A switched LQ regulator design in continuous time

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Problem formulation and preliminary results

Consider the class of linear switched systems in continuous time:

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u_{\sigma(t)}(t) \quad x(0) = x_0 \tag{1}$$

where

- $\sigma : [0, +\infty) \to S = \{1, \cdots, s\}$.
- $(A_i, B_i) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m_i}$, $i \in S$,
- $u_i(t) \in \mathbb{R}^{m_i}$, $0 \le m_i \le n$

Objective : Design a state feedback switching law (i.e. $x \mapsto (\sigma(x), u_{\sigma(x)}(x))$) that approaches the optimal solution of the following optimization problem:
Problem 1: Minimize the switched quadratic criterion:

$$\min_{\sigma, u_\sigma} \frac{1}{2} \int_0^\infty x^{\mathrm{T}} Q_\sigma x + u_\sigma^{\mathrm{T}} R_\sigma u_\sigma \mathrm{d}t \tag{2}$$

where $Q_i = Q_i^{\mathrm{T}} > 0$, $R_i = R_i^{\mathrm{T}} > 0$, $i \in S$

Up to now the exact solution is not available and only approximation via dynamic programming and (open loop) numerical solutions are available.

Problem formulation and preliminary results

Framework: Reformulate Problem 1 into
Problem 2: Minimize the quadratic criterion:,

$$\min_{\lambda(\cdot),\, u_i(\cdot)} \frac{1}{2} \int_0^\infty \sum_{i=1}^{s} \lambda_i (x^T Q_i x + u_i^T R_i u_i) dt$$

subject to

$$\dot{x} = \sum_{i=1}^{s} \lambda_i (A_i x + B_i u_i), \quad x(0) = x_0, \quad \lambda(t) \in \Lambda = \left\{ \lambda \in \mathbb{R}^s : \sum_{i=1}^{s} \lambda_i = 1 \quad \lambda_i \geq 0 \right\}.$$

Three reasons justify the convexification of the problem:

1. The solutions are well defined [Fillipov, 1988]
2. The density of the switched system trajectories into the trajectories of its relaxed version [Ingalls - Sontag 2002]
3. The existence of *singular* optimal solutions are taking into account [Patino-Riedinger 2009, Bengea-Decarlo 2005].

Problem formulation and preliminary results

To apply Pontryagin Maximum Principle (PMP) for Problem 1 or its relaxed version, the Hamiltonian function is defined as follow:

$$\mathcal{H}(x,\lambda,u,p) = \sum_{i=1}^{s} \lambda_i \mathcal{H}_i(x,u_i,p) \quad (3)$$

with $\mathcal{H}_i(x,u_i,p) = p^T(A_i x + B_i u_i) + \frac{1}{2}(x^T Q_i x + u_i^T R_i u_i)$ and where p defines the co-state.

Theorem (1)

Suppose that (λ^*, u^*) is optimal with the corresponding state x^*. Then, there exists an absolutely continuous function p^*, named co-state, such that:

1. $p^* \not\equiv 0$,
2. $\dot{p}^* = \sum_{i=1}^{s} \lambda_i^*(t)(-A_i^T p^* - Q_i x^*)$ *for almost all* $t \in \mathbb{R}^+$,
3. $(\lambda^*(t), u^*(t)) \in \arg\min_{(\lambda\in\Lambda,u)} \mathcal{H}(x^*(t),\lambda,u,p^*(t))$,
4. $\mathcal{H}(x^*(t),\lambda^*(t),u^*,p^*(t)) = 0$.

As the minimum of $\mathcal{H}$ with respect to the u_i's is clearly independent of the value of λ, Theorem 1 can be simplified :

Lemma

The optimal value of the u_i's are given by $u_i^*(t) = -R_i^{-1}B_i^T p^*(t)$ and λ^* satisfies:

$$\lambda^*(t) \in \arg\min_{\lambda\in\Lambda} \sum_{i=1}^{s} \lambda_i \mathcal{H}_i(x^*, -R_i^{-1}B_i^T p^*, p^*). \qquad (4)$$

Thus, optimal controls λ^* satisfy the complementarity constraints :

$$0 \leq \lambda_i^* \perp \mathcal{H}_i(x^*, -R_i^{-1}B_i^T p^*, p^*) \geq 0, \; i \in S$$

the sign $x \perp y$ means $xy = 0$.

Numerical resolution

Major drawback in the numerical resolution: The existence of singular controls.

