

**Observation et inversion à gauche des systèmes dynamiques hybrides:  
Système linéaire, non linéaire, parcimonie & Zénon.**

**Jean-Pierre Barbot**  
**Quartz-EA-7393-ENSEA**

- 1 Une brève épistémologie d'une théorie de l'observation**
- 2 Des exemples**
- 3 Le cas des systèmes hybrides enjeux et questions**
- 4 Quelques résultats**
- 5 Phénomène de Zénon**
- 6 Le cas des deux Bacs**
- 7 Conclusion**

- En linéaire l'observabilité se détermine classiquement par une condition de Rang.
- Les observateurs classiques pour de tels systèmes sont

### L'observateur de Luenberger

$$\begin{cases} \dot{\hat{X}} = A\hat{X} + BU + L(Y - \hat{Y}) \\ \hat{Y} = C\hat{X} \end{cases}$$

### L'observateur de Kalman

$$\begin{cases} \dot{\hat{X}} = AX + BU + H(Y - \hat{Y}) \\ Y = CX \\ H = \Sigma C' \Theta^{-1} \\ 0 = A\Sigma + \Sigma A' + D\Gamma D' - \Sigma C' \Theta^{-1} C\Sigma \end{cases}$$

Si on fait une analogie hasardeuse avec les philosophes grecs et notamment les stoïciens (fondateur Zénon) 'Rien n'est plus parfait que le monde' Cicéron à Épicure

- En non linéaire l'observabilité se détermine de multiple façons mais la pensée classique conduit à une condition de rang avec de petits arrangement entre amis :

$$\text{Rang} \left\{ \begin{array}{c} dh \\ dL_f h \\ \vdots \\ dL_f^j h \\ \vdots \end{array} \right\} = \dim X$$

- Parmi les problèmes :

Le bruit n'intervient plus linéairement

Observabilité fonction de l'entrée (**H Hammouri, JP Gauthier,...**)

Singularité d'observabilité (**A. Rapaport, C. Letellier,...**)

Différente notion d'observabilité

.....

Saint Augustin « ..., j'avais répandu mon âme sur l'instabilité d'un sable mouvant,..<sub>4</sub> »

Les observateurs classiques pour de tels systèmes sont :

Observateur de Kalman étendu (**D. Bestle, M. Zeitz , V. Fromion,...**)

Observateur avec injection de sortie (**A. Isidori, G. Besancon, F. Plestan,...**)

Observateur Grand Gain (**J-P Gauthier, H. Hammouri, H. Khalil, K. Busawon,..**)

Observateur adaptatif (**J de Leon, A. Glumineau, R. Marino, M. Ghanes,...**)

Observateur Ensembliste (**L. Jaulin, E. Walter, Ch. Combastel,..**)

Observateur en temps fini avec retard (**R. Engel, G. Kreislmeier, F. Allgower,.)**

Certains observateurs sliding (**V. Utkin, C. Edwards, L. Frideman, S. Drakunov,..**)

....

Ils sont tous basés sur  $\dot{e} = Ae$

**B. Pascal, « Le cœur a ses raisons que la raison ne connaît point » , « C'est en manquant de preuves qu'ils ne manquent pas de sens ».**

Le caractère systémique de chercher  $\dot{e} = Ae$  est remis en question par des observateurs du type:

Observateur réduit (D. Koenig, S. Mammari, H. Razik, C. Jung, ....)

Certains observateurs numériques (J Grizzle, S. Diop, M. Fliess, W. Kang, ...)

Certains observateurs sliding (M. Saif, M. Djemai, N. Manamanni,...)

Observateur invariant (N. Aghannan, S. Bonnabel, P. Rouchon, ...)

Observateur avec immersion (P. Bernard, V. Andrieu, L. Praly,...)

....

« Tout est en morceaux, toute cohérence disparue,  
plus de rapport juste, rien ne s'accorde plus » J. Donne

La rupture se fait de façon définitive avec la définition algébrique de l'observabilité

(S. Diop, M. Fliess)

$$X = F(y, \dot{y}, \dots, y^j, u, \dot{u}, \dots, u^k)$$

La conséquence directe de ceci est l'observateur numérique de type ALIEN (Non-A) :

$$p_1(t) = a_0 + a_1 t,$$

Dans le domaine symbolique

$$P_1 = \frac{a_0}{s} + \frac{a_1}{s^2}.$$

Qui après dérivation par  $s$  qui correspond à une multiplication par  $-t$  en temporel

$$s^2 \frac{dP_1}{ds} + 2sP_1 = a_0$$

D'où

$$a_0 = \frac{3!}{t^3} \left( - \int_0^t \tau P_1 d\tau + 2 \int_0^t \int_0^\tau P_1 d\theta d\tau \right)$$

Quelques remarques :

- Filtrage
- Fenêtre
- Dérivée à droite et à gauche
- Applications : observation, détection de défaut, identification, détection de rupture, ..

Une autre approche en rupture totale avec  $\dot{e} = Ae$  l'observateur à mode glissant d'ordre supérieur étape par étape qui est une succession d'équations algébriques résolues l'une après l'autre.

La déconstruction F. Nietzsche, M. Heidegger, G. Deleuze, ...



L'approche par fonction de Lyapunov homogène peut aussi être employée Perruquetti, Moulay, Moreno, Fridman, Ghanes,...

Observateur homogène du 2<sup>ème</sup> Ordre avec adaptation de l'exposant (Une réconciliation entre filtrage et observation)

$$D : \left\{ \begin{array}{l} \dot{z}_1 = z_2 + k_1 \mu |e_{1m}|^\alpha \text{sign}(e_{1m}) \\ \dot{z}_2 = k_2 \alpha \mu^2 |e_{1m}|^{2\alpha-1} \text{sign}(e_{1m}) \\ \dot{z}_3 = -\tau z_3 + \tau |y_{mhf}| \\ \alpha = 0.5 \left( 1 + \frac{z_3}{z_3 + \epsilon} \right) \\ y_{mhf} = \frac{s'^4}{(s'^2 + 0.7654s' + 1)(s'^2 + 1.8478s' + 1)} y_m \\ s' = \frac{s}{w_{cut}} \\ \hat{y} = z_1 \\ e_{1m} = y_m - z_1 \end{array} \right. \quad (2)$$

## Le compromis :

- Pour  $\alpha = 1$  on a un observateur Linéaire
- Pour  $\alpha = 0.5$  on a le différentiateur de A. Levant

**La réconciliation est-elle totale ?**

**Il n'y a pas de retard !**

**Mais le problème d'un signal bruité n'intervenant pas linéairement n'est pas résolu !**

**La généralisation aux systèmes de plus grande dimension n'est pas faite.**

**Mahatma Gandhi « tout compromis repose sur des concessions mutuelles »**

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Soit le système chaotique suivant :

Avec  $y_1 = x_1, y_2 = x_2$

$$\dot{x}_1 = a(x_2 - x_1) + x_2x_3 + \mu_1(x_1, x_2, x_3)m_1$$

$$\dot{x}_2 = b(x_1 + x_2) - x_1x_3$$

$$\dot{x}_3 = -cx_3 - ex_4 + x_1x_2 + \mu_2(x_1, x_2, x_3)m_1$$

$$\dot{x}_4 = -dx_4 + fx_3 + x_1x_3 + \mu_3(x)m_1 + \mu_4(x)m_2$$

On obtient les résultats suivants :

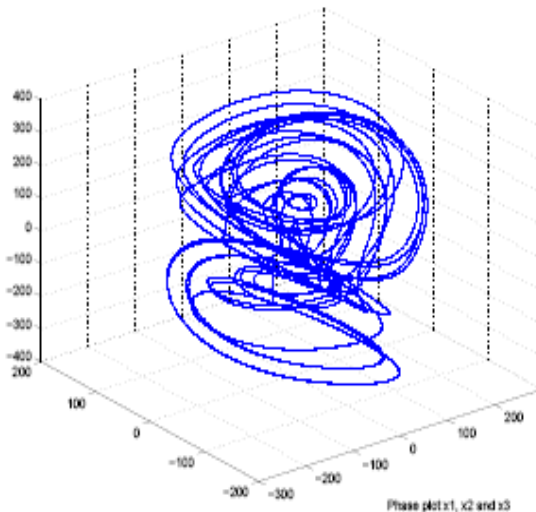


Fig. 5. Phase plot

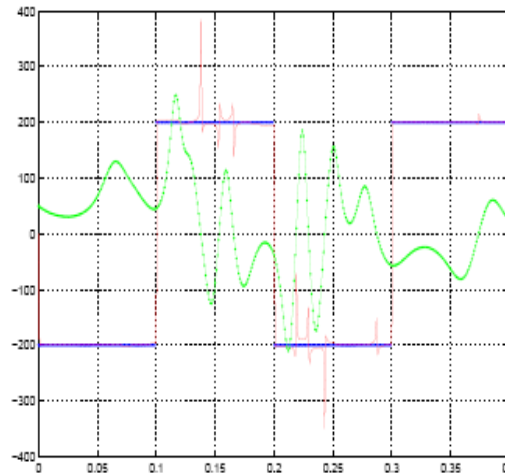


Fig. 8.  $m_1$  in blue, its estimate in red and  $x_1$  in green

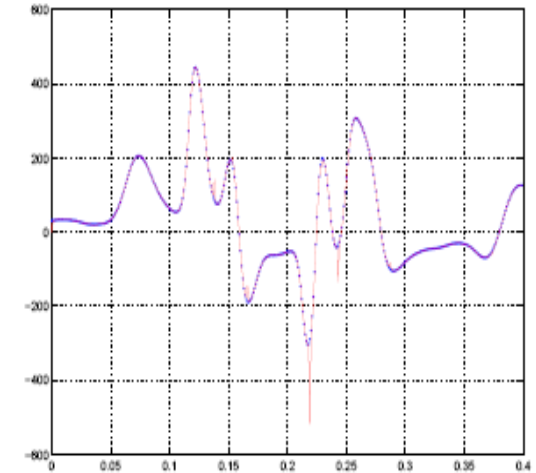


Fig. 7.  $x_4$  in blue and its estimate in red

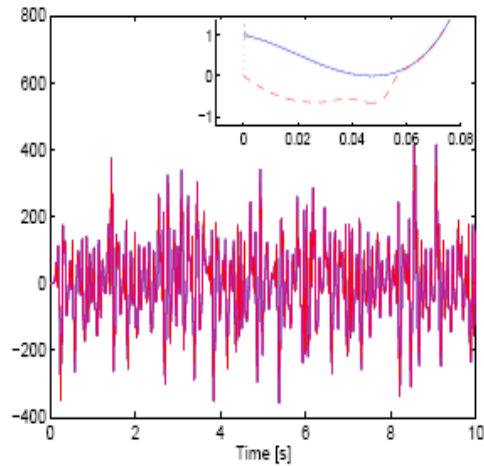


Fig. 2.  $x_3$  (solid line) and its estimation  $\hat{x}_3$  using the super-twisting (dot line) and adaptive (dash line) observers.

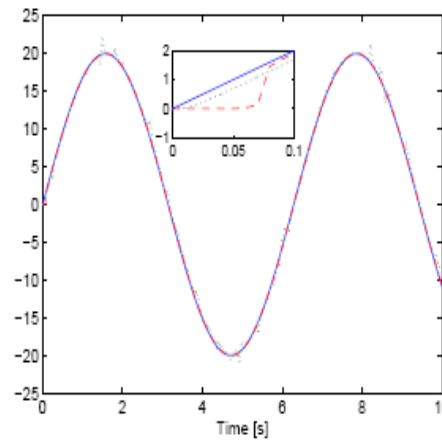


Fig. 3. Message  $m_1$  (solid line) and its estimation  $\hat{m}_1$  using the super-twisting (dot line) and adaptive (dash line) observers.

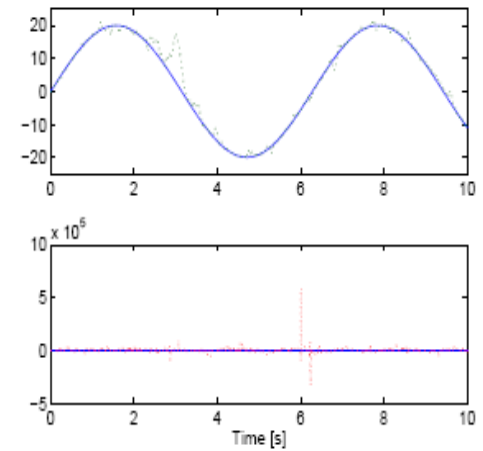


Fig. 4. Message  $m_1$  (solid line) and its estimation  $\hat{m}_1$  (dot line) for the system with 0.01% of uncertainty in the parameters. Above with the super-twisting observer, below with the adaptive observer.

**F. Bejarano, M. Ghanes, ...**

### Le cas d'un double intégrateur avec une entrée inconnue et du bruit de mesure

#### L'entrée inconnue

$$u(t) = \begin{cases} 0 & \text{if } t \in [150s, 300s], [450s, 600s], [750s, 900s] \\ \sin(0.05t) & \text{else.} \end{cases}$$

#### Le bruit de mesure

- by 0 for  $t \in [300, 500s]$  and  $[900, 1000s]$
- by  $\frac{0,1}{35}$  between 500 and 600 s
- by  $\frac{0,15}{35}$  for  $t \in [0, 300s], [800s, 900s]$
- and finally by  $\frac{0,25}{35}$  for  $t \in [600s, 800s]$ .

