Ecole MACS 2019 - Switched Systems and Observers

GTs MOSAR, SDH & SYNOBS

Program

June 3rd, 14h00 – 17h00 : Marc JUNGERS – *Stability and Stabilization I*

June 4th, 8h30 – 12h30 : Marc JUNGERS – *Stability and Stabilization II*

June 4th, 14h00 – 17h00 : Mihaly PETRECZKY – *Structural Properties*

June 5th, 8h30 – 12h30 : Jean-Pierre BARBOT – *Observers & Applications*

Introduction to (discrete-time linear) switched systems: stability and stabilization.

Ecole MACS, Talence

June 3rd-4th, 2019

Marc JUNGERS

UMR 7039

Outline of the talk

What is a switched system ?

About stability

Sufficient conditions for stability

Stabilization without constraints

Stabilization constrained by a language

Outline of the tutorial

What is a switched system ?

- Definition and link with hybrid systems
- Illustrations and motivation

About stability

Sufficient conditions for stability

Stabilization without constraints

Stabilization constrained by a language

Definition of switched systems

Definition :
Switched systems are the association of a finite set of dynamical systems (modes) and a switching law $\sigma(\cdot)$ that indicates at each time which mode is active.

Let $\mathcal{I}_N = \mathbb{N}_N = \{1; \cdots; N\}$, where $N \in \mathbb{N}$ is the number of modes.

Continuous-time

$$\dot{x}(t) = f_{\sigma(t)}(x(t), u(t), t), \quad \forall t \in \mathbb{R}^+,$$

where

- $x(t) \in \mathbb{R}^n$ is the state,
- $u(t)$ the input.
- σ the switching law

$$\sigma : \mathbb{R} \to \mathcal{I}_N.$$

Discrete-time

$$x_{k+1} = f_{\sigma(k)}(x_k, u_k, k), \quad \forall k \in \mathbb{N},$$

where

- $x_k \in \mathbb{R}^n$ is the state,
- u_k the input.
- σ the switching law

$$\sigma : \mathbb{N} \to \mathcal{I}_N.$$

Assumptions for the switching law

Several assumptions :

- $\sigma(\cdot)$ is arbitrary.
  $\sigma(\cdot)$ is seen as a perturbation. The results should be true for all the switching laws. Useful when the generation of the signal $k \mapsto \sigma(k)$ could be very difficult to take into account.
- $\sigma(\cdot)$ is state dependent.
  Here we have $\sigma(k) = g(x_k)$.
- $\sigma(\cdot)$ is time dependent or has time constraints.
  This is for instance the case when $\sigma(\cdot)$ is periodic, or has a time constraint such a dwell time.
- $\sigma(\cdot)$ is a control input with or without constraints.
  The issue here is to design the switching law $\sigma(\cdot)$.

Particular case of hybrid systems

Hybrid system :
Heterogenous interaction between continuous and discrete dynamics :

$$\begin{cases} \text{If } z(t) \in \mathcal{C}, \quad \dot{z}(t) \in F(z(t), u(t)), \text{ (flow map)} \\ \text{If } z(t) \in \mathcal{D}, \quad z(t^+) \in G(z(t), u(t)), \text{ (jump map).} \end{cases}$$

For continuous-time switched systems, we have :

$$\mathcal{C} = \mathcal{D} = \mathbb{R}^n \times \mathcal{I}_N, \quad z(t) = \begin{pmatrix} x(t) \\ \sigma(t) \end{pmatrix} \in \mathbb{R}^{n+1},$$

$$F(z(t)) = \begin{pmatrix} \{f_i(x(t), u(t))\}_{i \in \mathcal{I}_N} \\ 0 \end{pmatrix}; \quad G(z(t)) = \begin{pmatrix} x(t) \\ \mathcal{I}_N \end{pmatrix}.$$

