# Robust stability analysis of discrete-time systems with parametric and switching uncertainties

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- Analysis of discrete-time polytopic systems $x_{k+1} = A(\theta_k)x_k$ where

$$A(\theta_k) = \sum_{v=1}^{\bar{v}} \theta_{k,v} A^{[v]} \ : \ \theta_k \in \Xi_{\bar{v}} = \left\{ \sum_{v=1}^{\bar{v}} \theta_{k,v} = 1 \ , \ \theta_{k,v} \geq 0 \right\}$$

❶ “Quadratic stability" test [Bar85] $\theta_{k+1} \neq \theta_k$

$$\exists P \succ 0 \ : \ A^{[v]T} P A^{[v]} - P \prec 0 \ \ \forall v = 1 \dots \bar{v}$$

❷ “Slack variables" test [PABB00] assuming $\theta_k = \theta \ \ \forall k \in \mathbb{N}$

$$\exists P^{[v]} \succ 0, G \ : \ \begin{bmatrix} P^{[v]} & 0 \\ 0 & -P^{[v]} \end{bmatrix} \prec \left\{ G \begin{bmatrix} I & -A^{[v]} \end{bmatrix} \right\}^{\mathcal{S}} \ \ \forall v = 1 \dots \bar{v}$$

❸ “Switching" test [DB01] $\theta_{k+1} \neq \theta_k$

$$\exists P^{[v]} \succ 0 \ : \ A^{[v]T} P^{[w]} A^{[v]} - P^{[v]} \prec 0 \ \ \forall v = 1 \dots \bar{v} \ \ \forall w = 1 \dots \bar{v}$$

- ❷⇐❶ and ❸⇐❶ but ❷??❸

- Any rationally dependent model admits an LFT representation

$$x_{k+1} = A(\theta_k)x_k \quad : \quad A(\theta) = A + B\Delta(\theta)(I - D\Delta(\theta))^{-1}C$$

where $\Delta(\theta)$ is linear in $\theta$.

- Also admits a reduced size descriptor representation

$$E_x(\theta_k)x_{k+1} + E_\pi(\theta_k)\pi_k = F(\theta_k)x_k$$

where $E_x(\theta)$, $E_\pi(\theta)$, $F(\theta)$ are affine in $\theta$

- Proof: take

$$\begin{bmatrix} I \\ 0 \end{bmatrix} x_{k+1} + \begin{bmatrix} -B\Delta(\theta_k) \\ I - D\Delta(\theta_k) \end{bmatrix} \pi_k = \begin{bmatrix} A \\ C \end{bmatrix} x_k$$

- Remark: $\begin{bmatrix} E_x(\theta) & E_\pi(\theta) \end{bmatrix}$ is full column rank if LFT is well-posed.

- Example: $a_k y_{k+2} + {b_k}^2 y_{k+1} + a_k b_k y_k = 0$

- LFT model: $\Delta(\theta)$ is at least $4 \times 4$ (i.e. model with 8 exogenous signals), e.g.

$$A(a,b) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Delta (I - \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Delta)^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

with $\Delta = \begin{bmatrix} bI_2 & 0 \\ 0 & (a-1)I_2 \end{bmatrix}$

- Descriptor model with 1 exogenous signal

$$\begin{bmatrix} a_k & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} x_{k+1} + \begin{bmatrix} b_k \\ 0 \\ 1 \end{bmatrix} \pi_k = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ b_k & a_k \end{bmatrix} x_k$$

- Polytopic descriptor system with uncertain and witching parameters:

$$E_x(\phi,\theta_k)x_{k+1} + E_\pi(\phi,\theta_k)\pi_k - F(\phi,\theta_k)x_k = M(\phi,\theta_k)\begin{pmatrix} x_{k+1} \\ \pi_k \\ x_k \end{pmatrix} = 0$$

$$M(\phi,\theta_k) = \sum_{\mu=1}^{\bar{\mu}}\sum_{v=1}^{\bar{v}} \phi_\mu \theta_{k,v} M^{[\mu,v]} \quad : \quad \phi \in \Xi_{\bar{\mu}} \;\; , \;\; \theta_k \in \Xi_{\bar{v}}$$

- System assumed well-posed: $\begin{bmatrix} E_x(\phi,\theta) & E_\pi(\phi,\theta) \end{bmatrix}$ is strict full column rank.
- Stability is assed if $\exists P^{[\mu,v]} \succ 0, G^{[v]}$:

$$\begin{bmatrix} P^{[\mu,w]} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -P^{[\mu,v]} \end{bmatrix} \prec \left\{ G^{[w]} M^{[\mu,v]} \right\}^{\mathcal{S}} \quad \begin{array}{l} \forall \mu = 1 \ldots \bar{\mu} \\ \forall v = 1 \ldots \bar{v} \\ \forall w = 1 \ldots \bar{v} \end{array}$$

- Contains all previous results as special cases. High numerical complexity.

$$\begin{bmatrix} P^{[\mu,w]} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -P^{[\mu,v]} \end{bmatrix} \prec \left\{ G^{[w]} M^{[\mu,v]} \right\}^{\mathcal{S}} \quad \begin{matrix} \forall \mu = 1 \ldots \bar{\mu} \\ \forall v = 1 \ldots \bar{v} \\ \forall w = 1 \ldots \bar{v} \end{matrix}$$

- Numerical complexity can re reduced if
  - some of the columns of $\begin{bmatrix} E_x(\phi,\theta) & E_\pi(\phi,\theta) \end{bmatrix}$ are independent of $\phi, \theta$
  - or some of the rows of $M(\phi,\theta)$ are independent of $\phi, \theta$.