Singular controls : there exist at least two indices $(i, j) \in S^2$ such that on a non empty time interval (a, b),

$$\mathcal{H}_i = \mathcal{H}_j = 0, \forall t \in (a, b)$$

Then all values satisfying $\lambda_i + \lambda_j = 1$ are potential candidate for optimality

- PMP is inconclusive concerning the value of λ^* (Additional NC are required)
- λ is not admissible for the switched systems (not at the vertices of Λ) but could be approximated by chattering (Thanks to density theorem).

Numerical consequences:

- Indirect methods like shooting methods are inoperative
  - the uniqueness of the solution of Hamiltonian system is lost (bifurcations)
  - the solution structure (regular -singular) is required
- Direct methods (NLP) yield to bad numerical results due to the insensitivity of the Lagrangian w.r.t. the control

Numerical resolution

Idea: Take implicitly into account the singular arcs using the necessary condition of the PMP and the Hamiltonian systems and then solve directly an augmented constraint optimization problem.

Denote by $z = (x, p)$

Problem 2: Minimize (using NLP):

$$\min_{\lambda(\cdot)} \frac{1}{2} \int_0^\infty \sum_{i=1}^{s} \lambda_i (x^{\mathrm{T}} Q_i x + p^{\mathrm{T}} B_i R_i^{-1} B_i^T p) \mathrm{d}t \tag{5}$$

$$\text{subject to } \dot{z} = \sum_{i=1}^{s} \lambda_i \begin{pmatrix} A_i & -B_i R_i^{-1} B_i^T \\ -Q_i & -A_i^T \end{pmatrix} z \tag{6}$$

$$0 \leq \lambda_i \perp \mathcal{H}_i(x, -R_i^{-1} B_i^T p, p) \geq 0, \quad i \in S \tag{7}$$

$$\lambda(t) \in \Lambda, \quad x(0) = x_0$$

where the sign $x \perp y$ means $xy = 0$.

Special issue: Discontinuous Differential Systems : Theory and Numerical Methods

P. Riedinger, C. Morarescu, A numerical framework for optimal control of switched input affine nonlinear systems subject to path constraint, Mathematics and Computers in Simulation, January 2014

Lyapunov based switching law

Insight: Lyapunov function as a tight upper bound on the value function (may coincide at some points)

- Consider the family of Riccati equations parametrized by $\lambda \in \Lambda$:

$$A(\lambda)^T P_\lambda + P_\lambda A(\lambda) - P_\lambda B(\sqrt{\lambda}) R^{-1} B(\sqrt{\lambda})^{\mathrm{T}} P_\lambda + Q(\lambda) = 0. \tag{8}$$

  corresponding to the LQ subproblem obtained for a fixed λ, if exists.
- $A(\lambda) = \sum_{i \in S} \lambda_i A_i$,
- $B(\sqrt{\lambda}) = [\sqrt{\lambda_1} B_1 | \sqrt{\lambda_2} B_2 | \ldots | \sqrt{\lambda_s} B_s]$
- $Q(\lambda) = \sum_{i \in S} \lambda_i Q_i$ and $R = \mathrm{diag}([R_1, R_2, \cdots, R_s])$.

Lemma

If the pair $(A(\lambda), B(\sqrt{\lambda}))$ is stabilizable and $Q(\lambda)$ is positive definite, then there exists a positive definite solution to the parametrized Riccati equation Eq. (8).

Lyapunov based switching law

We denote by Λ^+ the set

$$\Lambda^+ = \left\{ \lambda \in \Lambda \mid \text{the pair } (A(\lambda), B(\sqrt{\lambda})) \text{ is stabilizable and } \max \text{spec}(P_\lambda) \leq \nu_{\max} \right\}$$

where $\text{spec}(P_\lambda)$ denotes the spectrum of P_λ and $\nu_{\max}$ an arbitrary large number.

Λ^+ satisfies the following property.

Lemma

The matrices Q_i being positive definite, if one can find $\lambda^0 \in \Lambda$ such that $(A(\lambda^0), B(\sqrt{\lambda^0}))$ is controllable, then, for every $\nu_{\max}$ large enough, set Λ^+ is compact and its interior is not empty in Λ.

Moreover, the two following real numbers, α_m and α_M, defined as

$$\alpha_m = \min_{\lambda \in \Lambda^+} \min(\text{spec}(P_\lambda)) \qquad \alpha_M = \max_{\lambda \in \Lambda^+} \max(\text{spec}(P_\lambda))$$

are positive.

Lyapunov based switching law

Let us now introduce the following Lyapunov function

$$V_m(x) := \inf_{\lambda \in \Lambda^+} x^{\mathrm{T}} P_\lambda x \tag{9}$$

where P_λ denotes the solution of Ricccati equation (8).