## Un deuxième exemple (suite)

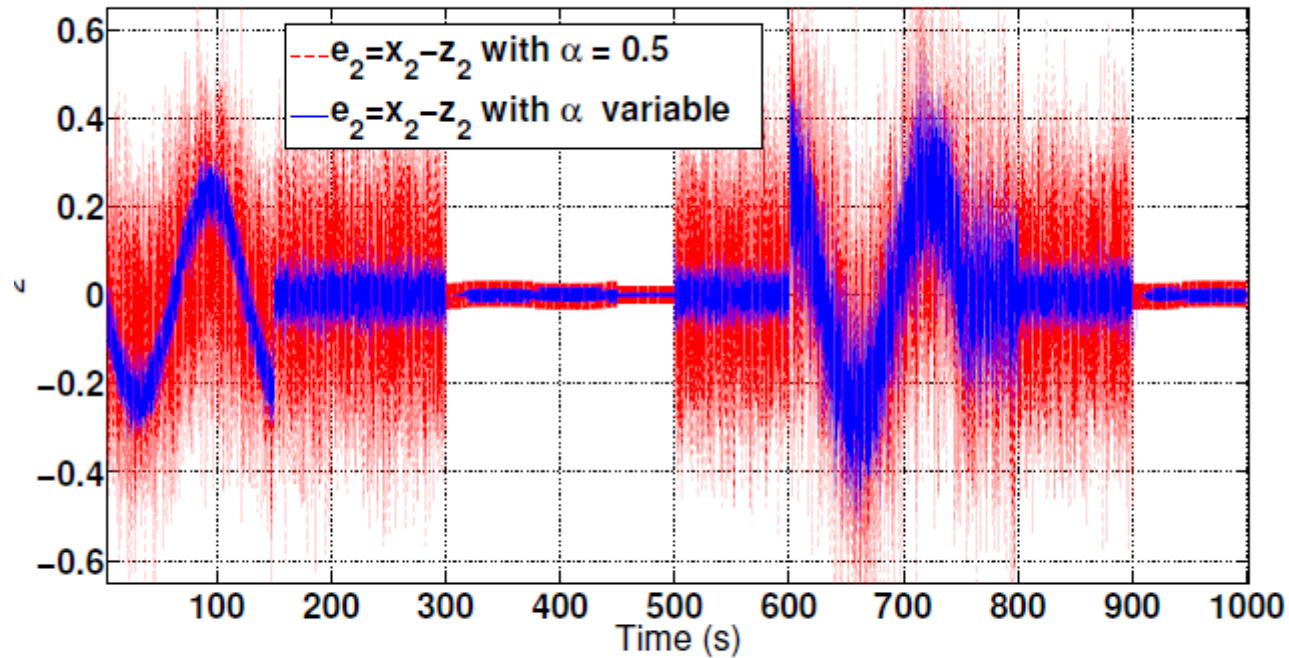


Fig. 1. Estimation error  $e_2$  with  $\alpha$  variable and fixed to 0,5

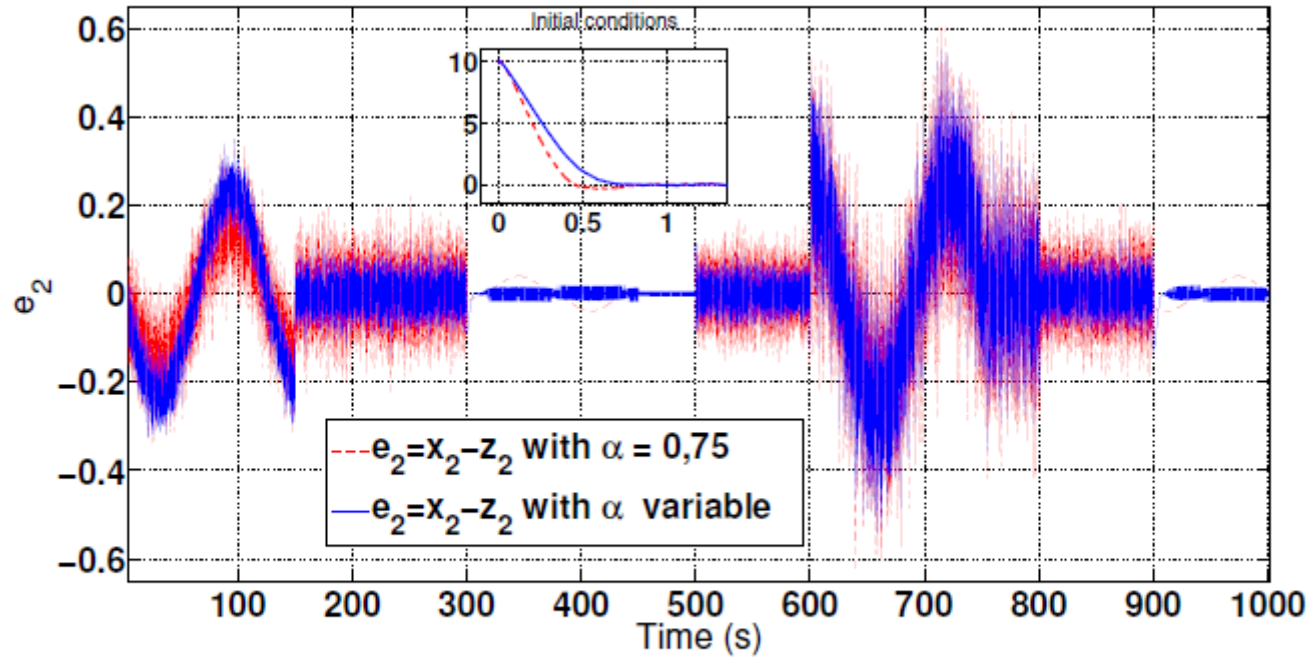


Fig. 2. Estimation error  $e_2$  with  $\alpha$  variable and fixed to 0,75



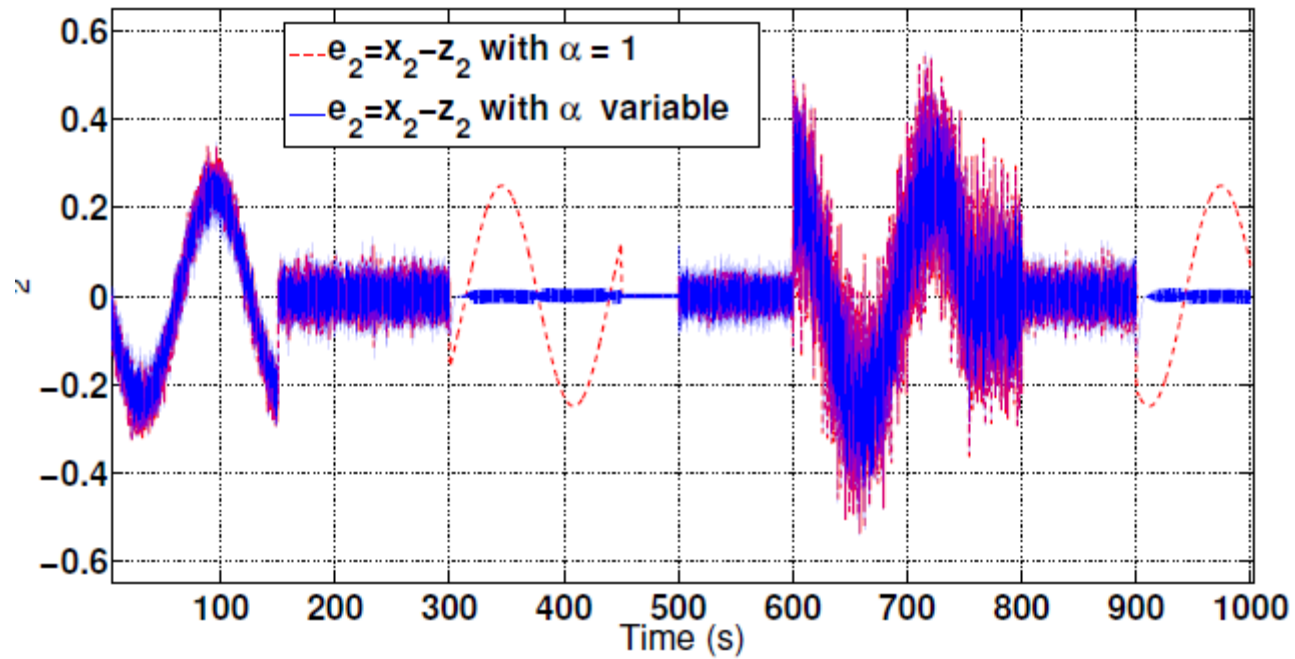


Fig. 3. Estimation error  $e_2$  with  $\alpha$  variable and fixed to 1

# Un deuxième exemple (suite)

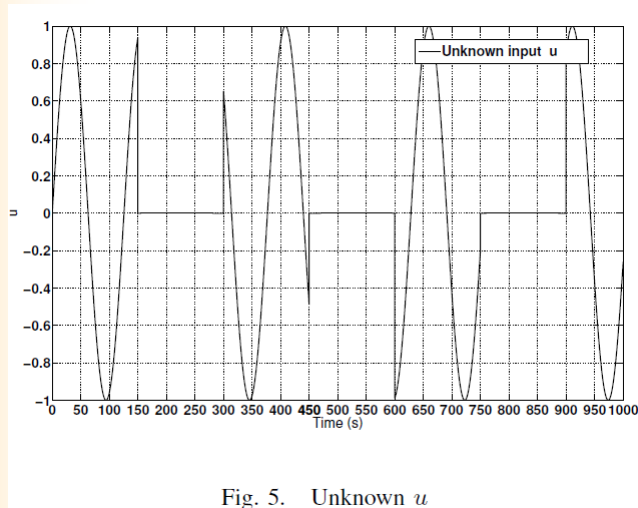
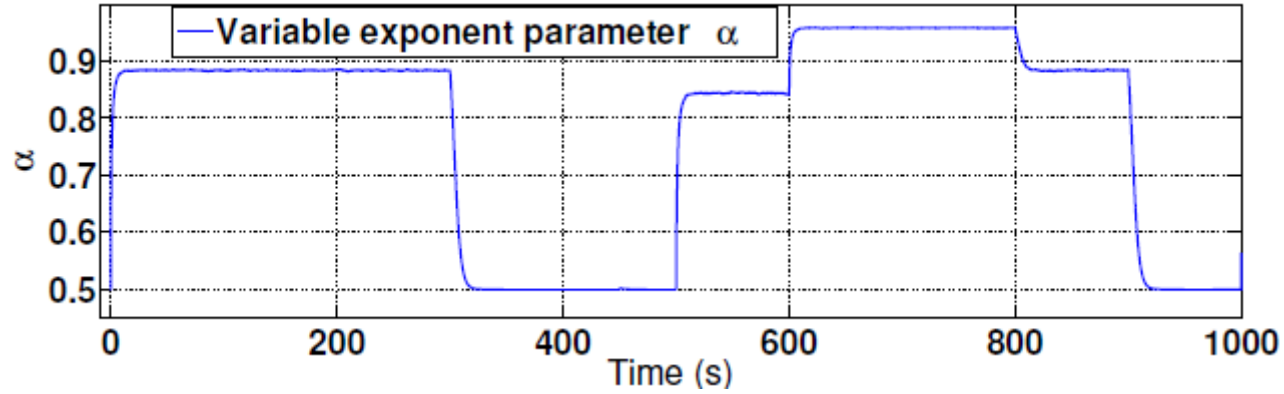


Fig. 5. Unknown  $u$

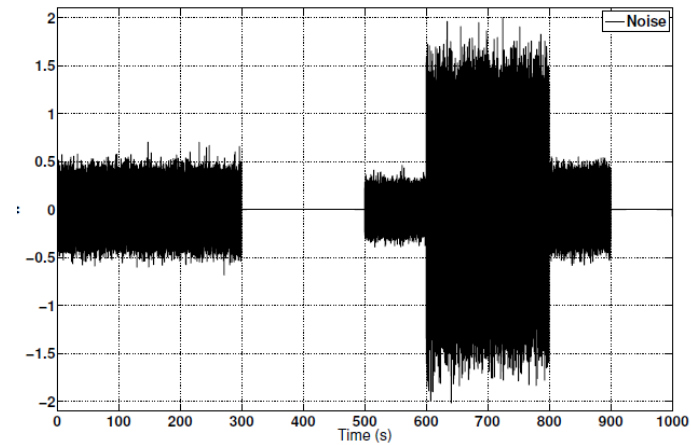


Fig. 6. Noise (added to the state  $x_1$ )

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Soit la définition suivante **J. Lygeros et al.** :

**Definition:** (Hybrid Automaton)

A hybrid automaton  $H$  is a collection  $H = (Q, X, f, Init, D, E, G, R)$ , where

- $Q$  is a finite set of discrete variables
- $X$  is a finite set of continuous variables
- $f$  is a vector field from  $Q \times X$  to  $TX$
- $Init$  is the initial set and  $Init \subseteq Q \times X$
- $D$  is the continuous time domain with respect to the discrete variables (i.e.  $D : Q \rightarrow P(X)$ )
- $E$  is the set of possible discrete transition (  $E \subseteq Q \times Q$  )
- $G$  is a guard condition, the discrete transition  $E$  is possible if the guard transition is verified ( $G : E \rightarrow P(X)$ )
- $R$  is the reset map, or more exactly the continuous state transition ( $R : E \times X \rightarrow P(X)$ )

- **Quelles Définitions d'observabilité ?**
- **la partie discrète**
- **Peut on observer des systèmes possédants des phénomènes de Zénon ?**
- **La convergence asymptotique a-t-elle un sens pour les SHD ?**
- **SDH Synchrones et asynchrones ?**
- **Quel capteur ?**
- **Peut on observer les systèmes sous Zenon ?**
- **...**
- **L'importance des SDH, Systèmes embarqués, systèmes impulsionnels, systèmes sous échantillonnage, systèmes à structure variable.**
- **Domaine d'application processus chimiques, biologiques, l'automobile, l'aviation les robots, l'électronique de puissance, la mécanique,....**

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### Observabilité pour une classe de système commuté :

$$\begin{cases} \dot{x} = f_1(x) \text{ and } y = c_1(x) \text{ if } h(x) \leq 0 \\ \dot{x} = f_2(x) \text{ and } y = c_2(x) \text{ if } h(x) > 0 \end{cases}$$

Les  $c_i(x)$ ,  $f_i(x)$  et  $h(x)$  sont suffisamment lisses.