- Example of the switching ordinary system $M(\theta) = \begin{bmatrix} I & -A(\theta) \end{bmatrix}$
  - lossless reduced size LMIs (no need for slack variables)

$$A^{[v]T} P^{[w]} A^{[v]} - P^{[v]} \prec 0 \quad \forall v = 1 \ldots \bar{v} \quad \forall w = 1 \ldots \bar{v}$$

- Large size LMI for the numerical example

$$\mathcal{P}(P^{[\mu,w]},P^{[\mu,v]}) \prec \left\{ G^{[w]} \left[ \begin{array}{cc|c|cc} a^{[\bullet]} & 0 & b^{[\star]} & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -b^{[\star]} & -a^{[\bullet]} \end{array} \right] \right\}^{\mathcal{S}}$$

- lossless reduced size LMIs (both in nb of rows and nb of vars)

$$\widehat{\mathcal{P}}(P^{[\mu,w]},P^{[\mu,v]}) \prec \left\{ \widehat{G}^{[w]} \left[ \begin{array}{cccc} a^{[\bullet]} & 0 & b^{[\star]} & 0 \\ 0 & -b^{[\star]} & 1 & -a^{[\bullet]} \end{array} \right] \right\}^{\mathcal{S}}$$

- $\bullet = \mu$ if $a$ is parametric (constant) and $\bullet = v$ if $a_k$ is switching
- $\star = \mu$ if $b$ is parametric (constant) and $\star = v$ if $b_k$ is switching

- Numerical results for $a \in [\,1,\ 2\,]$ and $b \in [\,-0.5,\ \beta\,]$.
- Goal: find maximal $\beta$ such that LMIs are feasible

| | $\beta$ (nb vars/nb rows) |
|---|---|
| $a_k, b_k$ | 0.81094 (44/64) |
| $a, b_k$ | 0.89027 (28/32) |
| $a_k, b$ | 0.82658 (28/32) |
| $a, b$ | 0.98059 (20/16) |

- Considered example, model before augmentation:

$$a_k y_{k+2} + {b_k}^2 y_{k+1} + a_k b_k y_k = 0$$

- After one step ahead augmentation:

$$\left\{ \begin{array}{l} a_k y_{k+2} + {b_k}^2 y_{k+1} + a_k b_k y_k = 0 \\ a_{k+1} y_{k+3} + {b_{k+1}}^2 y_{k+2} + a_{k+1} b_{k+1} y_{k+1} = 0 \end{array} \right.$$

- Corresponding descriptor representation

$$\begin{bmatrix} a_{k+1} & 0 & 0 \\ 0 & a_k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tilde{x}_{k+1} + \begin{bmatrix} b_{k+1} & 0 \\ 0 & b_k \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tilde{\pi}_k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ b_{k+1} & a_{k+1} & 0 \\ 0 & b_k & a_k \end{bmatrix} \tilde{x}_k$$

- Results applied to augmented model are guaranteed to be less conservative

- Numerical results for $a \in [\,1,\ 2\,]$ and $b \in [\,-0.5,\ \beta\,]$.
- Goal: find maximal $\beta$ such that LMIs are feasible

| $\beta$ (nb vars/nb rows) | original model | augmented | upper limit |
|---|---|---|---|
| $a_k, b_k$ | 0.81094 (44/64) | 0.84677 (480/1536) | ? |
| $a, b_k$ | 0.89027 (28/32) | 0.90293 (144/192) | ? |
| $a_k, b$ | 0.82658 (28/32) | 0.85375 (144/192) | ? |
| $a, b$ | 0.98059 (20/16) | 0.99519 (48/24) | 1 |

- New LMI test
  - Combines existing techniques w.r.t. uncertain & switching parameters
  - Extends results to descriptor-like models (represent any rationally-dependent system)
  - Contributes to conservatism reduction thanks to the system augmentation technique
  - Numerical complexity is increased ... but can be controlled

- Prospective work
  - Continuous time case
  - Time-varying parameters with bounded rate
  - Extension to design problems (state & output feedback, robust observers...)

- Springer monograph by Y. Ebihara & D. Peaucelle to be published in 2014-2015

# References

[Bar85] B.R. Barmish, *Necessary and sufficient condition for quadratic stabilizability of an uncertain system*, J. Optimization Theory and Applications **46** (1985), no. 4.

[DB01] J. Daafouz and J. Bernussou, *Parameter dependent lyapunov functions for discrete time systems with time varying parametric uncertainties*, Systems & Control Letters **43** (2001), 355–359.

[PABB00] D. Peaucelle, D. Arzelier, O. Bachelier, and J. Bernussou, *A new robust D-stability condition for real convex polytopic uncertainty*, Systems & Control Letters **40** (2000), no. 1, 21–30.