some notation for the dual representation

The usual description of the state partition for a PWA control laws use the half space representation. Consider the sets involved in these relations:

$$X = \{x : Fx \leq w,\ F \in \mathbb{R}^{p\times n},\ w \in \mathbb{R}^p\} \tag{15}$$

$$X_i = \{x : F_i x \leq w_i,\ F \in \mathbb{R}^{p_i\times n},\ w_i \in \mathbb{R}^{p_i}\} \tag{16}$$

Let us denote the tuple of matrices

$$\mathcal{H} = (H_1, H_2, \dots, H_i, \dots) \text{ with } H_i \in \mathbb{R}_+^{p\times p_i}, i \in \mathcal{I}_N \tag{17}$$

The elements of the tuple $\mathcal{H}$ are indexed by the set $\mathcal{I}_N$ and the dimensions of each matrix-element H_i will depend on the value of p_i. If $s = p \times (\sum_{i\in\mathcal{I}_N} p_i)$ then one can note $\mathcal{H} \in \mathbb{R}^s$.

hs_

Main result – the half-space representation

Theorem

Let a nominal LTI system affected by a time-varying, bounded, polytopic uncertainty. For a given piecewise affine control law, the robustness margin is obtained as the projection

$$\Omega^{\alpha}_{rob} = Proj_{\,\mathcal{S}_L}\mathcal{P} \tag{18}$$

where $\mathcal{P}$ represents the polyhedral set:

$$\mathcal{P} = \Big\{(\alpha, \mathcal{H}) \in \mathbb{R}^L \times \mathbb{R}^s_+ \Big| \; \sum_{j=1}^{L} \alpha_j F(A_j + B_j G_i) = H_i F_i \; H_i w_i \le w - \sum_{i=1}^{L} F_i \alpha_j B_j g_i \Big\} \tag{19}$$

pro

pro

Numerical example

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0.9 & 0.5 \end{bmatrix} \quad A_2 = \begin{bmatrix} 2 & 1 \\ 1.5 & 1.5 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1.5 & 1 \\ 3.8 & 1 \end{bmatrix}$$
$$B_1 = B_2 = B_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{20}$$

In presence of constraints on the control variable and the output variable

$$\begin{aligned} -0.2 \leq u_k \leq 0.2, \\ -0.5 \leq y_k \leq 0.5, \end{aligned} \tag{21}$$

with the nominal model chosen to synthesize a PWA control law

$$\begin{aligned} A_0 &= 0.5A_1 + 0.3A_2 + 0.2A_3 \\ B_0 &= B1. \end{aligned} \tag{22}$$

Numerical example

The partition of the PWA control law (obtained via predictive control design).

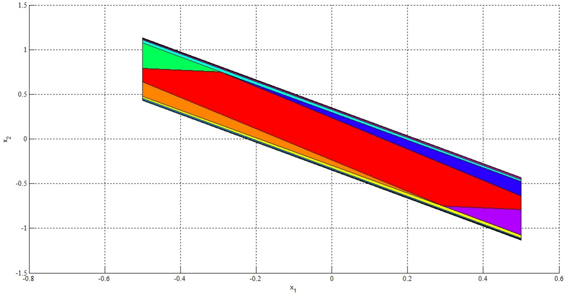


Figure: Partition of the PWA control law.

g:1

Numerical example

Ω_{rob}^{α} projected on the plane $[\alpha_1 \;\; \alpha_2]$ computed via the vertex representation approach. Note that $\alpha_3 = 1 - \alpha_1 - \alpha_2$

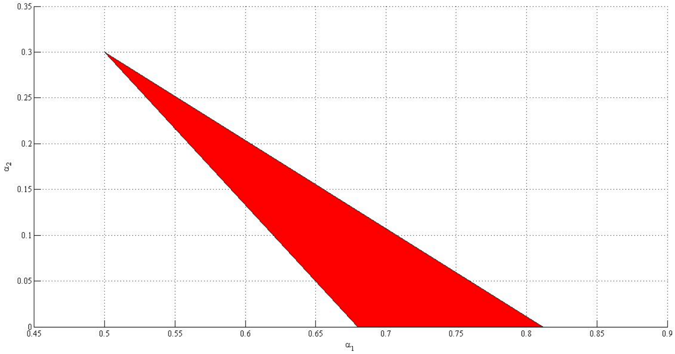


Figure: Robustness margin obtained via vertex representation of the PWA partition.