**Dans la formulation de Lygeros et al. Ceci nous donne :**

- $D(1) = \{x \in \mathbb{R}^n / h(x) \leq 0\}$
- $D(2) = \{x \in \mathbb{R}^n / h(x) > 0\}$
- $E = \{(1, 2), (2, 1)\}$
- $G(1, 2) = \{x \in \mathbb{R}^n / q = 1, h(x) > 0 \text{ and } x(t^-) \in P_1(x)\}$
- $G(2, 1) = \{x \in \mathbb{R}^n / q = 2, h(x) = 0 \text{ and } x(t^-) \in P_2(x)\}$
- $Init = \{1\} \times \{x \in D(1)\} \cup \{2\} \times \{x \in D(2)\}$
- $R(1, 2) = R(2, 1) = x$

**On transforme les systèmes sous forme canonique d'observabilité (M. Fliess) :**

$$\begin{cases} \dot{z}_i^1 = z_{i+1}^1 \text{ for } i=1:n-1 \\ \dot{z}_n^1 = g_1(z_1^1, z_2^1, \dots, z_n^1) \end{cases}$$

if  $h_1 := h(z_1^1, z_2^1, \dots, z_n^1) \leq 0$ , and

$$\begin{cases} \dot{z}_i^2 = z_{i+1}^2 \text{ for } i=1:n-1 \\ \dot{z}_n^2 = g_2(z_1^2, z_2^2, \dots, z_n^2) \end{cases}$$

if  $h_2 := h(z_1^2, z_2^2, \dots, z_n^2) > 0$ .

**Soit les sous variétés :**

$$\mathcal{M} = \{v \in \mathbb{R}^n / g_1(v) = g_2(v)\}$$

$$\mathcal{S} = \{v \in \mathbb{R}^n / h_1(v) = h_2(v)\}$$

$$\mathcal{L} = \{x \in \mathbb{R}^n / f_1(x) = f_2(x) = 0\}$$

**Alors, on a le résultat suivant :**



### Proposition 2 (D. Boutat et al.) :

1. Si  $M$  est un ensemble (topologique) discret alors le système est observable pour toute fonction  $h$  qui vérifie  $h(\mathcal{L}) \leq 0$  ou  $h(\mathcal{L}) > 0$ .
2. Si les dynamiques sont transverses à  $M$  à l'exception d'un ensemble (topologique) discret alors le système est observable pour tout  $h$  qui vérifie  $h(\mathcal{L}) \leq 0$  ou  $h(\mathcal{L}) > 0$ .
3. Si  $\mathcal{S} = \mathbb{R}^n$  alors le système est observable.

Il est à noter un travail en terme de fonction de transfert et sans connaissance de  $h$  par W. Perruquetti et al. et des nombreuses définitions d'observabilité: partie continue, partie discrète, en terme d'inversion à gauche, observabilité uniforme, observabilité partielle,...

### Synthèse d'un observateur :

Soit le système hybride suivant (A. Bemporad et al., A. Birouche et al. en discret):

$$\dot{x} = F(x, Z(k)) \quad (1a)$$

$$Z(k+1) = A_d Z(k) + \beta_d(v(k), y(k)) \quad (1b)$$

$$y = y_c = H(x) = [h_1(x), \dots, h_p(x)] \quad (1c)$$

On suppose la partie continue du système  $(F, H)$  au moins localement faiblement observable. De plus, on fait l'hypothèse que la liste des indices d'observabilité est invariante en fonction de  $Z$ .

Alors, le difféomorphisme suivant

$$\zeta = \Phi_Z(x) = (\zeta_1^T, \zeta_2^T, \dots, \zeta_p^T)^T = (h_1(x), L_F h_1, \dots, L_F^{k_1-1} h_1, \dots, h_p(x), L_F h_p, \dots, L_F^{k_p-1} h_p)$$

Transforme le système sous forme canonique d'observabilité et on souhaite à partir de l'information continu retrouver les états discrets. Pour ceci on a besoin des hypothèses suivantes:

## Hypothèses

i)  $\exists$  at least  $j \in \{1, \dots, p\}$  such that

$$\frac{\partial \left( L_{F(x=\Phi_Z^{-1}(\zeta), Z)}^{k_j} h_j \right) \Big|_{\zeta}}{\partial Z} \neq 0,$$

ii) For all  $i = 1, \dots, p$ , we have

$$\frac{\partial^2 \left( L_{F(x=\Phi_Z^{-1}(\zeta), Z)}^{k_i} h_i \right) \Big|_{\zeta}}{\partial Z^2} = 0,$$

On obtient alors

$$\left\{ \begin{array}{l} \dot{\zeta}_{1,i} = \dot{\zeta}_{2,i} \\ \dot{\zeta}_{2,i} = \dot{\zeta}_{3,i} \\ \vdots \\ \dot{\zeta}_{k_i,i} = \left( L_{F(x,Z)}^{k_i} h_i(x) \right) \Big|_{x=\Phi_Z^{-1}(\zeta)} \\ \quad = \Gamma_i(\zeta) + \Lambda_i(\zeta) Z(k) = \Gamma_i(\zeta) + \sum_{j=1}^m \Lambda_{ij}(\zeta) z_j(k) \\ y_i = \dot{\zeta}_{1,i} \quad i = 1, \dots, p \end{array} \right.$$

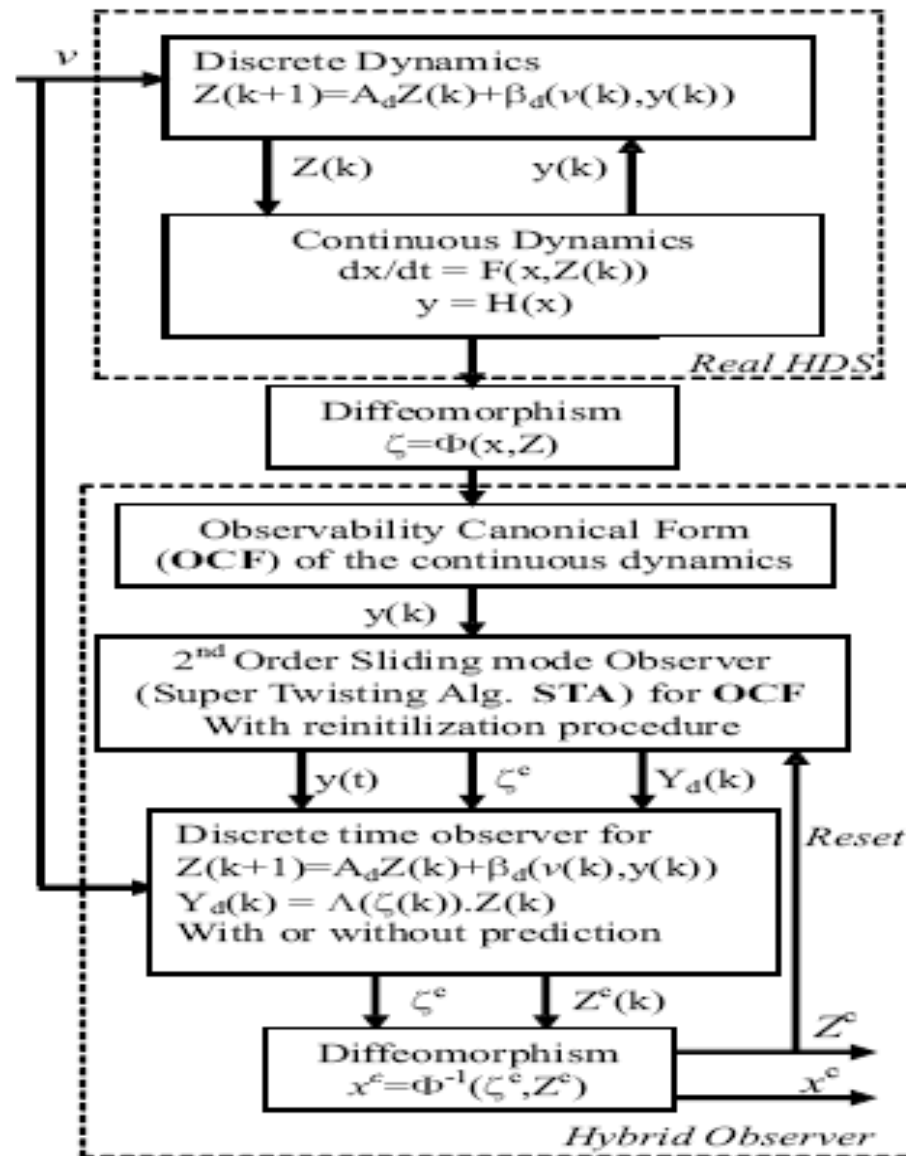
## Proposition (N. Manamanni, M. Djemai,...):

Le système discret est observable en  $Z(\mu) \in M$  pour un état continu initial fixé  $\zeta(\mu) \in \mathcal{N}$  et pour un ensemble d'entrées fixées  $v(k), k > \mu$ , si et seulement si le couple  $(\Lambda(\cdot), A_d)$  vérifie la condition suivante :

Il existe un entier borné  $\tilde{N}$  tel que

$$\text{rank} \left\{ \bar{\Lambda}_\mu = \begin{pmatrix} \Lambda(\zeta_\mu) \\ \Lambda(\zeta_{\mu+1})A_d \\ \vdots \\ \Lambda(\zeta_{\mu+\tilde{N}})A_d^{\tilde{N}} \end{pmatrix} \right\} = m$$

Observateur Hybride



Soit le système (1) avec le champ de vecteur suivant pour la partie continue :

$$F(x, Z(k)) = \begin{pmatrix} x_2 + z_1(k) \\ -10x_2 - 5 \sin(0.5x_3) - x_1^2 z_1(k) \\ x_4 + z_1(k) \\ -x_3^2 + 5x_1 + x_1 z_2(k) \end{pmatrix}$$

Et la dynamique suivante pour la partie discrète :

$$A_d = \begin{pmatrix} 0.50 & 0 & 1 \\ -0.2 & 0.1 & 0 \\ -0.20 & 0 & -0.1 \end{pmatrix}, \beta_d = \begin{pmatrix} y_1(k) \\ 0 \\ y_2(k) \end{pmatrix}$$

A l'aide du difféomorphisme

$$\zeta = \Phi_Z(x) \quad \text{avec} \quad \zeta_1 = x_1; \quad \zeta_2 = x_2 + z_1(k); \quad \zeta_3 = x_3; \quad \text{et} \quad \zeta_4 = x_4 + z_1(k).$$

On obtient :

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= -10\zeta_2 - 5 \sin(0.5\zeta_3) - \zeta_1^2 z_1(k) + 10z_1(k) \\ \dot{\zeta}_3 &= \zeta_4; \\ \dot{\zeta}_4 &= -\zeta_3^2 + 5\zeta_1 + \zeta_1 z_2(k) - z_1(k) \\ Z(k+1) &= A_d Z(k) + \beta_d(y(k)) \\ y_1 &= \zeta_1; \quad \text{and} \quad y_2 = \zeta_3 \end{aligned}$$

On a une procédure de réinitialisation

$$\begin{aligned} \hat{\zeta}_1^+(t_k) &= \hat{\zeta}_1(t_k^-); & \hat{\zeta}_2^+(t_k) &= \hat{\zeta}_2(t_k^-) - z_1(k-1) + z_1(k) \\ \hat{\zeta}_3^+(t_k) &= \hat{\zeta}_3(t_k^-); & \hat{\zeta}_4^+(t_k) &= \hat{\zeta}_4(t_k^-) - z_1(k-1) + z_1(k) \end{aligned}$$

