Convex conditions for robust stability analysis and stabilization of linear aperiodic impulsive systems with applications to asynchronous sampled-data systems

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Introduction

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Impulsive systems

Linear case

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t),\ t \notin \{t_k\}_{k\in\mathbb{N}_0} \\ x(t) & = & Jx(t^-),\ t \in \{t_k\}_{k\in\mathbb{N}_0} \\ x(0) & = & x_0 \end{array} \qquad (1)$$

where $x(t^-) = \lim_{s \uparrow t} x(s)$.

- A continuous part
- A discrete part
- A set of impulse instants $\{t_k\}_{k\in\mathbb{N}_0}$, $t_0 = 0$.

Jumping rule

- State-dependent jumping instants, e.g. when x enters some sets (internal)
- Time-dependent jumping instants (external)

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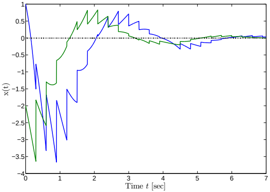
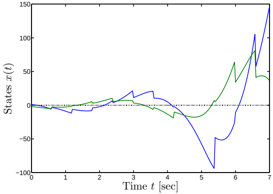
Problems

- Stability depends on the matrices of the system but also on the set of impulse instants!





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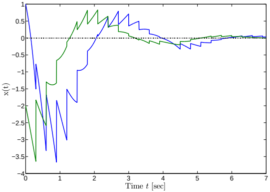

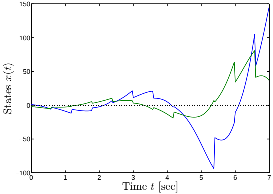

- How can we characterize stability in an efficient/accurate/tractable way?

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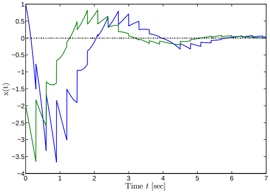

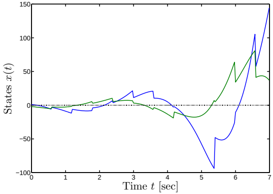

- How can we characterize stability in an efficient/accurate/tractable way?
- How can we derive tractable conditions for control design?

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Stability of impulsive systems

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Dwell-times

Definition

The dwell-time T_k is defined as $T_k = t_{k+1} - t_k$, i.e. $t_{k+1} = t_k + T_k$.

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Average dwell-time[1]

- The number of impulses in any time interval
- Asymptotic notion

1 J. P. Hespanha, et al. Lyapunov conditions for input-to-state stability of impulsive systems, *Automatica*, 2008

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Average dwell-time[1]

- The number of impulses in any time interval
- Asymptotic notion

Minimum/maximum/range dwell-time[2]

- Minimum dwell-time: $T_k \geq \bar{T}$ for some $\bar{T} > 0$, $k \in \mathbb{N}_0$
- Maximum dwell-time: $T_k \leq \bar{T}$ for some $\bar{T} > \varepsilon > 0$, $k \in \mathbb{N}_0$
- Minimum dwell-time: $T_k \in [T_{min}, T_{max}]$,for some $0 < T_{min} \leq T_{max} < \infty$, $k \in \mathbb{N}_0$
- Non-asymptotic notion

[1] J. P. Hespanha, et al. Lyapunov conditions for input-to-state stability of impulsive systems, *Automatica*, 2008

[2] C. Briat et al. A looped-functional approach for robust stability analysis of linear impulsive systems, *Systems & Control Letters*, 2012

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Average dwell-time - General result

Theorem ([1])

Assume that there exist $P \in \mathbb{S}^n_{\succ 0}$ *and scalars* $c, d \in \mathbb{R}$, $d \neq 0$, *such that*

$$\begin{array}{rcl} A^T P + PA + cP & \prec & 0 \\ J^T PJ - e^{-d} P & \prec & 0. \end{array} \quad (2)$$

Then, the system is stable provided that the number of impulses $N(t, s)$ *over the interval* $(s, t]$ *satisfies*

$$-dN(t, s) - (c - \lambda)(t - s) \leq \mu, \text{ for all } t \geq s$$

for some arbitrary constants $\lambda, \mu > 0$.