- We show that V_m is a positive definite function, homogeneous of degree 2, proper and locally Lipschitz.
- Moreover, the directional derivative of $V_m(x; d)$ in direction d is given by [Furukawa 1983]:

$$V_m'(x; d) = \lim_{h \to 0; h > 0} \frac{V_m(x + hd) - V_m(x)}{h} = 2 \inf_{\lambda \in \ell(x)} d^{\mathrm{T}} P_\lambda x.$$

where $\ell(x)$ denotes the subset of $\lambda \in \Lambda^+$ such that $V_m(x) = x^T P_\lambda x$.

Lyapunov based switching law

Theorem (Main result)

Assume that

1. $Q_i > 0,\ i \in S$
2. $\exists \lambda_0$ *s.t.* $(A(\lambda_0), B(\sqrt{\lambda_0}))$ *is controllable.*

For every $x \in \mathbb{R}^n$*, we choose*

$$(i(x), \lambda(x)) \in \arg \min_{(i,\lambda) \in S \times \ell(x)} \left(2x^{\mathrm{T}} M_i(\lambda) P_\lambda x + x^{\mathrm{T}} N_i(\lambda) x\right).$$

where
$M_i(\lambda) := A_i - B_i K_i(\lambda)$,
$K_i(\lambda) := R_i^{-1} B_i^T P_\lambda$
$N_i(\lambda) := Q_i + K_i(\lambda)^T R_i K_i(\lambda)$.
Then, the feedback

$$\begin{aligned} \sigma &= i(x) \\ u_{i(x)} &= -K_{i(x)}(\lambda(x))x = -R_{i(x)}^{-1} B_{i(x)}^T P_{\lambda(x)} x \end{aligned}$$

stabilizes the switched system (1) *with a cost smaller than* $\frac{1}{2} V_m(x_0)$*.*
Exponential convergence rate is greater than $\beta = \frac{\eta_0}{\alpha_1}$ *where* η_0 *and* α_1 *are given by:*

$$\eta_0 = \min_{i \in S} \inf_{x \in S^{n-1}} \inf_{\lambda \in \ell(x)} x^T N_i(\lambda) x, \quad \alpha_1 = \max_{x \in S^{n-1}} V_m(x)$$

Lyapunov based switching law

Sketch of the proof:

- Riccati eq. (8) can be rewritten as a convex combination:

$$\sum_{i \in S} \lambda_i \Big(2x^{\mathrm{T}} M_i^{\mathrm{T}}(\lambda) P_\lambda x + x^{\mathrm{T}} N_i(\lambda) x \Big) = 0,$$

- For every $(x, \lambda) \in \mathbb{R}^n \times \Lambda^+$,

$$\min_{i \in S} \Big(2x^{\mathrm{T}} M_i^{\mathrm{T}}(\lambda) P_\lambda x + x^{\mathrm{T}} N_i(\lambda) x \Big) \leq 0$$

- Then, from the directional derivative of V_m, for every $(x, \lambda^0) \in \mathbb{R}^n \times \ell(x)$, there exists $i(x, \lambda^0)$ such that in direction $d = M_{i(x,\lambda^0)}(\lambda^0) x$

$$V_m'(x; M_i(\lambda^0)x) \leq 2x^{\mathrm{T}} M_i^{\mathrm{T}}(\lambda^0) P_{\lambda^0} x \leq -x^{\mathrm{T}} N_i(\lambda^0) x$$

- Therefore, for any initial condition x_0,

$$V_m(x(t)) + \int_0^t x^{\mathrm{T}} (Q_{i(x)} + K_{i(x)}(\lambda(x))^{\mathrm{T}} R_{i(x)} K_{i(x)}(\lambda(x)) x d\tau \leq V_m(x_0), \quad \forall t \geq 0.$$

As $Q_i > 0,\ \forall i \in S$, it follows that: $x(t) \to 0$ when $t \to +\infty$.

Discussion concerning the switching law and its optimality

Why do we claim that the Lyapunov function can be a tight upper bound on the value function?

- The value $\frac{1}{2}V_m(x)$ is the best cost related to every constant convex combination that stabilizes the relaxed system (In infinite number !).
- If all subsystems are stabilizable, then $\frac{1}{2}V_m(x) \leq \min_{i \in S} \frac{1}{2}x^T P_i x$

When $\frac{1}{2}V_m(x)$ is optimal?
"Along the part of trajectories where the optimal control λ^* is constant to reach the origin".

- if the number of switchings is finite
- if the trajectory is steered to the origin by a constant singular control λ for which $P_\lambda > 0$.
  $\rightarrow$ Singular controls in dimension $n = 2$ are constant.