Outline

Problem statement of the fragility problem

Given a nominal system:

$$x_{k+1} = Ax_k + Bu_k, \tag{23}$$

eqs

stabilized by a PWA control law $f_{pwa} : \mathcal{X} = \bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i \to \mathbb{R}^m$:

$$u_k = f_{pwa}(x_k) = G_i x_k + g_i \quad \text{for} \quad x_k \in \mathcal{X}_i. \tag{24}$$

pwa

The fragility margin problem aims to determine the set of parametric errors Δ_i of the PWA control law for each region $\mathcal{X}_i$ such that:

$$(A + B(G_i + \delta_{G_i}))x_k + B(g_i + \delta_{g_i}) \subset \mathcal{X} \quad \text{for} \quad \forall x_k \in \mathcal{X}_i. \tag{25}$$

for all $\left[vec(\delta_{G_i})^T \ \delta_{g_i}^T \right]^T \in \Delta_i$.
The largest such set is called fragility margin.

Preliminary results

Theorem

The set Δ_i is convex for $\forall i \in \mathcal{I}_N$.

Notation

Consider the compact notation:

$$V_i = [\mathcal{V}(\mathcal{X}_i)] \quad (26)$$

Vi

$$U_i = [f_{pwa}(\mathcal{V}(\mathcal{X}_i))] \quad (27)$$

Ui

$$V = [\mathcal{V}(\mathcal{X})] \quad (28)$$

$\mathbf{1}$ denotes the one vector with appropriate dimension

Main result–vertex representation

Theorem

Consider a linear discrete-time system (23) stabilized by a given piecewise affine state feedback (24) over a bounded polyhedral set $\mathcal{X} = \bigcup_{i\in\mathcal{I}_N} \mathcal{X}_i$. The fragility margin of the controller defined over $\mathcal{X}_i$ is obtained as

$$\Delta_i = Proj_{\,(\delta_{G_i},\delta_{g_i})}\, \mathcal{F}_i, \tag{29}$$

where $\mathcal{F}_i$ represents the polyhedral set:

$$\begin{aligned}\mathcal{F}_i = \Big\{(\delta_{G_i}, \delta_{g_i}, \Gamma_i) \in \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times 1} \times \mathbb{R}_+^{q\times q_i} \mid \\ \mathbf{1}^T\Gamma_i = \mathbf{1}^T, \text{ and} \\ \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} V_i \\ U_i \end{bmatrix} + B\delta_{G_i} V_i + B\delta_{g_i}\mathbf{1}^T = V\Gamma_i \Big\}.\end{aligned} \tag{30}$$

Notation

Recall the half space representation:

$$\mathcal{X} := \{x \in \mathbb{R}^n \mid Fx \leq w\}$$
$$F \in \mathbb{R}^{r \times n}, \text{ and } w \in \mathbb{R}^{r \times 1} \tag{31}$$

$$\mathcal{X}_i := \{x \in \mathbb{R}^n \mid F_i x \leq w_i\}$$
$$F_i \in \mathbb{R}^{r_i \times n}, \text{ and } w_i \in \mathbb{R}^{r_i \times 1}, \text{ for } i \in \mathcal{I}_N \tag{32}$$

Main result–half-space representation

Theorem

For a given piecewise affine control law (24) stabilizing the discrete LTI system (23), the fragility margin of this controller is obtained as the projection

$$\Delta_i = Proj_{\,(\delta_{G_i},\delta_{g_i})}\mathcal{Q}_i \tag{33}$$

where $\mathcal{Q}_i$ represents the polyhedral set:

$$\begin{aligned}\mathcal{Q}_i = \Big\{(\delta_{G_i},\delta_{g_i},H_i) \in \mathbb{R}^{m\times n}\times\mathbb{R}^{m\times 1}\times\mathbb{R}_+^{r\times r_i} \mid \\ F(A+B(G_i+\delta_{G_i})) = H_iF_i \ \textit{ and} \\ H_iw_i \leq w - FB(g_i+\delta_{g_i})\Big\}.\end{aligned} \tag{34}$$

Numerical example

- The example considered in the robustness margin section is revisited here. The fragility margin can be computed for each reagion independently as long as the parameters' influence is local.
- The fragility margin for the green region $\mathcal{X}_3$ in Fig.1.

$$\mathcal{X}_3 := \begin{bmatrix} 0.84484 & 0.53502 \\ -1 & 0 \\ -0.17014 & -0.98542 \end{bmatrix} x \leq \begin{bmatrix} 0.15458 \\ 0.5 \\ -0.69202 \end{bmatrix} \qquad (35)$$

$$u_k = \begin{bmatrix} -1.4911 & -0.9443 \end{bmatrix} x + 0.0728$$

Numerical example

The set of error δ_{G_3} via the vertex representation by considering that there does not exist the error on g_3:

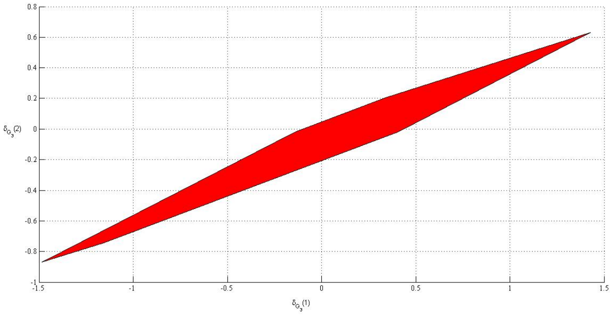


Figure: Fragility margin on the controller associated with $\mathcal{X}_3$

The same result can be obtained via the half-space representation.

Outline

Problem statement of explicit additive disturbance margin

Consider a PWA controller $f_{pwa} : \mathcal{X} \to \mathbb{R}^m$:

$$\mathcal{X} = \bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i, \qquad u_k = f_{pwa}(x) = G_i x_k + g_i \text{ for } \forall x \in \mathcal{X}_i, \tag{36}$$

pwa

which stabilizes a nominal dynamic:

$$x_{k+1} = Ax_k + Bu_k, \tag{37}$$

eqs

The explicit additive disturbance problem determines the largest set of an additive disturbance w such that the positive invariance property of the feasible set $\mathcal{X}$ holds:

$$Ax + B(G_i x + g_i) + w \in \mathcal{X}, \quad \forall x \in \mathcal{X}. \tag{38}$$

This largest set of additive disturbance is called *additive disturbance margin*.

Main result

Note that:

- $W = [\mathcal{W}]\,, \quad \mathcal{W} = \bigcup_{i \in \mathcal{I}_N} \mathcal{V}(\mathcal{X}_i)$
- $U = [f_{pwa}(\mathcal{W})]\,, \quad p = Card(\mathcal{W})$
- $\mathcal{X} := \{x \in \mathbb{R}^n \mid Fx \leq h\}$
- $\mathbf{I}$ denotes the identity matrix of appropriate dimension.

Theorem

Given a LTI dynamic (37) stabilized by a given PWA control law (36), the biggest set of additive disturbance w which perturbs the nominal dynamic and preserves the positive invariance property of $\mathcal{X}$ is described in a the half-space representation by:

$$(\mathbf{1} \otimes F)w \leq (\mathbf{1} \otimes h) - (\mathbf{I} \otimes F \begin{bmatrix} A & B \end{bmatrix}) vec\left(\begin{bmatrix} W \\ U \end{bmatrix}\right). \qquad (39)$$

Numerical example

The same example is considered, the additive disturbance margin is depicted as follow:

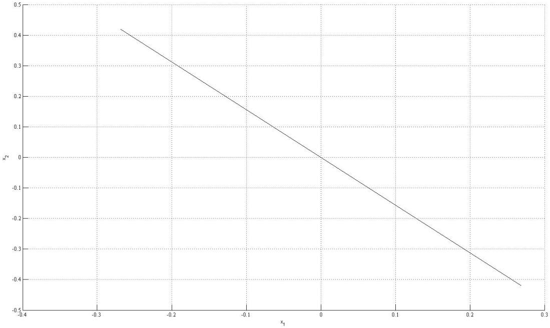


Figure: Additive disturbance margin

Outline

Problem statement

Given a nominal system:

$$x_{k+1} = Ax_k + Bu_k, \tag{40}$$

eqs

and the stabilizing PWA control law:

$$\begin{aligned} \mathcal{X} &= \bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i, \\ u_k &= f_{pwa}(x) = G_i x_k + g_i \text{ for } \forall x \in \mathcal{X}_i, \end{aligned} \tag{41}$$

pwa

Consider the vertex representation of $\mathcal{X}_i$:

$$\widetilde{\mathcal{X}}_i := conv\left\{w_{i1} + \delta_{i1}, \ldots, w_{iq_i} + \delta_{iq_i}\right\}. \tag{42}$$

The explicit fragility of polyhedral partition problem aims to determine the largest set of uncertainties $\delta_i = \left[\delta_{i1}^T \ldots \delta_{iq_i}^T\right]^T, i \in \mathcal{I}_N$ for each region $\mathcal{X}_i$ of the polyhedral partition $\mathcal{X}$ such that *$\mathcal{X}$ is positively invariant*.