[1] J. P. Hespanha, et al. Lyapunov conditions for input-to-state stability of impulsive systems, *Automatica*, 2008

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Average dwell-time

Theorem ([1])

Assume that there exist $P \in \mathbb{S}^n_{\succ 0}$ *and scalars* $c > 0$, $d < 0$, *such that*

$$\begin{array}{rcl} A^TP + PA + cP & \prec & 0 \\ J^TPJ - e^{-d}P & \prec & 0. \end{array} \quad (3)$$

Then, the system is stable provided that the number of impulses $N(t,s)$ *over the interval* $(s,t]$ *satisfies*

$$N(t,s) \leq \frac{t-s}{\tau^*} + N_0, \text{ for all } t \geq s.$$

[1] J. P. Hespanha, et al. Lyapunov conditions for input-to-state stability of impulsive systems, *Automatica*, 2008

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Reverse average dwell-time

Theorem ([1])

Assume that there exist $P \in \mathbb{S}^n_{\succ 0}$ *and scalars* $c < 0$, $d > 0$*, such that*

$$\begin{array}{rcl} A^T P + PA + cP & \prec & 0 \\ J^T PJ - e^{-d}P & \prec & 0. \end{array} \quad (4)$$

Then, the system is stable provided that the number of impulses $N(t, s)$ *over the interval* $(s, t]$ *satisfies*

$$N(t, s) \geq \frac{t - s}{\tau^*} - N_0, \text{ for all } t \geq s.$$

[1] J. P. Hespanha, et al. Lyapunov conditions for input-to-state stability of impulsive systems, *Automatica*, 2008

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Minimum dwell-time

Theorem ([1])

Assume that there exist $P \in \mathbb{S}^n_{\succ 0}$ *and a scalar* $\bar{T} > 0$ *such that the conditions*

$$\begin{array}{rcl} A^T P + PA & \prec & 0 \\ J^T e^{A^T \bar{T}} P e^{A\bar{T}} J - P & \prec & 0 \end{array} \qquad (5)$$

hold.
Then, the system is stable provided that $T_k \geq \bar{T}$*; i.e.* $t_{k+1} \geq t_k + \bar{T}$, $k \in \mathbb{N}_0$.

- A must be Hurwitz
- Stable continuous-time dynamics, potentially unstable discrete-time dynamics
- If we let $\bar{T} \to 0$, then we obtain a condition for arbitrary impulse times (but we must deal with Zeno behavior)
- Easy to check

[1] C. Briat et al. A looped-functional approach for robust stability analysis of linear impulsive systems, *Systems & Control Letters*, 2012

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Maximum dwell-time

Theorem ([1])

Assume that there exist $P \in \mathbb{S}^n_{\succ 0}$ *and a scalar* $\bar{T} > 0$ *such that the conditions*

$$\begin{array}{rcl} A^T P + PA & \succ & 0 \\ J^T e^{A^T \bar{T}} P e^{A\bar{T}} J - P & \prec & 0 \end{array} \quad (6)$$

hold.
Then, the system is stable provided that $0 < \varepsilon < T_k \leq \bar{T}$*; i.e.* $t_{k+1} \leq t_k + \bar{T}$*,* $k \in \mathbb{N}_0$*.*

- A must be anti-Hurwitz
- Anti-stable continuous-time dynamics, stable discrete-time dynamics
- Easy to check

[1] C. Briat et al. A looped-functional approach for robust stability analysis of linear impulsive systems, *Systems & Control Letters*, 2012

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Range dwell-time

Discretization

- We consider here the discrete-time system

$$x(t_{k+1}^-) = e^{AT_k}Jx(t_k^-),\ k \in \mathbb{N}_0 \tag{7}$$

where $t_0 = 0$ and $T_k \in [T_{min}, T_{max}]$.