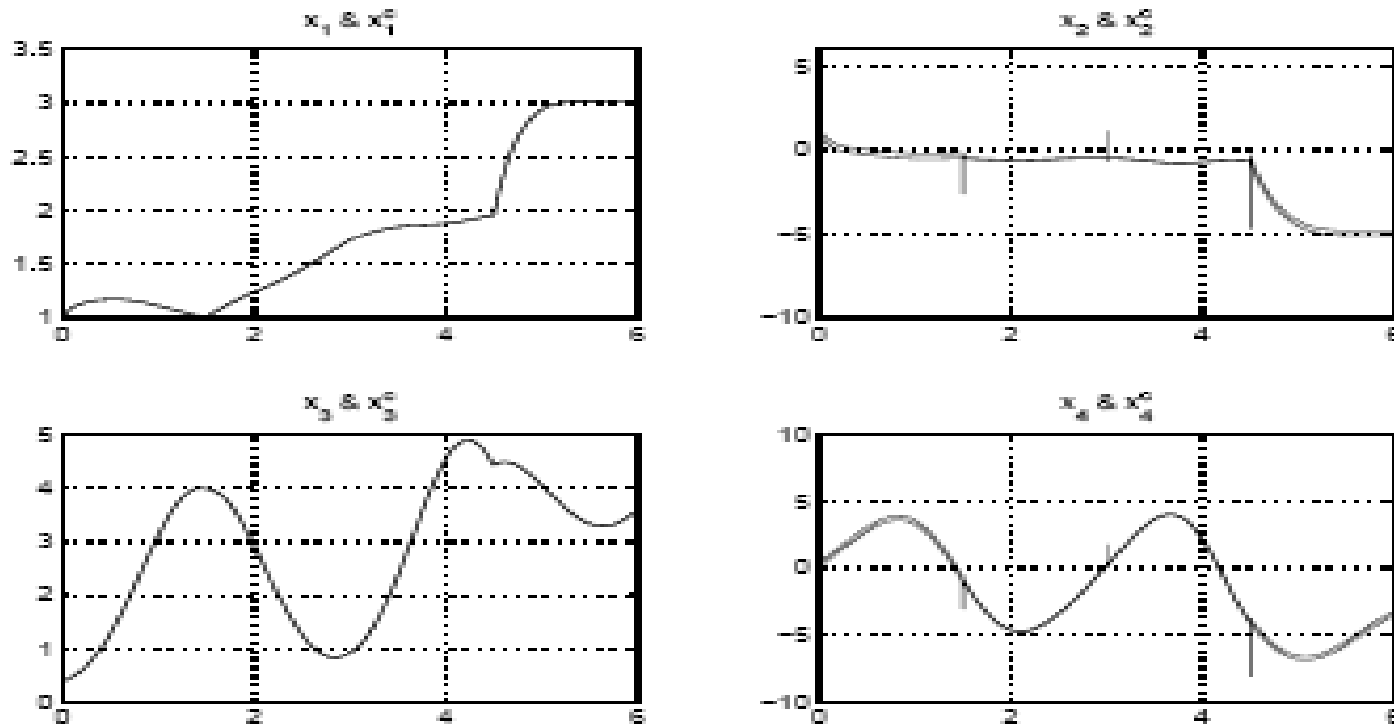


Figure 2 : Real  $x$  and observed  $\hat{x}$  continuous states



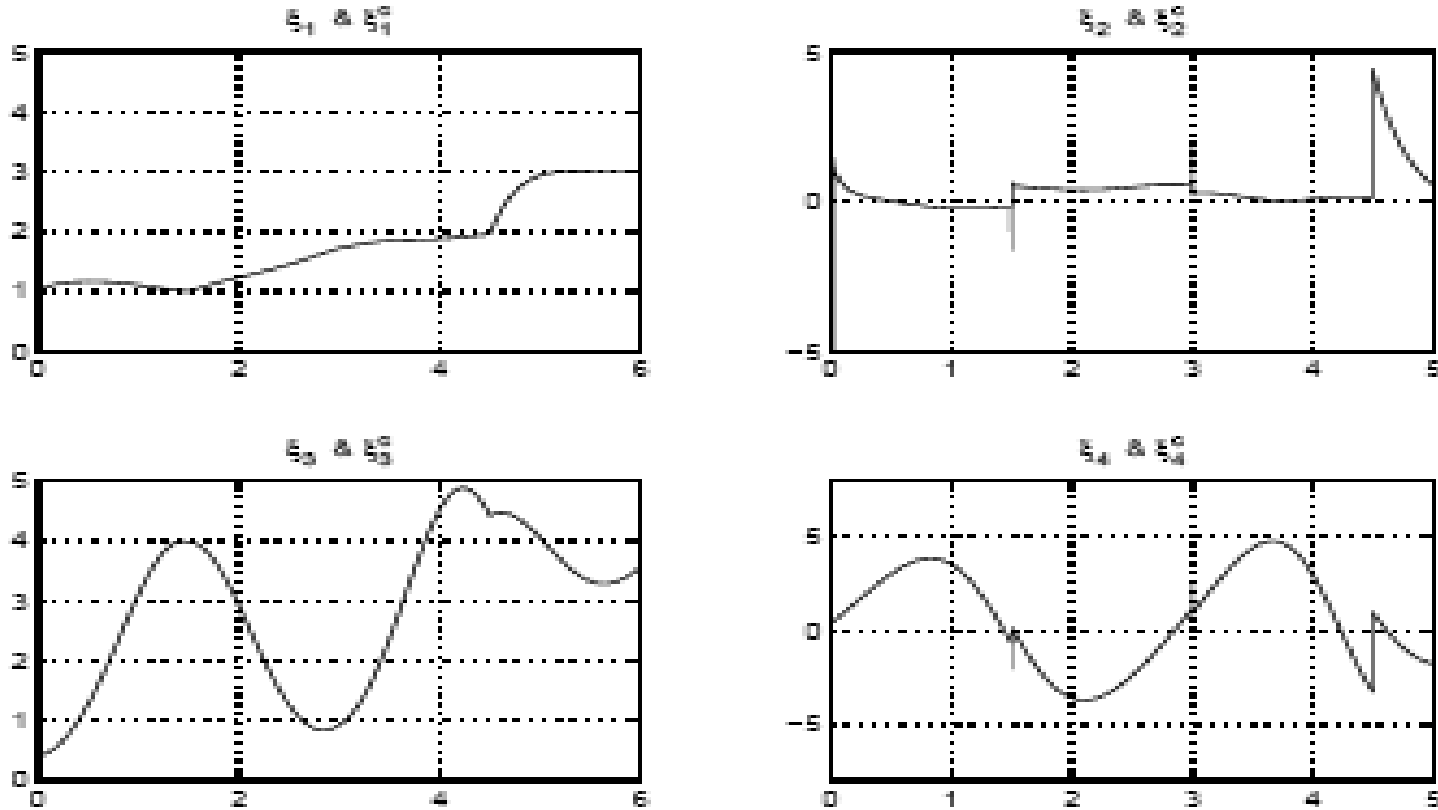


Figure 3 : Real  $\xi$  and observed  $\hat{\xi}$  continuous states

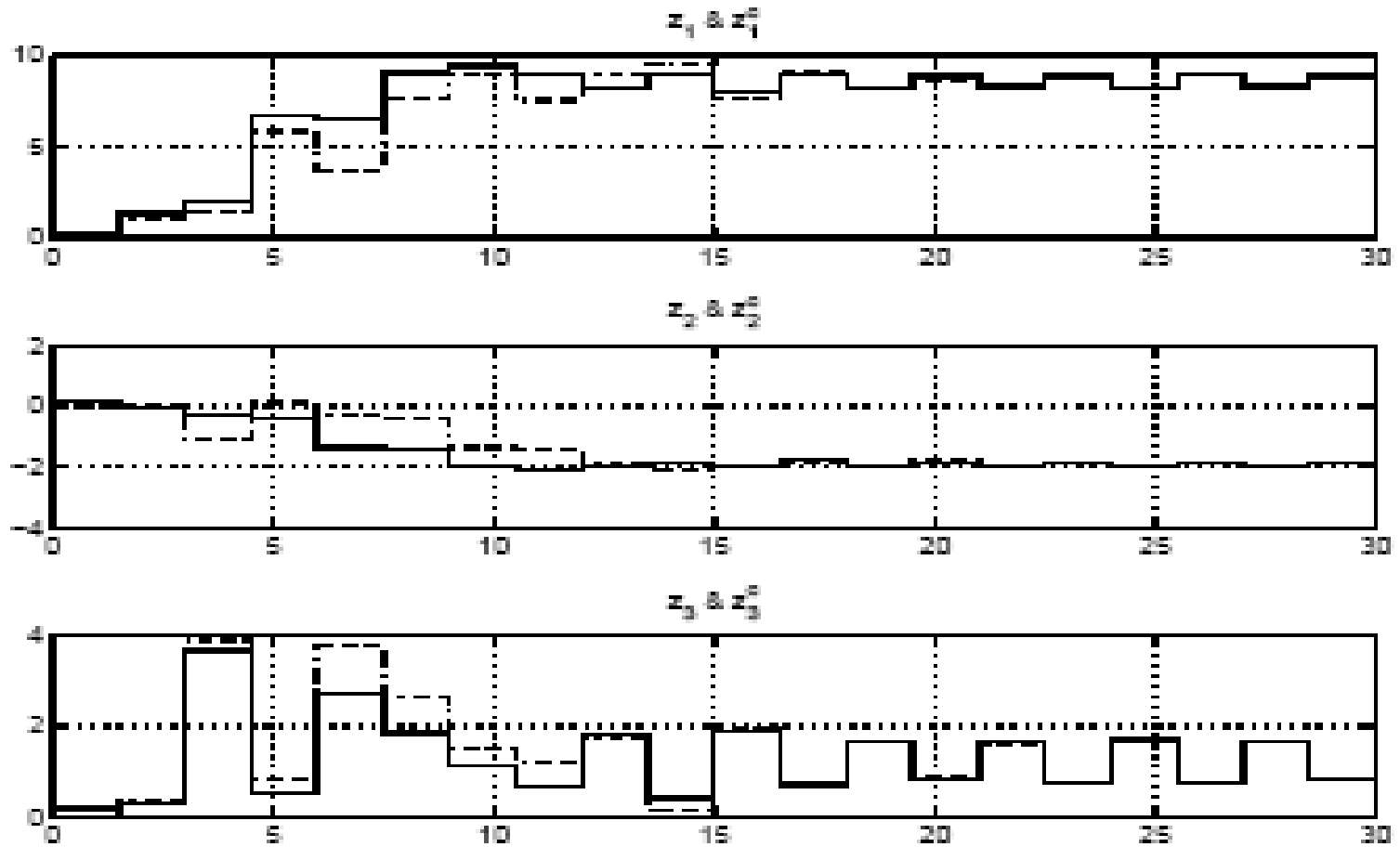


Figure 4 : Real and observed discrete states

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- Observation of Hybrid system is an open problem
- Observation of hybrid system with Zeno phenomena is a particularly difficult problem
- What is the Zeno phenomena? (Bouncing ball example), Chattering Zeno and Genuinely Zeno.
- Particular case switched system without jump.
- Mains ideas are to used: sliding mode method, left invertibility method, particular output injection form and the Kurzweil-Henstock integral.

## Definition

A dynamical system is called Zeno if  $t_\infty := \sum_{i=0}^{\infty} (\tau_{i+1} - \tau_i)$  is bounded.

- ▶ **Chattering Zeno:** There exist a finite number  $C \in \mathbb{N}$  and a bounded value  $t_\infty$  such that:

$$t_\infty = \sum_{i=0}^{\infty} (\tau_{i+1} - \tau_i) \quad \text{and} \quad \forall j > C, \quad [t_{j+1} - t_j] = 0.$$

- ▶ **Genuinely Zeno:** There exist  $C \in \mathbb{N}$  and a bounded value  $t_\infty$  such that:

$$t_\infty = \sum_{i=0}^{\infty} (\tau_{i+1} - \tau_i) \quad \text{and} \quad \forall j > C, \quad [t_{j+1} - t_j] > 0.$$

## PROBLEM STATEMENT

Let us consider the following class of switched system:

$$\dot{x} = f_q(x), \quad q \in \{1, 2\} \quad (0.1)$$

$$y = h(x) \quad (0.2)$$

where  $x \in U \subset \mathbb{R}^n$  is the state,  $y \in \mathbb{R}$  is the measured output and functions  $f_q : U \rightarrow \mathbb{R}^n$ ,  $q \in Q$  and  $h : U \rightarrow \mathbb{R}$  are smooth for each  $q$ . And  $q$  is the discrete state which may be driven by a switching function  $\sigma$ , where  $\sigma : \mathbb{R} \rightarrow Q = \{1, 2\}$ .

## Assumption 1

For each  $q \in Q$  the pair  $(y, f_q)$  is regularly weakly locally observable, that  $\text{rank}\{dL_{f_q}^j h, \quad j = 0 : n - 1\} = n$  on  $U$ .

## First observability form

We will exhibit the necessary and sufficient conditions for the existence of diffeomorphism  $\xi = \psi(x)$ , which transforms the system (0.1)-(0.2) into the following form:

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \vdots \\ \dot{\xi}_n \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \vdots \\ \xi_n \end{pmatrix} + \begin{pmatrix} \tilde{f}_q \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (0.3)$$

$$y = \tilde{h}(\xi) = \xi_n \quad (0.4)$$

where  $q \in Q$  is the discrete state.

## Second observability form

The second case considered in the next section is to exhibit sufficient condition for the existence of diffeomorphism  $z = \phi(x)$  such that the system (0.1)-(0.2) is transformed into the following form:

$$\dot{z} = \alpha_0(z) + \gamma_q(y), \quad q \in Q \quad (0.5)$$

$$y = z_n \quad (0.6)$$

**Remark 1** *The local weakly observability property of system (0.5-0.6) doesn't depend on the term of  $\gamma_q(y)$ . Thus, it is equivalent to the observability of the pair  $(\alpha_0(z), z_n)$ .*



## MAIN RESULTS

Hereafter, the structural conditions is analyzed in order to transform by diffeomorphism the system (0.1)-(0.2) into one of both cases previously introduced. For this consider:

$$f_{\Delta}(x) = f_1(x) - f_2(x)$$

## CONDITIONS FOR THE EXISTENCE OF DIFFEOMORPHISM FOR THE FIRST CASE

We give the necessary and sufficient conditions for the existence of diffeomorphism  $\xi = \psi(x)$  such that system (0.1)-(0.2) is transformed into the form (0.3)-(0.4).

**Proposition 1** *Under assumption 1, there exist a diffeomorphism  $\xi = \psi(x)$  which transform the system (0.1)-(0.2) into the form (0.3)-(0.4) if and only if the following conditions are fulfilled:*

$$L_{f_{\Delta}} L_{f_i}^{j-1} h = 0 \quad \forall i \in Q \quad \text{and} \quad \forall j \in \{1, \dots, n-1\} \quad (0.7) \quad 41$$

**Remark 2** *If  $f_{\Delta} \neq 0$  then assumption 1 and condition (0.7) implied that  $L_{f_{\Delta}} L_{f_i}^{n-1} h \neq 0$ . Note that, this condition is closed to the notion of relative index and observability matching condition. Moreover, under this condition and for a switch  $\sigma$  without Zeno we can estimate the discrete state  $q$ .*

**Remark 3** *From the observability concept introduced by M. Fliess and S. Diop in the theoretical frames of differential algebra, proposition 1 is equivalent to the existence of a common function  $F$  (i.e. independent of  $q$ ), for both systems, which verify the following algebraic equation:*

$$x = F(y, \dots, y^{(n-1)})$$

*where  $y^{(j)}$  denotes the  $j^{\text{th}}$  derivative of  $y$ .*

## CONDITIONS FOR THE EXISTENCE OF DIFFEOMORPHISM FOR THE SECOND CASE

Hereafter, it is exhibited the conditions for the existence of diffeomorphism  $\phi$  for the second case, which transforms the system (0.1)-(0.2) into the system (0.5)-(0.6).

**Theorem 1** *Under assumption 1, a sufficient and necessary conditions for the existence of diffeomorphism  $\phi(x)$  such that the system (0.1)-(0.2) is transformed into system (0.5)-(0.6) is :*

*There exist a family of independent vector fields  $\{\tau_1, \dots, \tau_n\}$  such that:*

- a)–  $dh \cdot \tau_n = 1$
- b)–  $dh \cdot \tau_i = 0 \quad \forall i \in \{1, \dots, n-1\}$
- c)–  $[\tau_i, \tau_j] = 0 \quad \forall (i, j) \in \{1, \dots, n\} \times \{1, \dots, n\}$
- d)–  $[\tau_i, f_\Delta] = 0 \quad \forall i \in \{1, \dots, n-1\}$

## EXAMPLE

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -(1 + \cos x_1) \sin(x_2 - x_1^2) \\ x_1(1 - 2(1 + \cos x_1) \sin(x_2 - x_1^2)) \end{pmatrix} + f_q$$

$$y = x_2 - x_1^2$$

$$\text{where } f_q = \begin{cases} \begin{pmatrix} 0 \\ -(x_2 - x_1^2)^2 \end{pmatrix} & q = 1 \\ \begin{pmatrix} -(x_2 - x_1^2)^2 \\ (x_2 - x_1^2)^2 \end{pmatrix} & q = 2 \end{cases} .$$

Vector fields  $\tau_i$  are given as follow:  $\tau_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ,  $\tau_1 = \begin{pmatrix} 1 \\ 2x_1 \end{pmatrix}$

and as  $f_\Delta = y^2 \begin{pmatrix} 1 \\ 2(1 - x_1) \end{pmatrix}$  then  $[\tau_1, f_\Delta] = 0$ .