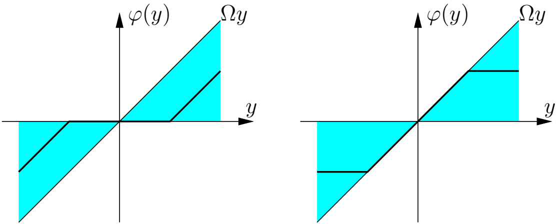
Illustrations

Saturated systems :
Let $x(t) \in \mathbb{R}^2$, with

$$\dot{x}(t) = \begin{bmatrix} -1 & 1 \\ 0 & -5 \end{bmatrix} x(t) + \begin{bmatrix} 0.2 \\ 1 \end{bmatrix} \mathrm{sat}\left(\begin{bmatrix} 1 & -1 \end{bmatrix} x(t) \right).$$

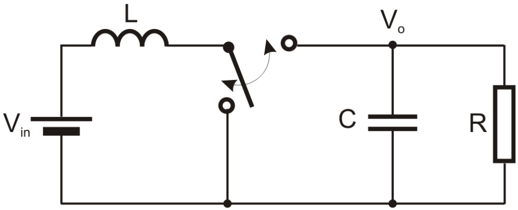
with

$$\mathrm{sat}(u) = \begin{cases} -1 \text{ if } u < -1, \\ +1 \text{ if } u > +1, \\ u \text{ if } -1 \leq u \leq +1. \end{cases}$$



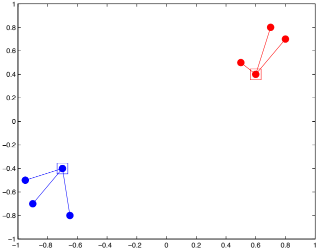
Boost converter :

$$
\begin{aligned}
C\frac{\mathrm{d}v_o}{\mathrm{d}t} &= (2-\sigma)i_L - \frac{1}{R}v_o, \qquad \sigma(t) \in \{1;2\} \\
L\frac{\mathrm{d}i_L}{\mathrm{d}t} &= v_{in} - (2-\sigma)v_o.
\end{aligned}
$$

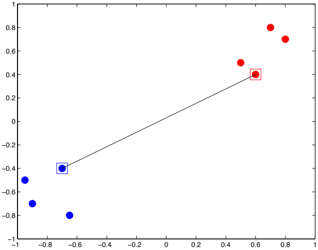


Illustrations

Multiagent systems :
The new position of each agent i is a mean of the position of agents, who are in the current neighborhood (depending on time k). Existence of a consensus $\lim_{k \to +\infty} x_k^{(i)} = x^*$?



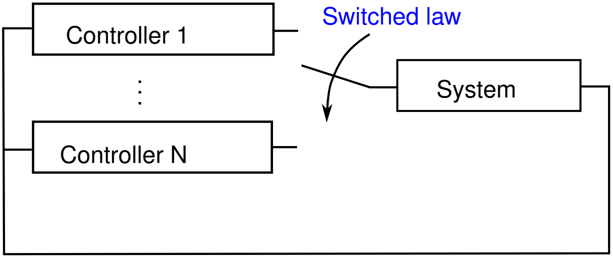
$$x_{k+1} = A_1 x_k$$



$$x_{k+1} = A_2 x_k$$

Illustrations

Switching controllers :

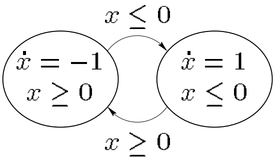


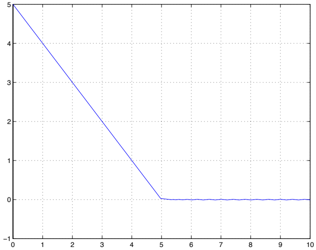
Illustrations

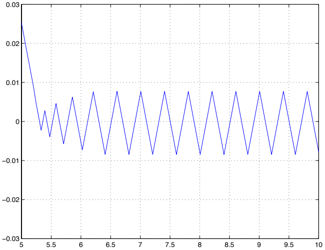
Sliding modes :