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Theorem ([1])

Assume that there exist $P \in \mathbb{S}^n_{\succ 0}$ such that the condition

$$J^T e^{A^T\theta} P e^{A\theta} J - P \prec 0 \tag{8}$$

holds for all $\theta \in [T_{min}, T_{max}]$.
Then, the system is stable provided that $T_k \in [T_{min}, T_{max}]$, $k \in \mathbb{N}_0$.

- Robust feasibility problem (due to parametric dependence)
- Not easy to check since non-convex in θ...

[1] C. Briat et al. A looped-functional approach for robust stability analysis of linear impulsive systems, *Systems & Control Letters*, 2012

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Difficulties

Robustness

- Robust LMIs are difficult to check

$$J^T e^{A^T\theta} P e^{A\theta} J - P \prec 0,\ \theta \in [T_{min}, T_{max}]$$

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Robustness

- Robust LMIs are difficult to check

$$J^T e^{A^T\theta} P e^{A\theta} J - P \prec 0,\ \theta \in [T_{min}, T_{max}]$$

- Difficult to extend to uncertain matrices A

$$J^T e^{(A+\Delta)^T \bar{T}} P e^{(A+\Delta)\bar{T}} J - P \prec 0$$

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$$J^T e^{(A+\Delta)^T \bar{T}} P e^{(A+\Delta)\bar{T}} J - P \prec 0$$

- Not directly applicable to systems with time-varying A

$$J^T \Phi(\bar{T})^T P \Phi(\bar{T}) J - P \prec 0$$

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$$J^T \Phi(\bar{T})^T P \Phi(\bar{T}) J - P \prec 0$$

Control Design

- Not convex

$$J^T e^{(A+BK)^T \bar{T}} P e^{(A+BK)\bar{T}} J - P \prec 0$$

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Convex conditions for periodic impulses

Theorem

Let us consider an impulsive system (A, J) *with periodic impulses, i.e.* $T_k = \bar{T}$, $k \in \mathbb{N}$. *Then, the following statements are equivalent:*

(a) *The impulsive system with* $\bar{T}$*-periodic impulses is asymptotically stable.*

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(a) *The impulsive system with $\bar{T}$-periodic impulses is asymptotically stable.*

(b) *There exists a matrix $P \in \mathbb{S}^n_{\succ 0}$ such that the LMI*

$$J^T e^{A^T \bar{T}} P e^{A\bar{T}} J - P \prec 0 \tag{9}$$

holds.

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holds.

(c) *There exist a differentiable matrix function* $R : [0, \bar{T}] \mapsto \mathbb{S}^n$, $R(0) \succ 0$, *and a scalar* $\varepsilon > 0$ *such that the LMIs*

$$A^T R(\tau) + R(\tau)A + \dot{R}(\tau) \preceq 0 \quad \text{and} \quad J^T R(0)J - R(\bar{T}) + \varepsilon I \preceq 0$$

hold for all $\tau \in [0, \bar{T}]$.

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hold for all $\tau \in [0, \bar{T}]$.

(d) *There exist a differentiable matrix function $S : [0, \bar{T}] \mapsto \mathbb{S}^n$, $S(\bar{T}) \succ 0$, and a scalar $\varepsilon > 0$ such that the LMIs*

$$A^T S(\tau) + S(\tau)A - \dot{S}(\tau) \preceq 0 \quad \text{and} \quad J^T S(\bar{T})J - S(0) + \varepsilon I \preceq 0$$

hold for all $\tau \in [0, \bar{T}]$.

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Convex conditions for range dwell-time

Theorem

Let us consider an impulsive system (A, J). Then, the following statements are equivalent:

(a) *There exists a matrix $P \in \mathbb{S}^n_{\succ 0}$ such that the LMI*

$$J^T e^{A^T \theta} P e^{A\theta} J - P \prec 0 \tag{10}$$

holds for all $\theta \in [T_{min}, T_{max}]$.