The following diffeomorphism

$$\phi(x) = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 - x_1^2 \end{pmatrix}$$

transforms the system in the form:

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{pmatrix} = \begin{pmatrix} -(1 + \cos \xi_1) \sin \xi_2 \\ \xi_1 \end{pmatrix} + f'_q$$

$$\tilde{y} = \xi_2$$

$$\text{where } f'_q(\tilde{y}) = \begin{cases} \begin{pmatrix} 0 \\ -\tilde{y}^2 \end{pmatrix} & q = 1 \\ \begin{pmatrix} -\tilde{y}^2 \\ \tilde{y}^2 \end{pmatrix} & q = 2 \end{cases} .$$

## SUFFICIENTLY OBSERVABILITY CONDITIONS

### CASE ONE

For system of the form (0.3)-(0.4) algebraic estimator or step by step sliding mode observer work well for the continuous state estimation, because the first equation of (0.3), (i.e.  $\dot{\xi}_1$ ) is never considered.

More precisely, in algebraic solution the output derivative is considered only until  $n - 1$  and in the step by step sliding mode observer the last step is a sliding mode observer of order one or it is requested that  $\dot{z}_1$  is bounded and the bound is known.

From these and proposition 1, it is possible to set the following corollary:

**Corollary 1** *Under assumption 1, a sufficient condition for the observability of the continuous state of system (0.5)-(0.6) with zeno phenomena is:*

$$L_{f_\Delta} L_{f_i}^{j-1} h = 0 \quad \forall i \in Q \quad \text{and} \quad \forall j = \{1, \dots, n - 1\}$$

## Definition

Consider a real function  $f$  defined on an interval  $[a, b] \subset \mathbb{R}$ . For any tagged partition  $P$  of  $[a, b]$  such as

$a = x_0 < x_1 < x_2 < \cdots < x_n = b$ , and  $\tau_i \in [x_{i-1}, x_i] \subset [a, b]$ ,

consider

$$f(P) := \sum_{i=1}^n f(\tau_i)(x_i - x_{i-1}).$$

For any given function  $\delta : [a, b] \rightarrow \mathbb{R}_+^*$ ,  $P$  is said to be a  $\delta$ -fine tagged partition of  $[a, b]$  if  $h_i := x_i - x_{i-1} < \delta(\tau_i)$  where  $\delta$  is called a gauge for each  $i$ .

## Definition

A number  $I$  is called the Riemann integral (respectively the H-K integral) of  $f : [a, b] \rightarrow R$  if for each constant  $\varepsilon > 0$ , there exists a constant  $\delta$  (respectively a function  $\delta : [a, b] \rightarrow R_+^*$ ), such that whenever  $P$  is a  $\delta$ -fine tagged partition of  $[a, b]$ , one has  $|I - f(P)| < \varepsilon$ .



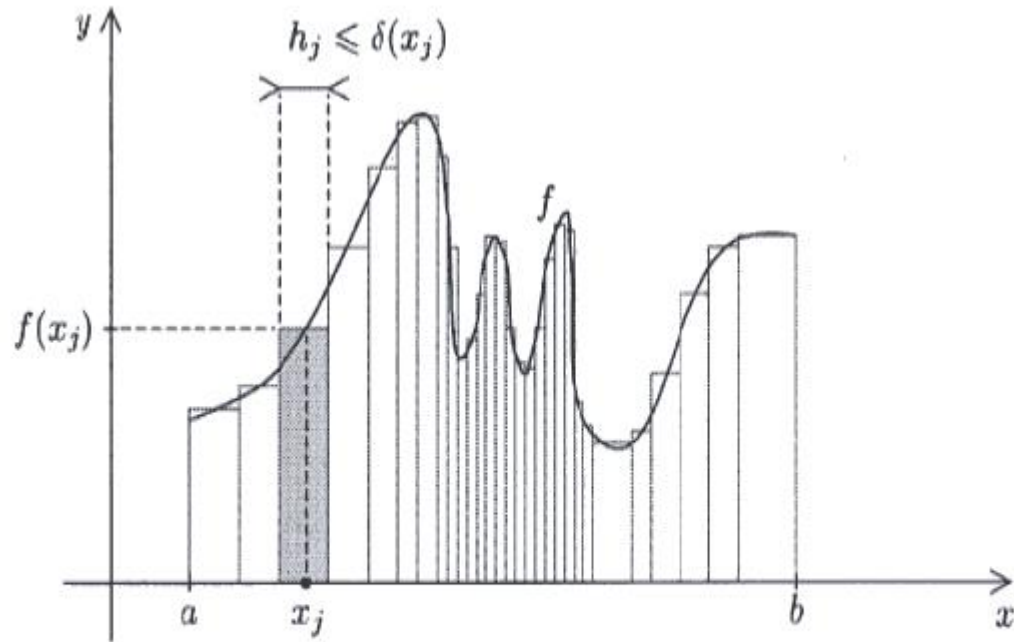


Figure: Illustrative explanation of H-K Gauge

## Example

To highlight the principle of the H-K integral, consider the two following examples.

Consider the function  $f$  defined on  $[0, 1]$  as follows:

$$f(x) = \begin{cases} x^{-1} \sin(x^{-2}) & \text{on } ]0, 1] \\ 0 & \text{for } x = 0. \end{cases}$$

This function is neither Riemann nor Lebesgue integrable, but H-K integrable, and this can be proved by using a special choice of gauge (that requires skill and effort) given as follows:

$$\delta(\tau) = \begin{cases} \sqrt{\varepsilon} & \text{if } \tau = 0 \\ \min \left\{ \frac{\tau}{2}, \frac{\varepsilon\tau^4}{24} \right\} & \text{if } 0 < \tau \leq 1. \end{cases}$$

## CASE TWO

In the form (0.5)-(0.6), the discrete state  $q$  is not considered as a perturbation and consequently the following assumption is requested:

**Assumption 2** *The discrete state  $q$  is known.*

More realistic assumption:

### **Assumption 3**

*a- The discrete state  $q$  is Henstock-Kurzweil integrable or NV-Integrable and its average is measured via a low pass filter of sufficiently large bandwidth on time interval  $[0, \alpha]$ .*

*b- The system (0.1)-(0.2) is affine with respect to the discrete state  $q$ .*

The last condition and the conditions of theorem 1 imply that there exists a diffeomorphism such that system (0.1)-(0.2) is transformed into:

$$\dot{z} = \alpha_0(z) + \gamma_0(y)q \quad (0.8)$$

$$y = \bar{h}(z) = z_n \quad (0.9)$$

So practically, it is only possible to obtain the discrete state after filtering  $q_f$  (instead of the real discret state  $q$ ).

Now, let us consider the system (0.8)-(0.9) reduced to the common dynamics:

$$\dot{z} = \alpha_0(z) \quad (0.10)$$

$$y = \bar{h}(z) = z_n \quad (0.11)$$

and assuming:

**Assumption 4** *For the system (0.10)-(0.11), there exist an observer such that the continuous state observation error (i.e. the difference between the continuous state and its estimate) is exponentially stable.*

**Remark 6** *Under some specific assumptions as Lipschitz condition, persistent excitation,.. it is possible to use classical high gain observer,.. or adaptive observer,.. For these observers the exponential stability of the continuous state observation error is guaranteed.*

From the previous assumption and theorem 1, it is possible to set the following proposition:

## **Proposition 2**

- – A) *Assumptions 1, 2, 4 and conditions of theorem 1 are sufficient conditions for the continuous state observability of system (0.1)-(0.2) with zeno phenomena.*
- – B) *Assumptions 1, 3, 4 and conditions of theorem 1 are sufficient conditions for the practical continuous state observability of system (0.1)-(0.2) with zeno phenomena.*

## EXAMPLE FOR THE FIRST FORM

Considering the following simple academical system in the form (0.3)-(0.4)

$$\dot{z}_1 = f_q \quad (0.12)$$

$$\dot{z}_2 = z_1 \quad (0.13)$$

$$y = z_2 \quad (0.14)$$

with  $f_q = -z_1 - 2z_2 + q$  and  $q = \text{sign}(w)$  where  $w = N$  is a white noise.

This form of  $q$  theoretically leads to the Zeno phenomenon.

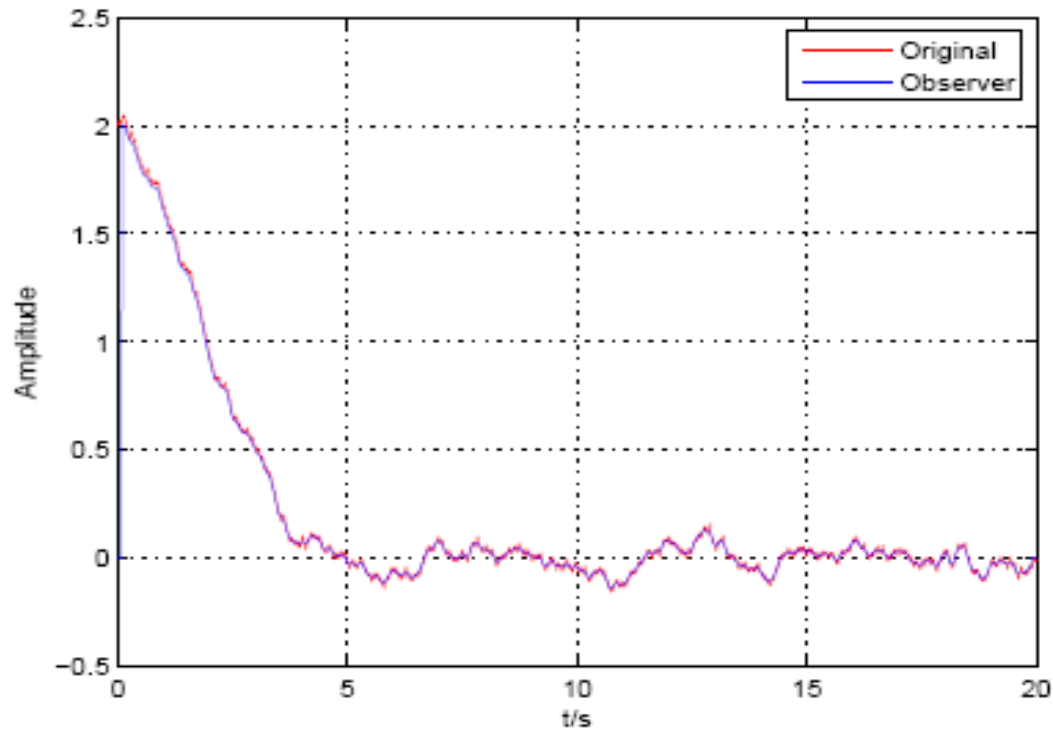


Figure 1: Result of ALIEN Observer without noise.



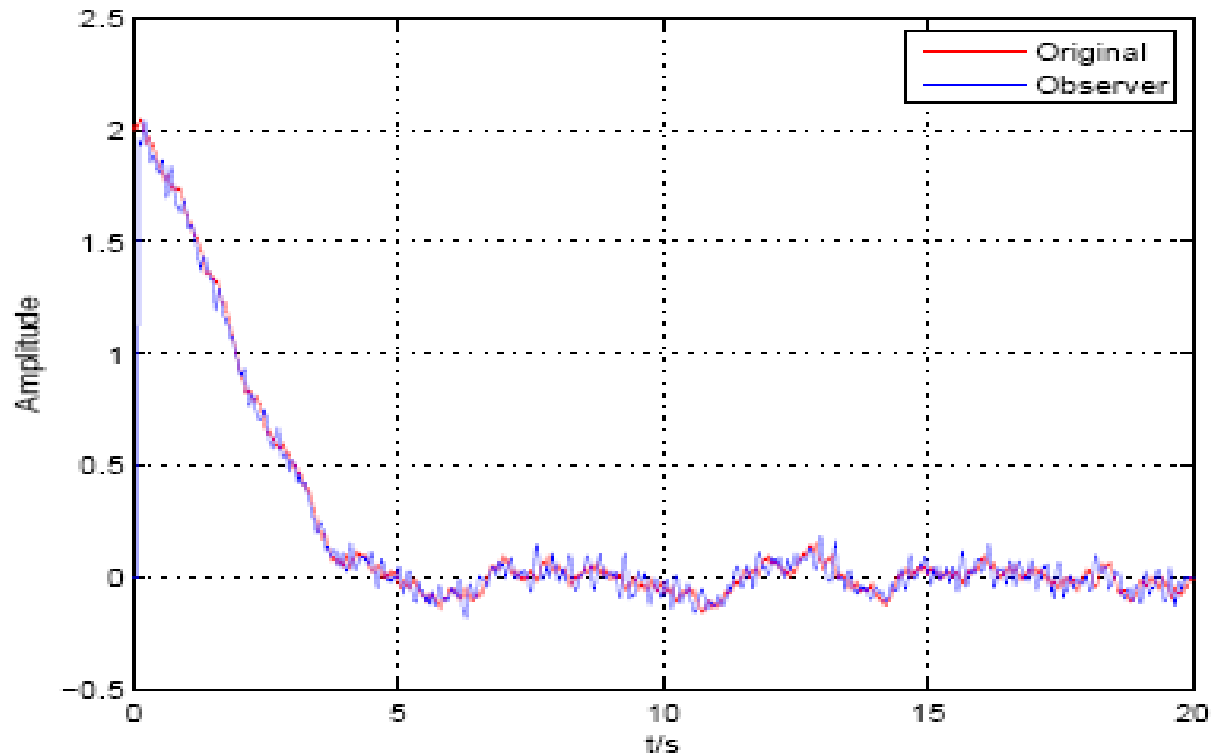


Figure 2: Result of ALIEN Observer with noise.