Let $x(t) \in \mathbb{R}$, with

$$\dot{x}(t) = -\mathrm{sign}(x(t)) = \begin{cases} -1 \text{ if } x(t) > 0, \\ +1 \text{ if } x(t) < 0, \\ \text{undefined if } x(t)=0. \end{cases}$$



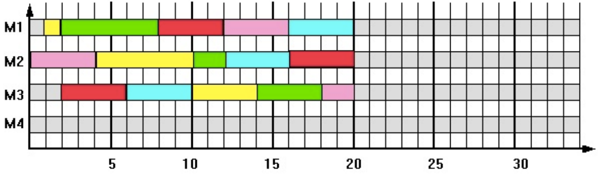




Difficulties : Well-posed solution ? Possible presence of Zeno phenomenon.

Typical examples of embedded systems

Last example : task scheduling



Framework of the talk

Consider discrete-time switched systems :

- Avoid well-posedness of solutions (different kinds of solutions : Filipov solution etc),
- Avoid Zeno phenomenon,
- Simplicity and richness of this class of systems.

Assume also for this talk :

- The modes are time invariant,
- The modes are autonomous (or already in their closed-loop form).

To sum up, we consider in the following (with distinct assumptions on $\sigma(\cdot)$) :

$$x_{k+1} = A_{\sigma(k)} x_k.$$

Outline of the tutorial

Definitions relative to stability

The definitions are relative to an equilibrium point. Here we assume that the equilibrium point is the origin $x^* = 0$. In addition, the following definitions are valid for linear switched systems, for which there does not exist finite time escape.

Global asymptotic stability (GAS) : ensure that, for a given $\sigma(\cdot)$:

$$\lim_{k \mapsto +\infty} x_k = 0, \quad \forall x_0 \in \mathbb{R}^n. \tag{1}$$

Global uniform asymptotic stability (GUAS) : ensure that

$$\lim_{k \mapsto +\infty} x_k = 0, \quad \forall x_0 \in \mathbb{R}^n, \quad \forall \sigma : \mathbb{N} \mapsto \mathcal{I}_N. \tag{2}$$

The term uniform means uniformly in $\sigma(\cdot)$.

Geometric approach

We recall stability results for the time invariant discrete-time linear system :

$$x_{k+1} = Ax_k, \quad \forall k \in \mathbb{N}. \tag{3}$$

The solution is given by

$$x_k = A^k x_0, \quad \forall k \in \mathbb{N}.$$

Theorem

The system (3) is GAS if and only if

$$\rho(A) = \max_{i \in \mathcal{I}_n} \|\lambda_i(A)\| < 1. \tag{4}$$

Lyapunov function approach

Theorem

Consider the system $x_{k+1} = Ax_k$ *and* $V : \mathbb{R}^n \to \mathbb{R}$*, such that*

- $V(x) \to +\infty$ *as* $\|x\| \to +\infty$*. (radially unbounded).*
- $V(0) = 0$ *and* $V(x) > 0$ *if* $x \neq 0$*. (positive definite).*
- $V(Ax) - V(x) < 0, \forall x \neq 0$*. (decreasing)*

Then the origin $x^* = 0$ *is GAS.*

The function V is called a Lyapunov function and is an extended energy of the system, which should decrease to zero along all trajectories.

Converse theorem

Theorem

If the origin $x^* = 0$ *is GAS for the system* $x_{k+1} = Ax_k$*, then there exists a Lyapunov function* $V(\cdot)$*.*

In such a case, the difficulty is to obtain the expression of the Lyapunov function $V(\cdot)$.