Discussion concerning the switching law and its optimality

Formally, we can justified the design of the switching law as follow.

- Assuming known the value function, one can write for any $T > 0$,

$$V^*(x_0) = min_\sigma \frac{1}{2} \int_0^T x^T Q_{\sigma(t)} x + u^T_{\sigma(t)} R_{\sigma(t)} u_{\sigma(t)} dt + V^*(x(T))$$

- The transversality condition of PMP implies at time T, $p^*(T) = \frac{\partial V(x(T))}{\partial x}$ (if exists).
- Now suppose that $V^*(x(T))$ is approximated by $V_m(x(T))$. Then, an approximation of $p^*(T)$ is given by $p^*(x(T)) \approx P_\lambda x(T)$ with $\lambda \in \ell(x)$.
- Thus, it is easy to check that the minimization of the Hamiltonian at time T leads to the provided switching law.
- As the problem is homogenous and if the approximation is "good", one can infer that $p^*(x) \approx P_{\lambda(x)} x$ with $\lambda(x) \in \ell(x)$ for every x.

Roughly speaking, the state feedback switching law matches the optimal one when $P_{\lambda(x)} x$ is a good approximation of p^*.

Example 1

Consider a two mode switched system with the following design parameters:

$$A_1 = \begin{pmatrix} -2.7 & 3.9 \\ 4.4 & -12.6 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -9.5 & -5.1 \\ -7.5 & -3.3 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 4.6 \\ 0 \end{pmatrix},$$

$Q_1 = Q_2 = \mathrm{Id}$, $R_1 = 1$ and $R_2 = 2$.

For each subsystem, an LQ design can be be performed separately.

Example 1 :

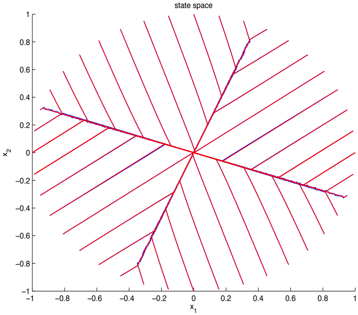

Figure: Ex. 1: State space trajectories: (red) optimal solution (NLP); (blue) switching law

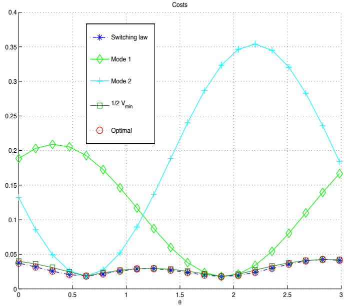

Figure: Ex. 1: Cost comparisons for different initial positions taken on the unit ball.

Example 2

For this second example, we have chosen two non stabilizable subsystems:

$$A_1 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -2 & 1 \\ 0 & 1 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$Q_1 = Q_2 = \mathrm{Id}$, $R_1 = 2$ and $R_2 = 1$

- There is no LQ design that can be defined separately for each subsystem.
- The set Λ^+ is non empty, the switching law presented in this paper can be applied.

Example 2 :

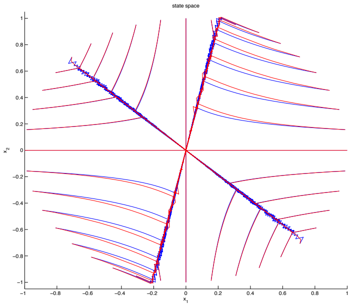

Figure: Ex. 2: State space trajectories: (red) optimal solution (NLP); (blue) switching law

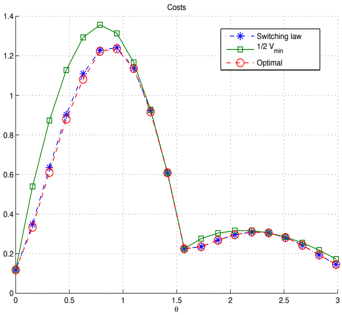

Figure: Ex. 2: Cost comparisons for different initial positions taken on the unit ball.

Conclusion

- A state feedback switching law for switched LQ regulator problems in continuous time.
- Applicable if a controllable convex combination of the subsystems exists
- The switching law can be optimal along arcs (singular or not) ending to the origin with a constant optimal control.
- In any case, a guarantee on the cost is provided by the upper bound $\frac{1}{2}V_{min}(x)$.
- Additional stability results in the paper for sampled switched law

Related papers:

- P. Riedinger, A switched LQ regulator design in continuous time, IEEE TAC to appear in May 2014.
- P. Riedinger, J-C. Vivalda, An LQ sub-optimal stabilizing feedback law for switched linear systems, HSCC 2014, Berlin.