Moreover, when one of the above statements holds, then the aperiodic impulsive system with ranged dwell-time $T_k \in [T_{min}, T_{max}]$ is asymptotically stable.

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holds for all $\theta \in [T_{min}, T_{max}]$.

(b) *There exist a differentiable matrix function $R : [0, T_{max}] \mapsto \mathbb{S}^n$, $R(0) \succ 0$, and a scalar $\varepsilon > 0$ such that the LMIs*

$$A^T R(\tau) + R(\tau)A + \dot{R}(\tau) \preceq 0 \tag{11}$$

and

$$J^T R(0)J - R(\theta) + \varepsilon I \preceq 0 \tag{12}$$

hold for all $\tau \in [0, T_{max}]$ and all $\theta \in [T_{min}, T_{max}]$.

Moreover, when one of the above statements holds, then the aperiodic impulsive system with ranged dwell-time $T_k \in [T_{min}, T_{max}]$ is asymptotically stable.

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Convex conditions for minimum dwell-time

Theorem (Minimum Dwell-Time)

Let us consider an impulsive system (A, J). Then, the following statements are equivalent:

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hold.

Moreover, when one of the above statements holds, the impulsive system is asymptotically stable under minimum dwell-time $\bar{T}$, i.e. for any sequence $\{t_k\}_{k\in\mathbb{N}}$ such that $T_k \geq \bar{T}$.

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hold.

(b) *There exist a differentiable matrix function $R : [0, \bar{T}] \mapsto \mathbb{S}^n$, $R(0) \succ 0$, and a scalar $\varepsilon > 0$ such that the LMIs*

$$A^T R(0) + R(0)A \prec 0$$

$$A^T R(\tau) + R(\tau)A + \dot{R}(\tau) \preceq 0 \quad \text{and} \quad J^T R(0)J - R(\bar{T}) + \varepsilon I \preceq 0$$

hold for all $\tau \in [0, \bar{T}]$.

Moreover, when one of the above statements holds, the impulsive system is asymptotically stable under minimum dwell-time $\bar{T}$, i.e. for any sequence $\{t_k\}_{k\in\mathbb{N}}$ such that $T_k \geq \bar{T}$.

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Pros and cons

Benefits

- Convex in the matrices of the system → robustness analysis possible

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- Convex in the matrices of the system $\rightarrow$ control design possible

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Pros and cons

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- Convex in the matrices of the system $\rightarrow$ robustness analysis possible
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- Applicable to systems with time-varying matrices

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Drawbacks

- Infinite-dimensional LMI problems

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Pros and cons

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Benefits

- Convex in the matrices of the system $\rightarrow$ robustness analysis possible
- Convex in the matrices of the system $\rightarrow$ control design possible
- Applicable to systems with time-varying matrices

Drawbacks

- Infinite-dimensional LMI problems
- Needs relaxation (piecewise linear approximation or SOS)

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Example 1 - Range dwell-time

Let us consider the system[1]

$$A = \begin{bmatrix} -1 & 0.1 \\ 0 & 1.2 \end{bmatrix}, \qquad J = \begin{bmatrix} 1.2 & 0 \\ 0 & 0.5 \end{bmatrix}. \tag{13}$$

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1 C. Briat et al. A looped-functional approach for robust stability analysis of linear impulsive systems, *Systems & Control Letters*, 2012

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Example 1 - Range dwell-time

Let us consider the system[1]

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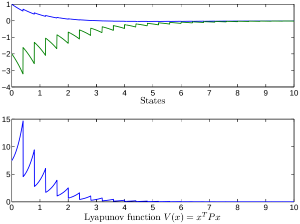
	d_R	T_{min}	T_{max}
Proposed method	2	0.1834	0.4998
	4	0.1824	0.5768
	6	0.1824	0.5776
Periodic case	–	0.1824	0.5776

- Finds the theoretical bounds
- Also holds in the aperiodic case



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Example 2 - Minimum dwell-time

Let us consider the system [1]

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}, \qquad J = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}. \qquad (14)$$