Main result

Assumption 1:

No uncertainty on the boundary of $\mathcal{X}$.

Assumption 1:

The uncertainty is considered exclusively with respect to the generators representation of the partition of $\mathcal{X}$.

Notation

Recall that:

- $\mathcal{X} = \{x \in \mathbb{R}^n \mid Fx \leq h\}$
- $V_i = [\mathcal{V}(\mathcal{X}_i)]$
- $U_i = [f_{pwa}(\mathcal{V}(\mathcal{X}_i))]$
- $q_i = Card(\mathcal{V}(\mathcal{X}_i))$

Main result

Theorem

Consider a polyhedral partition $\mathcal{X} = \bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i$ over which a continuous PWA controller (41) is defined. This controller stabilizes a LTI dynamic (40). The fragility margin of the vertex representation of $\mathcal{X}_i$ is described as:

pwactrl2 eqsys2

$$\begin{bmatrix} \mathbf{I} \otimes F \\ \mathbf{I} \otimes F(A + BG_i) \end{bmatrix} \delta_i \leq \begin{bmatrix} \mathbf{1} \otimes h - (\mathbf{I} \otimes F) vec(V_i) \\ \mathbf{1} \otimes h - (\mathbf{I} \otimes F \, [A \ B]) vec\left(\begin{bmatrix} V_i \\ U_i \end{bmatrix}\right) \end{bmatrix}, \tag{43}$$

d00

where $\mathbf{1} \in \mathbb{R}^{q_i}$ and $\mathbf{I} \in \mathbb{R}^{q_i \times q_i}$.

Numerical example

Consider the region containing the origin in the previous example. One of its vertices is $x = [-0.5 \quad 0.7886]^T$ and its fragility margin is shown below:

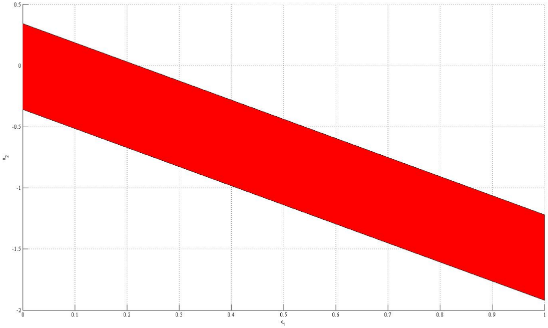


Figure: The fragility margin of the vertex $x = [-0.5 \quad 0.7886]^T$.

Conclusions

- The robustness margin has been studied with respect to the positive invariance for a PWA system
- The same methodology allows the extension to the fragility margin, additive disturbance margin and fragility margin of the state space partition.
- It is shown that these margins are represented by polyhedra in the parameter space and can be explicitly computed.

Further details and proofs:

Further details and proofs:

- N. A. Nguyen, S. Olaru, G. Bitsoris, P. Rodriguez-Ayerbe. "Explicit fragility margins for PWA control laws of discrete-time linear systems", ECC 2014.
- N. A. Nguyen, S. Olaru, P. Rodriguez-Ayerbe, G. Bitsoris, M. Hovd. "Explicit robustness margins for discrete-time linear systems with PWA control", ICSTCC 2013.

and for the use of the results for control synthesis:

- N. A. Nguyen, S. Olaru, P. Rodriguez-Ayerbe, M. Hovd. "An inverse optimality argument to improve robustness in constrained control", IFAC 2014.

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Workshop # 5

Prev Next

Interpolation-based techniques for constrained control: from improved vertex control to robust model predictive control alternatives.

Organizers: Hoai-Nam Nguyen, Sorin Olaru and Per-Olof Gutman

The present workshop intends to present the latest developments on a topic with a renewed interest in the last years: the interpolation based control. Here, the strength of interpolation based control will be demonstrated for a large class of fundamental control problems covering the state/output feedback control of uncertain linear systems, including time varying and parameter varying systems with a specific attention to the presence of constraints on the outputs, states, and control inputs. The one-day event will present historical elements of constrained control and bring gradually the auditorium to the recent challenges in the optimization-based control design related topic. The auditorium will be introduced to the theoretical foundations of a generic interpolation scheme and subsequently will be exposed to the analysis of structural implications and the computational aspects of the resulting control law for various classes of dynamical systems. The participants will have the opportunity to compare the features of novel methodologies with respect to classical alternatives as vertex control or Model Predictive Control in their implicit or explicit formulations. In addition, the participants will attain the basis to apply computational tools to the design of robust interpolating controllers for uncertain linear systems with state and control constraints.

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