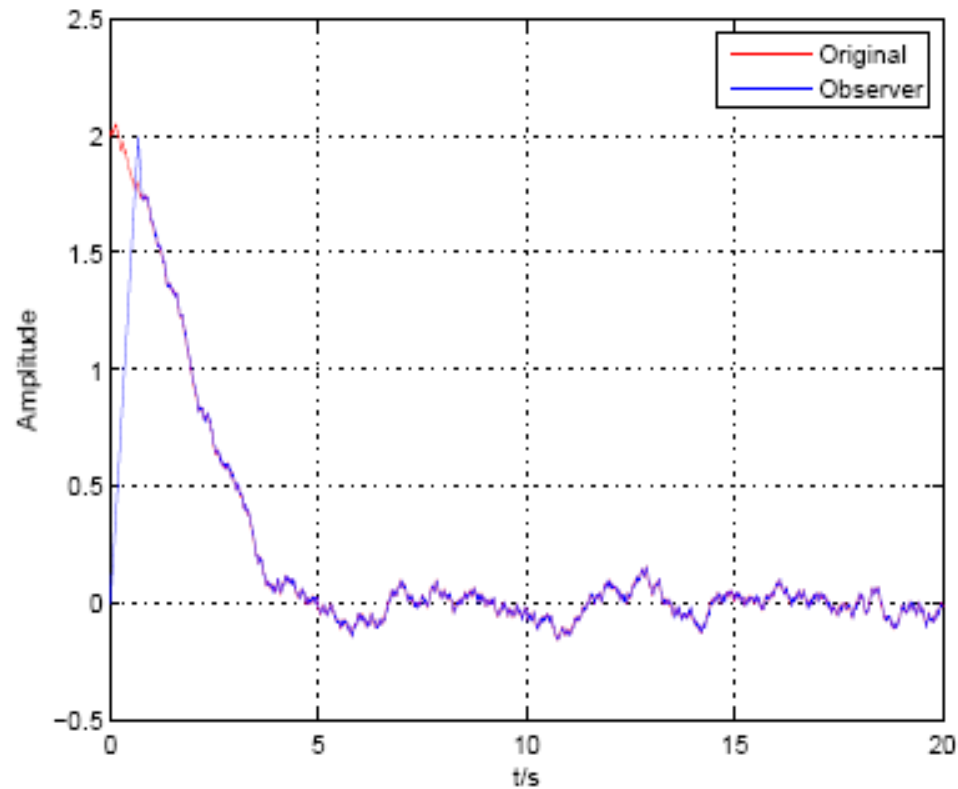


Figure 3: Result of second order sliding mode observer without noise.

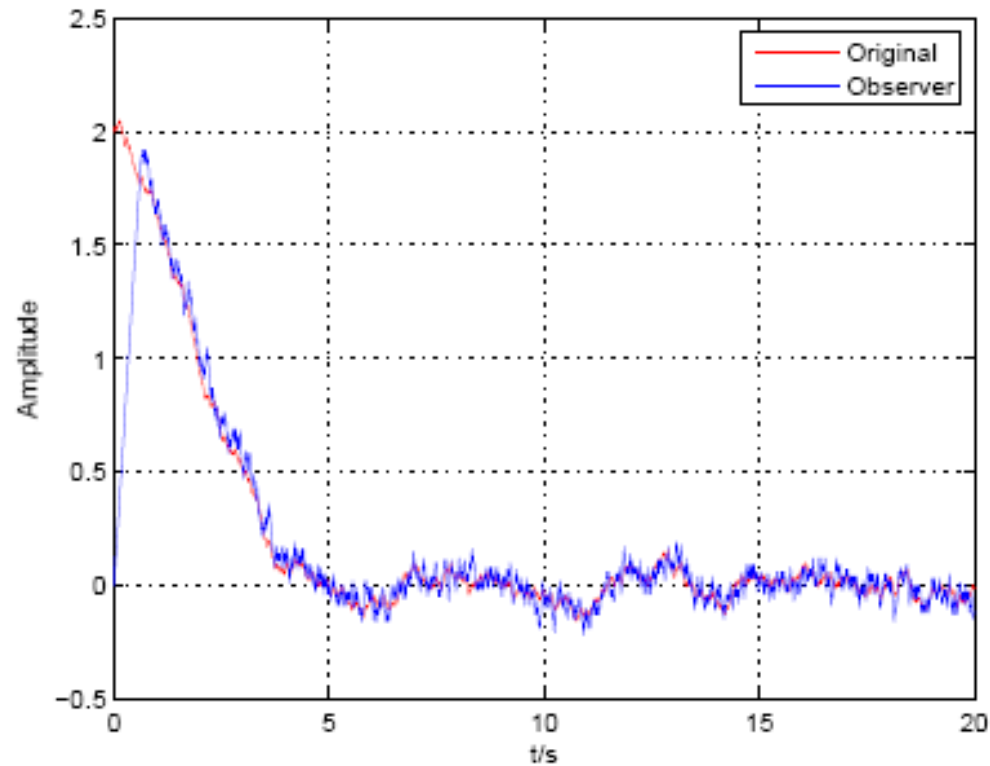


Figure 4: Result of second order sliding mode observer with noise.

## EXAMPLE FOR SECOND FORM

Considering another simple academical system in the form (0.5)-(0.6)

$$\dot{z}_1 = -z_2 \quad (0.15)$$

$$\dot{z}_2 = z_1 + (-z_2^3 + q) \quad (0.16)$$

$$y = z_2 \quad (0.17)$$

with  $q = 1.5 + 0.5\text{sign}(w)$  and  $w = N + e^{-t}$ , wherein  $N$  is a white noise. This form of  $q$  theoretically leads to infinite commutation in finite time. In this case, due to assumption 4 high-gain observer is adapted and used. For this system (0.15)-(0.17), the high-gain observer is designed as follows

$$\dot{\hat{z}} = A_0 \hat{z} + \phi(\hat{z}) - \Lambda(\lambda)^{-1} K_0 (C_0 \hat{z} - y) \quad (0.18)$$

wherein  $A_0 = (0, -1; 1, 0)$ ,  $C_0 = (0, 1)$  and  $\phi((\hat{z})) = (0, -\hat{z}_2^3 + q_f)^T$ .

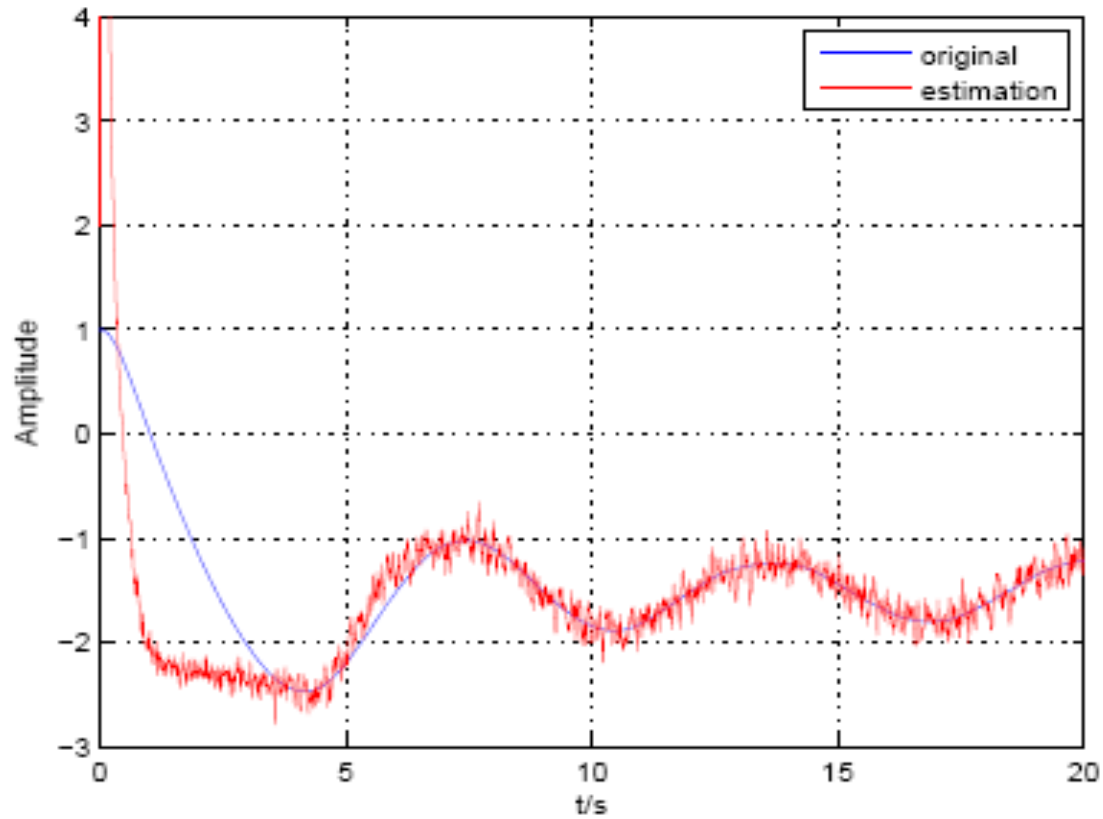
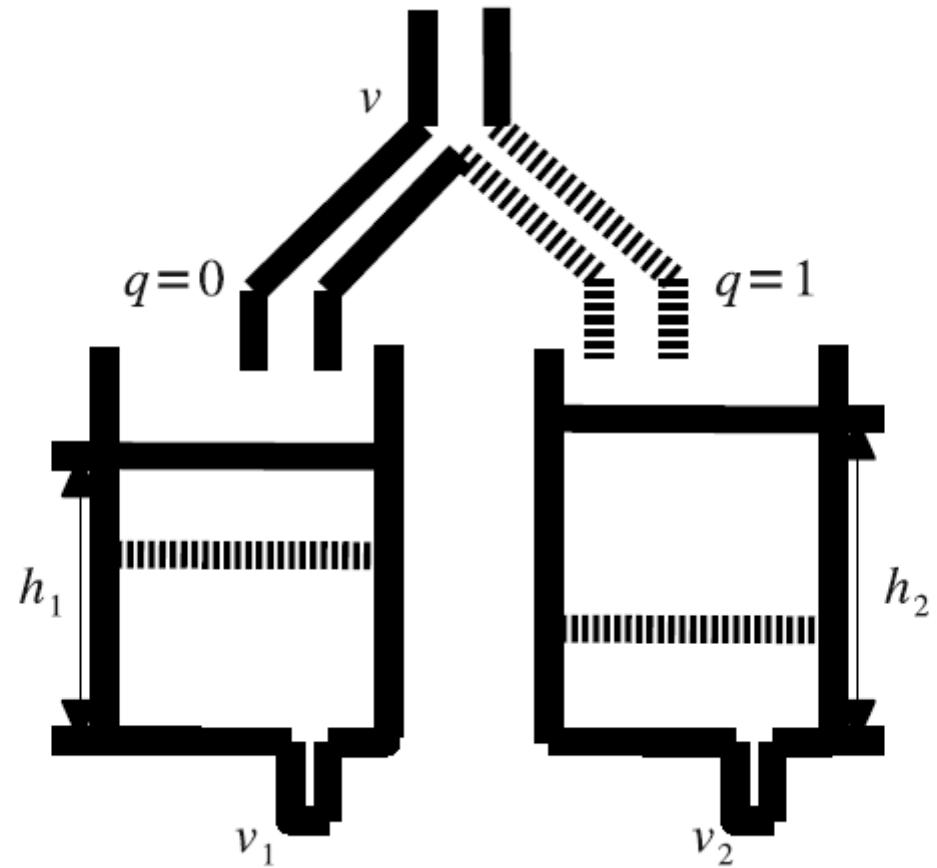


Figure 6: Result of High-Gain Observer with noise.

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## Le cas des deux Bacs



The simplified model of fig. 1 is given below:

$$\begin{pmatrix} \dot{h}_1 \\ \dot{h}_2 \end{pmatrix} = \begin{pmatrix} v - v_1 \\ -v_2 \end{pmatrix} + \begin{pmatrix} -v \\ v \end{pmatrix} q$$

where  $h_1, h_2$  are the water levels for each tanks,  $v_1, v_2$  are flows of water out of the tanks,  $v$  is a constant input flow of water which goes through a pipe and into either tank at any particular time point, and  $q$  is the switching state of the pipe which is determined by the transients conditions.



The case 1:

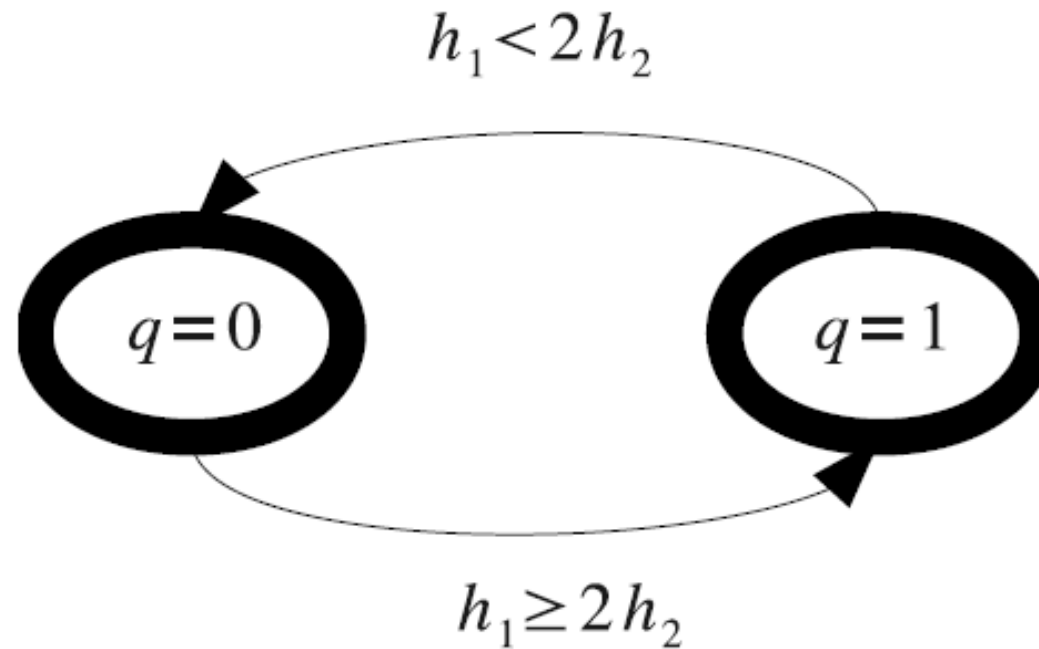


Figure: The transient of discrete state for the case 1 ( $h_1 = 2h_2$ ).