[1] C. Briat et al. A looped-functional approach for robust stability analysis of linear impulsive systems, *Systems & Control Letters*, 2012

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Example 2 - Minimum dwell-time

Let us consider the system [1]

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}, \qquad J = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}. \tag{14}$$

	d_R	T_{min}
Proposed approach	2	1.1883
	4	1.1408
	6	1.1406
Exponential LMI	–	1.1406
Periodic case	–	1.1406

- Non-conservative dwell-time result

[1] C. Briat et al. A looped-functional approach for robust stability analysis of linear impulsive systems, *Systems & Control Letters*, 2012

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Robustification

- Let us consider now the system

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t),\ t \notin \{t_k\}_{k\in\mathbb{N}_0} \\ x(t) & = & Jx(t^-),\ t \in \{t_k\}_{k\in\mathbb{N}_0} \\ x(0) & = & x_0 \end{array} \tag{15}$$

where

$$A \in \mathcal{A} := \mathbf{co}\{A_1, \ldots, A_N\},\ J \in \mathcal{J} := \mathbf{co}\{J_1, \ldots, J_N\}$$

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where

$$A \in \mathcal{A} := \mathbf{co}\left\{A_1, \ldots, A_N\right\},\ J \in \mathcal{J} := \mathbf{co}\left\{J_1, \ldots, J_N\right\}$$

- Define the set

$$\mathbf{\Phi}_{\bar{T}} := \left\{\Phi(\bar{T}) :\ \Phi(s) \text{ solves (16)},\ \lambda(s) \in \Lambda_N,\ s \in [0, \bar{T}]\right\}.$$

$$\frac{d\Phi(s)}{ds} = \left(\sum_{i=1}^{N} \lambda_i(s) A_i\right) \Phi(s),\ \Phi(0) = I. \tag{16}$$

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$$\frac{d\Phi(s)}{ds} = \left(\sum_{i=1}^{N} \lambda_i(s) A_i\right) \Phi(s),\ \Phi(0) = I. \tag{16}$$

- We can now consider the uncertain discrete-time system

$$x((k+1)\bar{T}) = \Psi J x(k\bar{T}), k \in \mathbb{N}_0 \tag{17}$$

where $\Psi \in \mathbf{\Phi}_{\bar{T}}$.

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Robustification

Theorem

Let us consider an uncertain (time-varying) impulsive system (A, J), $A \in \mathcal{A}$, $J \in \mathcal{J}$, *with* $\bar{T}$*-periodic impulses. Then, the following statements are equivalent:*

(a) *The uncertain (time-varying) impulsive system with* $\bar{T}$*-periodic impulses is quadratically stable*

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Robustification

Theorem

Let us consider an uncertain (time-varying) impulsive system (A, J), $A \in \mathcal{A}$, $J \in \mathcal{J}$, *with* $\bar{T}$*-periodic impulses. Then, the following statements are equivalent:*

(a) *The uncertain (time-varying) impulsive system with* $\bar{T}$*-periodic impulses is quadratically stable*

(b) *There exists a matrix* $P \in \mathbb{S}^n_{\succ 0}$ *such that the LMI*

$$J^T \Psi^T P \Psi J - P \prec 0$$

holds for all $(\Psi, J) \in \mathbf{\Phi}_{\bar{T}} \times \mathcal{J}$.

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Robustification

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Let us consider an uncertain (time-varying) impulsive system (A, J)*,* $A \in \mathcal{A}$*,* $J \in \mathcal{J}$*, with* $\bar{T}$*-periodic impulses. Then, the following statements are equivalent:*

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holds for all $(\Psi, J) \in \mathbf{\Phi}_{\bar{T}} \times \mathcal{J}$*.*

(c) *There exist a differentiable matrix function* $R : [0, \bar{T}] \mapsto \mathbb{S}^n$*,* $R(0) \succ 0$*, and a scalar* $\varepsilon > 0$ *such that the LMIs*

$$A_i^T R(\tau) + R(\tau) A_i + \dot{R}(\tau) \preceq 0, \text{ and } J_i^T R(0) J_i - R(\bar{T}) + \varepsilon I \preceq 0$$

hold for all $\tau \in [0, \bar{T}]$ *and all* $i = 1, \ldots, N$*.*

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Stabilization of impulsive systems

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Stabilization problem

System

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + B_c u_c(t),\ t \neq t_k \\ x(t) & = & Jx(t^-) + B_d u_d(t),\ t = t_k \end{array} \quad (18)$$

where $u_c \in \mathbb{R}^{m_c}$ and $u_d \in \mathbb{R}^{m_d}$ are the control inputs.

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Stabilization problem

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where $u_c \in \mathbb{R}^{m_c}$ and $u_d \in \mathbb{R}^{m_d}$ are the control inputs.

Control law

We consider the following class of control-laws:

$$\begin{aligned} u_c(t_k + \tau) &= K_c(\tau)x(t_k + \tau),\ \tau \in [0, T_k), \\ u_d(t_k) &= K_d x(t_k^-) \end{aligned} \tag{19}$$

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Stabilization problem

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Minimum dwell-time case

$$K_c(\tau) = \begin{cases} \tilde{K}_c(\tau) & \text{if } \tau \in [0, \bar{T}) \\ \tilde{K}_c(\bar{T}) & \text{if } \tau \in [\bar{T}, T_k) \end{cases} \quad (20)$$

where $T_k \geq \bar{T}$, $k \in \mathbb{N}$ and $\tilde{K}_c(\tau)$ is some matrix function to be determined.

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Minimum dwell-time result

Theorem (Minimum dwell-time)

Assume that here exist a differentiable matrix function $S : [0, \bar{T}] \mapsto \mathbb{S}^n$, $S(\bar{T}) \succ 0$, a matrix function $U_c : [0, \bar{T}] \mapsto \mathbb{R}^{m_c \times n}$, a matrix $U_d \in \mathbb{R}^{m_d \times n}$ and a scalar $\varepsilon > 0$ such that the LMIs

$$\mathrm{Sym}[AS(\bar{T}) + B_c U_c(\bar{T})] \prec 0, \tag{21}$$

$$\mathrm{Sym}[AS(\tau) + B_c U_c(\tau)] + \dot{S}(\tau) \preceq 0 \tag{22}$$

and

$$\begin{bmatrix} -S(0) + \varepsilon I & JS(\bar{T}) + B_d U_d \\ \star & -S(\bar{T}) \end{bmatrix} \preceq 0 \tag{23}$$

hold for all $\tau \in [0, \bar{T}]$. Then, the closed-loop system is asymptotically stable with minimum dwell-time $\bar{T}$ and suitable controller gains are retrieved using

$$\tilde{K}_c(\tau) = U_c(\tau) S(\tau)^{-1} \quad \text{and} \quad K_d = U_d S(\bar{T})^{-1}. \tag{24}$$

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Range dwell-time result

Theorem (Range dwell-time)

Assume that here exist a differentiable matrix function $S : [0, \bar{T}] \mapsto \mathbb{S}^n$, $S(0) \succ 0$, *a matrix function* $U_c : [0, \bar{T}] \mapsto \mathbb{R}^{m_c \times n}$, *a matrix* $U_d \in \mathbb{R}^{m_d \times n}$ *and a scalar* $\varepsilon > 0$ *such that the LMIs*

$$\mathrm{Sym}[AS(\tau) + B_c U_c(\tau)] + \dot{S}(\tau) \preceq 0 \tag{25}$$

and

$$\begin{bmatrix} -S(\theta) + \varepsilon I & JS(0) + B_d U_d \\ \star & -S(0) \end{bmatrix} \preceq 0 \tag{26}$$

hold for all $\tau \in [0, T_{max}]$ *and all* $\theta \in [T_{min}, T_{max}]$. *Then, the closed-loop system is asymptotically stable with range dwell-time* $[T_{min}, T_{max}]$ *and suitable controller gains are retrieved using*

$$\tilde{K}_c(\tau) = U_c(\tau) S(\tau)^{-1} \quad \text{and} \quad K_d = U_d S(0)^{-1}. \tag{27}$$

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Example

Let us consider the system with matrices

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } J = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \tag{28}$$

- We want to compute $\tilde{K}_c(\tau)$ such that the minimum dwell-time is, at most, $\bar{T} = 0.1$.