The case 2:

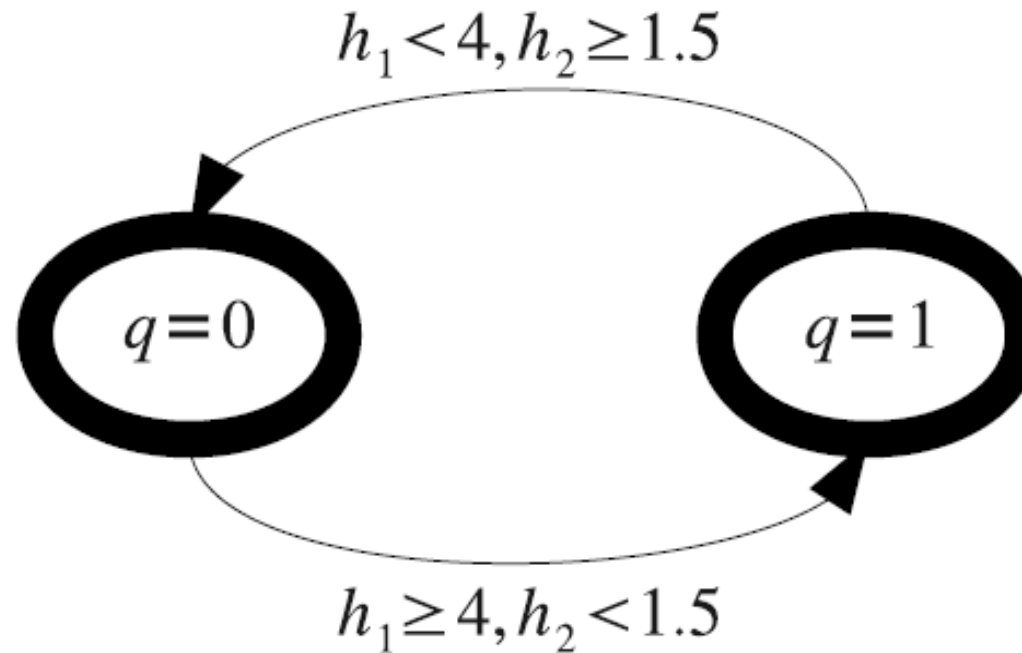


Figure: The transient of discrete state for the case 2 ( $h_1 = 4$  and  $h_2 = 1.5$ ).

By considering Torricelli's law, it has  $v_1 = \sqrt{2gh_1}$  and  $v_2 = \sqrt{2gh_2}$ , where  $g$  is the gravity. Then the dynamic can be rewritten as below.

$$\begin{pmatrix} \dot{h}_1 \\ \dot{h}_2 \end{pmatrix} = \begin{pmatrix} v - \sqrt{2gh_1} \\ -\sqrt{2gh_2} \end{pmatrix} + \begin{pmatrix} -v \\ v \end{pmatrix} q$$

Considering  $y = h_1 + h_2$  as its output, the system is regularly locally weakly observable:

$$\text{rank} \begin{pmatrix} dy \\ d\dot{y} \end{pmatrix} = \text{rank} \begin{pmatrix} 1 & 1 \\ \frac{-1}{\sqrt{2gh_1}} & \frac{-1}{\sqrt{2gh_2}} \end{pmatrix} = 2$$

excepted on the observability singularity set  $S_O$

$$S_O = \{(h_1, h_2)^T : h_1 = h_2\}$$

which is a manifold of dimension  $n - 1 = 1$ . Moreover, the system verifies the observability matching condition

$$\frac{\partial y}{\partial q} = \frac{\partial \dot{y}}{\partial q} = 0$$

Under the transformation  $\xi = \phi(h_1, h_2)$  with  $\phi_1(h_1, h_2) = h_1 + h_2$  and  $\phi_2(h_1, h_2) = v - \sqrt{2gh_1} - \sqrt{2gh_2}$ , we obtain locally:

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 & (4) \\ \dot{\xi}_2 &= -\sqrt{\frac{g}{2}} \left( \frac{(v - \sqrt{2gh_1} - vq)}{\sqrt{h_1}} + \frac{(vq - \sqrt{2gh_2})}{\sqrt{h_2}} \right) \\ y &= \xi_1 \end{aligned}$$

Where

$$\begin{aligned} h_1 &= \frac{1}{4} \left( \frac{v - \xi_2}{\sqrt{2g}} + \mu \sqrt{2\xi_1 - \frac{(v - \xi_2)^2}{2g}} \right)^2 \\ h_2 &= \frac{1}{4} \left( \frac{v - \xi_2}{\sqrt{2g}} - \mu \sqrt{2\xi_1 - \frac{(v - \xi_2)^2}{2g}} \right)^2 \end{aligned}$$

The proposed observer is the following:

$$\begin{aligned}\dot{\hat{\xi}}_1 &= \hat{\xi}_2 + \lambda|y - \hat{\xi}_1|^{\frac{1}{2}} \text{sign}(y - \hat{\xi}_1) \\ \dot{\hat{\xi}}_2 &= \alpha \text{sign}(y - \hat{\xi}_1) \\ \hat{y} &= \hat{\xi}_1\end{aligned}\tag{5}$$

Setting,  $e_1 = \xi_1 - \hat{\xi}_1$  and  $e_2 = \xi_2 - \hat{\xi}_2$ , the observation error dynamics is:

$$\begin{aligned}\dot{e}_1 &= e_2 - \lambda|e_1|^{\frac{1}{2}} \text{sign}(e_1) \\ \dot{e}_2 &= \dot{\xi}_2 - \alpha \text{sign}(e_1)\end{aligned}\tag{6}$$

Which is stable for appropriate values of  $\alpha$  and  $\lambda$  [28, 34].

For all our simulations, we have set  $\lambda = 5$ ,  $\alpha = 40$  and a computation step equal to  $10^{-5}s$  and use the ode5 of Matlab.

**The case 1** (fig. 2) corresponds to chattering Zeno when the state behavior reach the sliding manifold  $h_1 = 2h_2$  (approximately at time  $t = 0.2s$ ) and the system behavior slide on the switching until  $h_1 \simeq 2.5$  and  $h_2 \simeq 1.25$ .

# Le cas des deux bacs

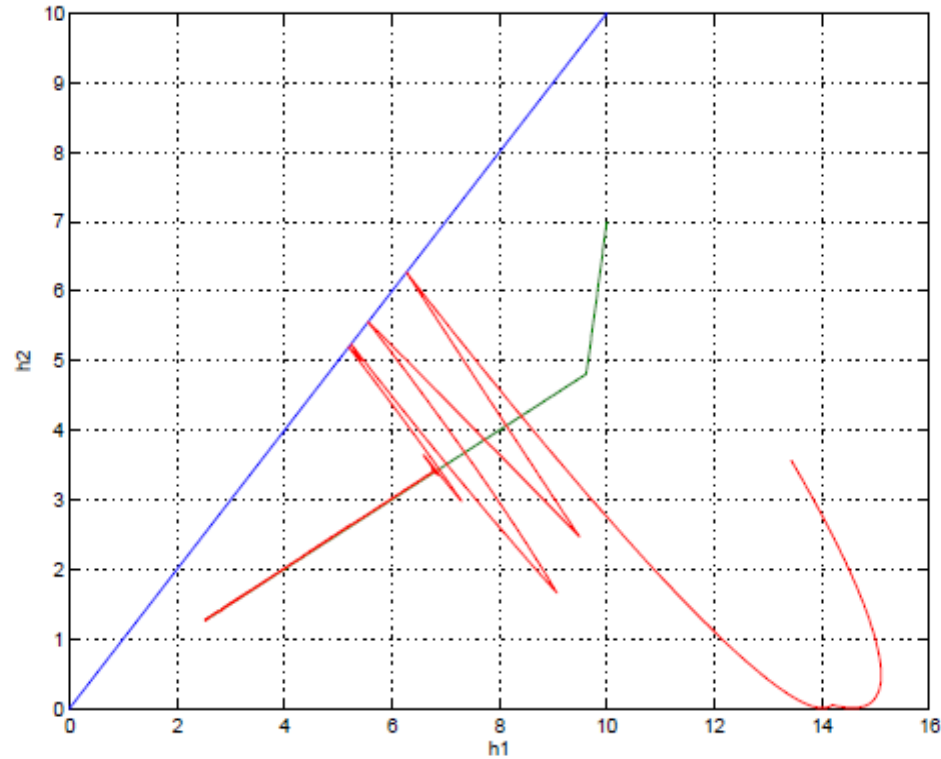


Figure: The Phase Map with the appropriate coordinates choice.



## Le cas des deux bacs

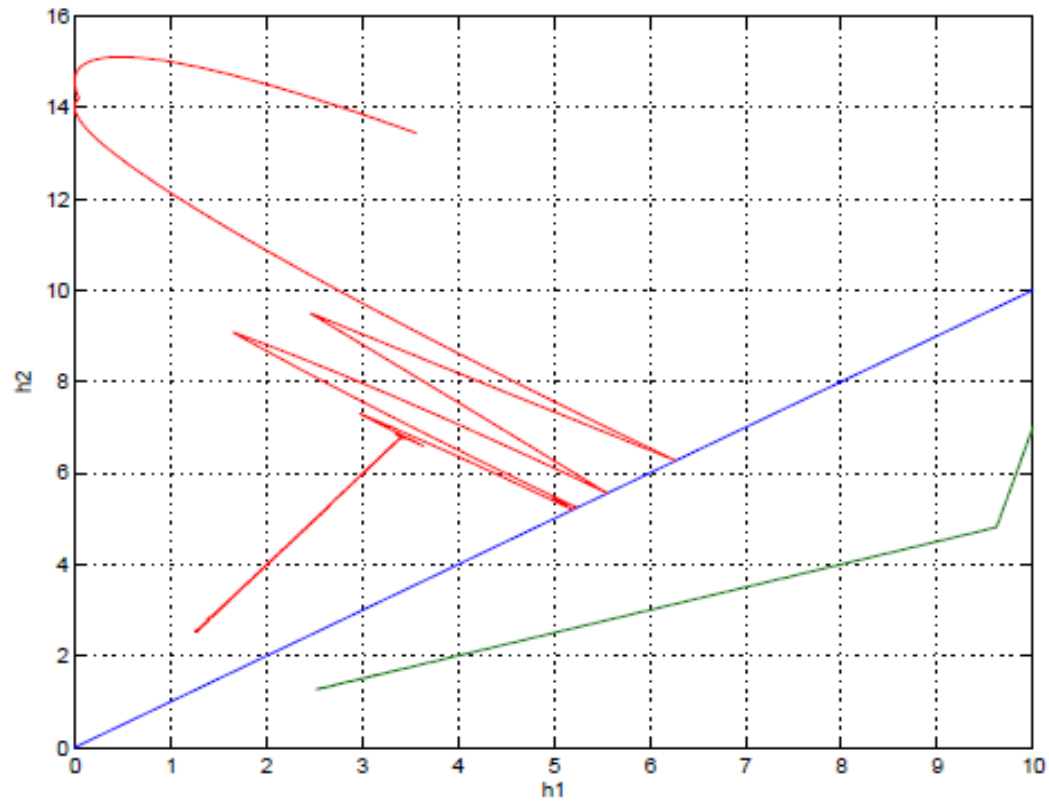


Figure: The Phase Map with a wrong coordinates choice.

## Le cas des deux bacs

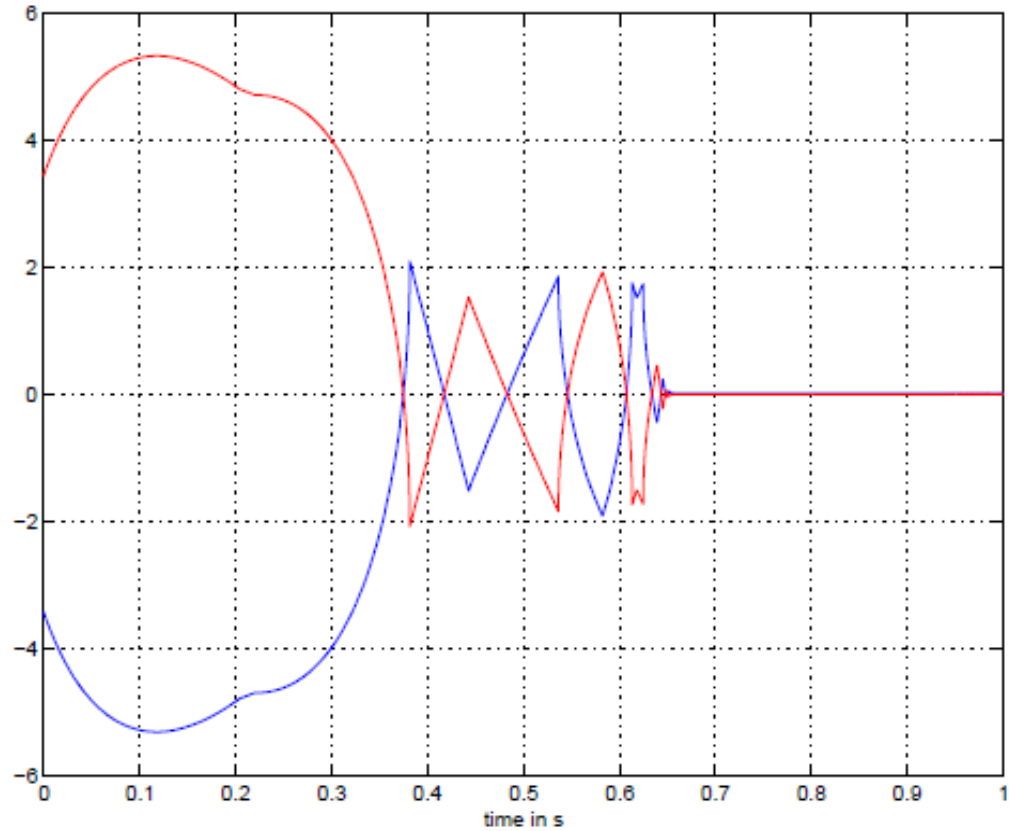


Figure: The observation errors in the original coordinates.

**The case 2** (fig. 3) corresponds to genuinely Zeno ( $I_{min}$  is nonstandard) and after  $t_{\infty} = 1.6s$  the Zeno time, there is life [3] see the green trajectory in the figure 8.

The Zeno time  $t_{\infty}$  is when  $h_1 = 4$  and  $h_2 = 1.5$  after this time the system converges to a limit cycle and the observer works well in both coordinates (original and canonical coordinates) because the limite cycle do not cross  $S_0$ .