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Example

Let us consider the system with matrices

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- We want to compute $\tilde{K}_c(\tau)$ such that the minimum dwell-time is, at most, $\bar{T} = 0.1$.
- We obtain

$$\tilde{K}_c(\tau) = \frac{1}{d(\tau)} \begin{bmatrix} 1.4750481 + 3.2714889\tau - 41.011914\tau^2 \\ 3.9063911 - 1.6733059\tau - 37.472443\tau^2 \end{bmatrix}^T$$

where $d(\tau) = -0.19767438 + 0.78454217\tau + 7.6562219\tau^2$.

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Example

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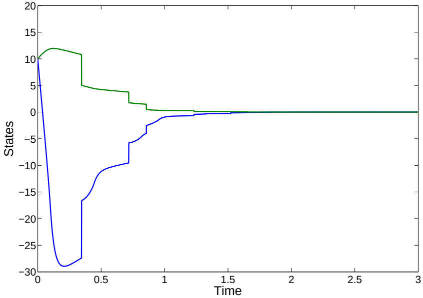

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Sampled-data systems

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Sampled-data systems

System

Let us consider now the continuous-time system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{29}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state of the system and the control input, respectively.

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Sampled-data systems

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$$\dot{x}(t) = Ax(t) + Bu(t) \tag{29}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state of the system and the control input, respectively.

Controller

The control input is assumed to be computed from a sampled-data state-feedback control law given by

$$u(t) = K_1 x(t_k) + K_2 u(t_{k-1}),\ t \in [t_k, t_{k+1}) \tag{30}$$

where $K_1 \in \mathbb{R}^{m \times n}$ and $K_2 \in \mathbb{R}^{m \times m}$ are the control gains to be determined.

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Sampled-data systems

System

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The control input is assumed to be computed from a sampled-data state-feedback control law given by

$$u(t) = K_1 x(t_k) + K_2 u(t_{k-1}),\ t \in [t_k, t_{k+1}) \tag{30}$$

where $K_1 \in \mathbb{R}^{m \times n}$ and $K_2 \in \mathbb{R}^{m \times m}$ are the control gains to be determined.

Objectives

Find a control law of the form (30) such that the closed-loop system is robustly stable for all sampling-periods in the range $[T_{min}, T_{max}]$.

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Sampled-data systems as impulsive systems

- Any sampled-data system can be equivalently reformulated as an impulsive system:

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} &= \underbrace{\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}}_{\bar{A}} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}, \ t \neq t_k \\ \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} &= \underbrace{\begin{bmatrix} I & 0 \\ K_1 & K_2 \end{bmatrix}}_{\bar{J}} \begin{bmatrix} x(t^-) \\ z(t^-) \end{bmatrix}, \ t = t_k \end{aligned} \tag{31}$$

  where $z(t) = u(t_k)$, $t \in [t_k, t_{k+1})$.
- Let $\bar{J} = J_0 + B_0 K$ where

$$J_0 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \ B_0 = \begin{bmatrix} 0 \\ I \end{bmatrix} \text{ and } K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}. \tag{32}$$

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Stabilization

Theorem (Aperiodic sampled-data systems)

The following statements are equivalent:

(a) *There exists a control law of the form* (30) *that quadratically stabilizes the system* (29) *for any aperiodic sampling instant sequence* $\{t_k\}$ *such that* $T_k \in [T_{min}, T_{max}]$.

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Stabilization

Theorem (Aperiodic sampled-data systems)

The following statements are equivalent:

(a) *There exists a control law of the form* (30) *that quadratically stabilizes the system* (29) *for any aperiodic sampling instant sequence* $\{t_k\}$ *such that* $T_k \in [T_{min}, T_{max}]$.

(b) *There exist a differentiable matrix function* $R : [0, T_{max}] \mapsto \mathbb{S}^{n+m}$, $S(0) \succ 0$, *a matrix* $Y \in \mathbb{R}^{m\times(n+m)}$ *and a scalar* $\varepsilon > 0$ *such that the conditions*

$$\bar{A}(\tau)S(\tau) + S(\tau)\bar{A}(\tau)^T + \dot{S}(\tau) \preceq 0 \tag{33}$$

and

$$\begin{bmatrix} -S(\theta) + \varepsilon I & J_0 S(0) + B_0 Y \\ \star & -S(0) \end{bmatrix} \preceq 0 \tag{34}$$

hold for all $\tau \in [0, T_{max}]$ *and all* $\theta \in [T_{min}, T_{max}]$.
Moreover, when this statement holds, a suitable stabilizing control gain can be obtained using the expression $K = YS(0)^{-1}$.

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Example 1

Let us consider the sampled-data system (29) with matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}. \tag{35}$$

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Example 1

Let us consider the sampled-data system (29) with matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}. \quad (35)$$

- Fixed control law: $K_1 = \begin{bmatrix} -3.75 & -11.5 \end{bmatrix}$ and $K_2 = 0$.

	d_R	System (35) T_{max}
Proposed result	4	1.7279
	6	1.7252
(Fridman et al., 2004)	–	0.869
(Naghshtabrizi et al., 2008)	–	1.113
(Fridman, 2010)	–	1.695
(Liu et al., 2010)	–	1.695
(Seuret, 2012)	–	1.723
(Seuret and Peet, 2013)	3	1.7294
	5	1.7294

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Example 1

Let us consider the sampled-data system (29) with matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}. \tag{35}$$

- Designed control law for some given $[T_{min}, T_{max}]$.

T_{min}	T_{max}	K_1	K_2	d_R
0.001	10	$\begin{bmatrix} -0.1145 & -0.8088 \\ -0.0202 & -0.1560 \end{bmatrix}$	-0.0024	2
	50		-0.0030	2
0.001	10	$\begin{bmatrix} -0.0310 & -0.3222 \\ -0.0259 & -0.2726 \end{bmatrix}$	0	3
	50		0	4

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Example 2

- Let us consider the following sampled-data system (29) with matrices

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (36)$$

- Let $K_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $K_2 = 0$.

	d_R	System (36)	
		T_{min}	T_{max}
Proposed result	4	0.4	1.6316
	6	0.4	1.8270
(Seuret, 2012)	–	0.400	1.251
(Seuret and Peet, 2013)	3	0.4	1.820
	5	0.4	1.828

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Example 3

Let us consider the uncertain sampled-data system (29) with matrices

$$A \in \mathcal{A} = \mathbf{co}\left\{ \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, \delta \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} \right\} \text{ and } B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (37)$$

where δ is a positive parameter.

δ	T_{min}	T_{max}	K_1	K_2	d_R
5	0.001	10	$\begin{bmatrix} -0.0757 & -0.7306 \end{bmatrix}$	-0.0006	2
5		20	$\begin{bmatrix} -0.0411 & -0.3835 \end{bmatrix}$	-0.0022	2
20	0.001	10	$\begin{bmatrix} -0.0578 & -0.5560 \end{bmatrix}$	-0.0025	2
20		20	$\begin{bmatrix} -0.0339 & -0.3121 \end{bmatrix}$	-0.0019	2

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Concluding remarks

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Conclusions

Concluding statements

- Robust stability under minimum, maximum and range dwell-time
- Robust stabilization possible
- Can be extended to homogeneous Lyapunov functions easily

Possible extensions

- Switched systems, time-dependent hybrid systems
- Dynamic output feedback?
- Nonlinear systems

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Thank you for your Attention