## Le cas des deux bacs

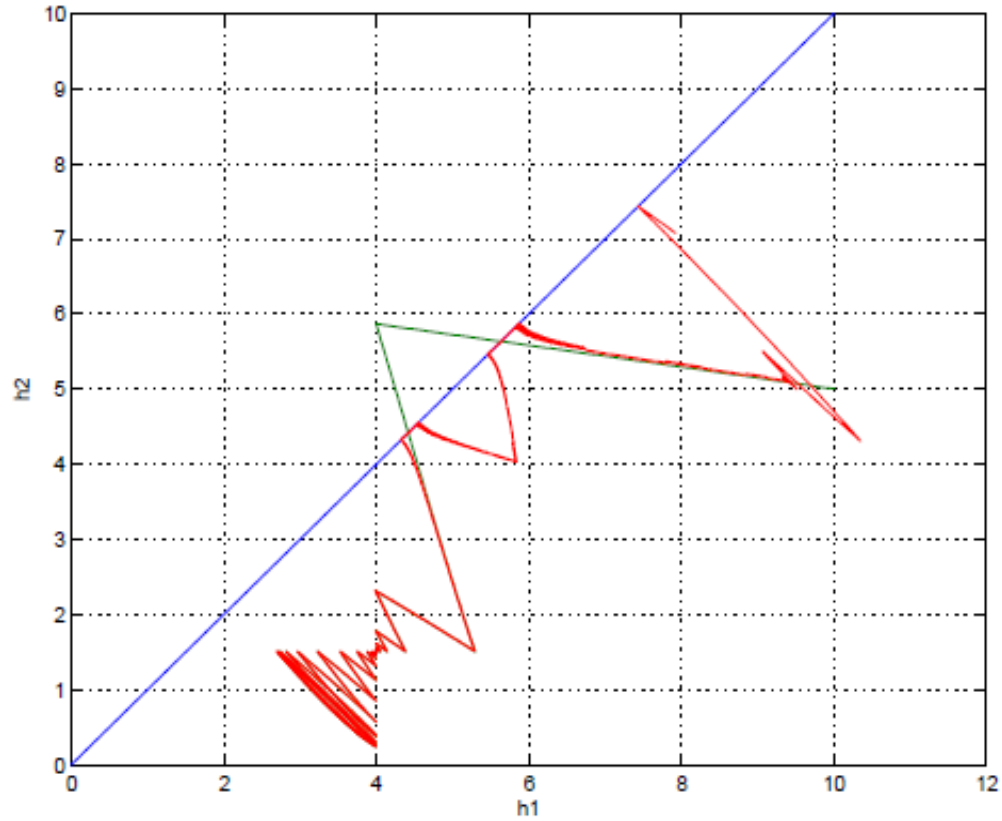


Figure: The Phase Map with the appropriate coordinates choice.

# Le cas des deux bacs

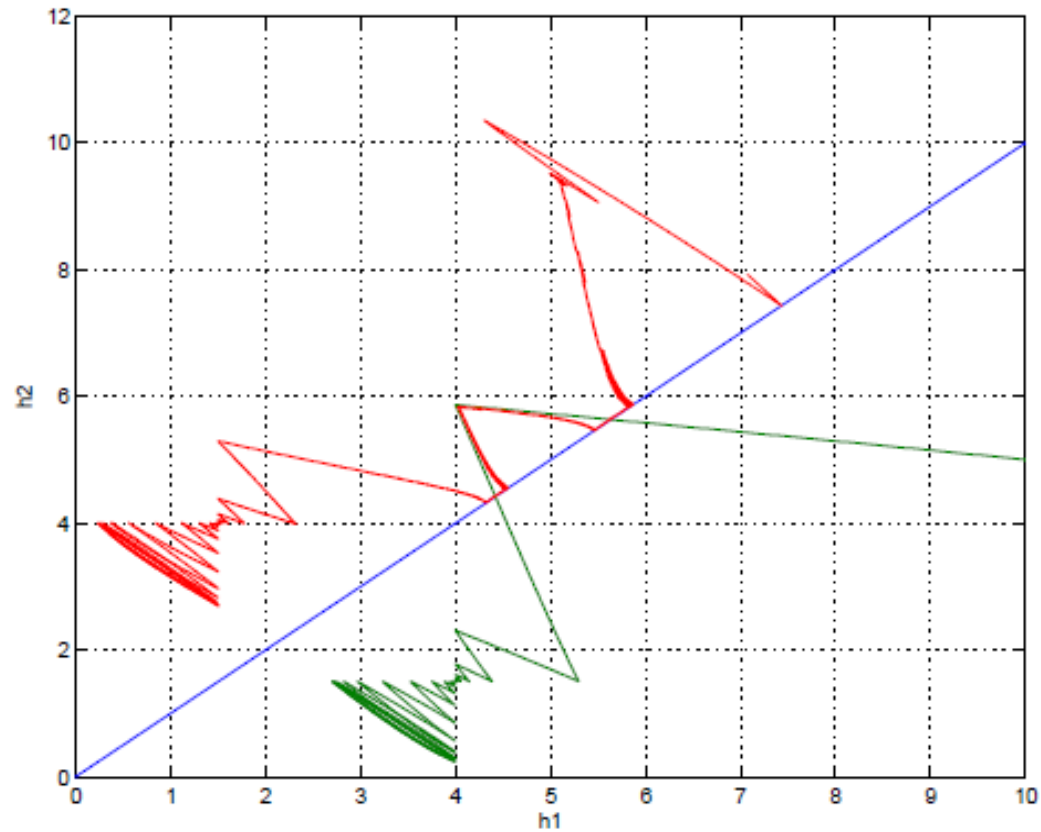


Figure: The Phase Map with a wrong coordinates choice.

## Le cas des deux bacs

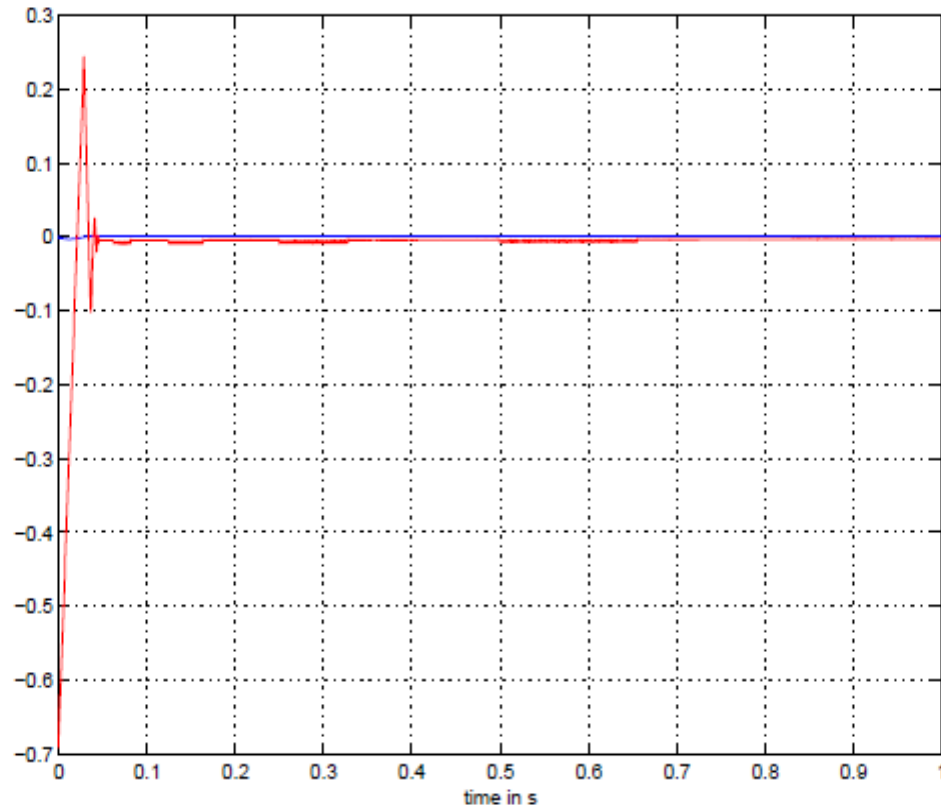


Figure: The observation errors in the canonical coordinates.

## Le cas des deux bacs

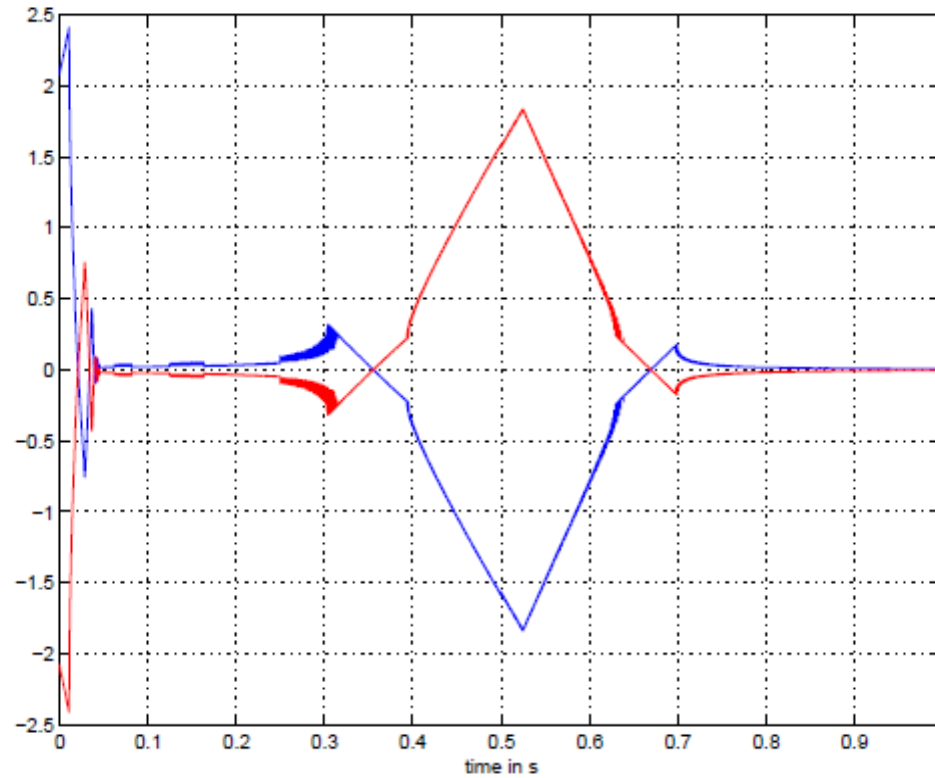


Figure: The observation error in the original coordinates.

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**Position du problème** (Ceci dépasse le cadre des SDHs, Cyber Attack, Diagnosis,...)

**Soit le système sous la forme :**

For  $j \in \{1, m\}$ ,  $h_j = \xi_{j,1}$  and we have

$$\begin{aligned}\dot{\xi}_{j,1} &= \xi_{j,2} \\ &\vdots \\ \dot{\xi}_{j,k_j-1} &= \xi_{j,k_j} \\ \dot{\xi}_{j,k_j} &= f_j(\xi) + \Phi_j(\xi)x\end{aligned}$$

with  $x \in \{0, 1\}^n$  and  $n \gg m$

**Après un temps  $t_1 > 0$  en utilisant par exemple un différentiateur homogène (Levant, Fridman, Perruquetti, Ghanes, Moreno,...) on obtient :**

Nous posons le problème de la façon suivante :

For  $j \in \{1, m\}$ , we set  $y_j = \xi_{j, k_j} - f_j(\xi)$ , this gives :

$$y = \Phi x$$

The Moore–Penrose pseudo inverse gives

$$\hat{x} = (\Phi^T \Phi)^{-1} \Phi y$$

It is not the expected solution, we haven't consider the fact que  $x$  is sparse.

Que faire ?

En tenant compte de la parcimonie le problème devient :

$$\hat{x} = \arg \min_{x \in \mathbb{R}^n} \{ \|y - \Phi x\|_2^2 + \lambda \|x\|_0 \}$$

but  $\|\cdot\|_0$  is not a norm

Nous devons imposer des conditions supplémentaires pour passer au problème suivant :

$$\hat{x} = \arg \min_{x \in \mathbb{R}^n} \{ \|y - \Phi x\|_2^2 + \lambda \|x\|_1 \}$$

**Assumption 1** (RIP [11]). *Matrix  $\Phi$  satisfies the  $s$ -order of RIP condition with constant  $\delta_s \in (0, 1)$ .*

The above assumption implies that for any  $s$  sparse signals  $x$ , i.e., vectors with at most  $s$  nonzero elements, the following condition is verified

$$(1 - \delta_s) \|x\|_2^2 \leq \|\Phi x\|_2^2 \leq (1 + \delta_s) \|x\|_2^2.$$

Denote  $\Gamma$  the index set of nonzero elements for  $x$ , it implies that

$$1 - \delta_s \leq \text{eig}(\Phi_\Gamma^T \Phi_\Gamma) \leq 1 + \delta_s$$

where  $\Phi_\Gamma$  denotes the submatrix of  $\Phi$  with active nodes.

Sous cette hypothèse l'algorithme suivant nous donne la solution

$$\begin{cases} \tau \dot{u}(t) = -[u(t) + (\Phi^T \Phi - I)a(t) - \Phi^T y]^\alpha \\ \hat{x}(t) = a(t) \end{cases}$$

with  $[\cdot]^\alpha$  being a function defined as

$$[\cdot]^\alpha = |\cdot|^\alpha \cdot \text{sgn}(\cdot)$$

And

$$a(t) = T_\lambda(u(t))$$

With

$$T_\lambda(u) = \max(|u| - \lambda, 0) \cdot \text{sgn}(u)$$

Voir TP 2

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- 8 Conclusion**

- L' « observabilité » (ou les observabilités) des systèmes dynamiques hybrides reste à définir.
- L'inversion à gauche n'a pas été abordée
- La convergence asymptotique a-t-elle encore un avenir pour ce type de système? (**A. Balluchi, L. Benvenuti, M.D. Di Benedetto and A.L. Sangiovanni-Vincentelli, CDC 2003**)
- La commande basée sur l'observateur a-t-elle un sens ?
- Si non quel retour de sortie dynamique ?
- Techniques de détection de rupture et instant de commutation.
- Phénomène de Zénon et analyse non standard
- Phénomène de Zénon sur plus de deux modes
- Phénomène de Zénon avec saut sur l'état.
- ...

**Merci**