Stability for linear systems with Lyapunov functions

Theorem : the following statements are equivalent :

1. The linear system $x_{k+1} = Ax_k$ is GAS.
2. There is a quadratic Lyapunov function

   $$V(x) = x^T Px, \tag{5}$$

   where P is a positive definite matrix $P > 0_n$ such that the following Lyapunov inequality (Linear Matrix Inequality LMI) is satisfied :

   $$A^T PA - P < 0. \tag{6}$$

3. There is a quadratic Lyapunov function

   $$V(x) = x^T Px, \tag{7}$$

   where P is the positive definite matrix $P > 0_n$ associated with any $Q > 0$ such that the following Lyapunov equation is satisfied.

   $$A^T PA - P = -Q. \tag{8}$$

Sketch of proof

3) $\Rightarrow$ 2) . Trivial

$$A^T PA - P = -Q < 0.$$

2) $\Rightarrow$ 1) If the inequality $A^T PA - P < 0$ has a positive definite solution $P > 0_n$, then there exists sufficient small $1 > \epsilon > 0$ such that

$$A^T PA - P < -\epsilon P < 0.$$

Then, by considering $V(x) = x^T Px$, and $x_k \neq 0$,

$$V(x_{k+1}) - V(x_k) = x_k^T (A^T PA - P)x_k < -\epsilon x_k^T Px_k < 0,$$

which implies, with $\lambda_{\min}(P)\|x\|^2 \leq x^T Px \leq \lambda_{\max}(P)\|x\|^2$, that

$$x_k^T Px_k \leq (1-\epsilon)^k V(x_0); \quad \|x_k\|^2 \leq \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\|x_0\|^2(1-\epsilon)^k.$$

1) $\Rightarrow$ 3) If the system $x_{k+1} = Ax_k$ is GAS, then the Grammian associated with the pair (Q, A), with any $Q > 0$ is well-defined (the sum converges).

$$\sum_{k \in \mathbb{N}} \left(A^T\right)^k QA^k,$$

and is a solution of the Lyapunov equation. To end the proof, we have only to prove that $P > 0$.

Distinct frameworks

Discrete-time switched linear system :

$$x_{k+1} = A_{\sigma(k)} x_k, \qquad (9)$$

with

- $x_k \in \mathbb{R}^n$ the state,
- A_i invertible, $i \in \mathbb{N}_q = \{1; \cdots, q\}$,
- $\sigma : \mathbb{N} \to \mathbb{N}_q$ is the switching law.

Several frameworks depending on σ :

σ is a perturbation : stability analysis and robustness

$A_1 = \begin{pmatrix} 0.5 & 5 \\ 0 & 0.5 \end{pmatrix}$, and $A_2 = \begin{pmatrix} 0.5 & 0 \\ 10 & 0.5 \end{pmatrix}$ are Schur, but
$A_1 A_2 = \begin{pmatrix} 50.25 & 2.5 \\ 5 & 0.25 \end{pmatrix}$ is not Schur ($\lambda(A_1 A_2) = \{50.49; 0.012\}$).

- Sufficient conditions for asymptotic stability. Switched Lyapunov function (*Daafouz et al. TAC 2002*) ;
- Necessary and sufficient condition for asymptotic stability. Existence of a polyhedral Lyapunov function (*Molchanov & Pyatnitskiy SCL 1989 ; Blanchini AUT 1995*) ; Joint Spectral Radius (*R.M. Jungers, Springer, 2009*).

Distinct frameworks

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Several frameworks depending on σ :

σ is a controlled input : stabilizability and stabilization.

$A_1 = \begin{pmatrix} 1.2 & 5 \\ 0 & 0.5 \end{pmatrix}$, and $A_2 = \begin{pmatrix} 0.5 & 0 \\ 0 & 1.4 \end{pmatrix}$ are not Schur, but $A_1 A_2 = \begin{pmatrix} 0.6 & 7 \\ 0 & 0.7 \end{pmatrix}$ is Schur.

- Sufficient conditions for stabilizability. Lyapunov–Metzler inequalities (*Geromel & Colaneri IJC 2006*) ;
- Necessary and sufficient condition for stabilizability. Geometric approach (*Fiacchini & Jungers, Automatica 2014*) and comparison with other approaches (*Fiacchini, Girard, Jungers, TAC 2016*).

Distinct frameworks

Discrete-time switched linear system :

$$x_{k+1} = A_{\sigma(k)} x_k, \qquad (9)$$

with

- $x_k \in \mathbb{R}^n$ the state,
- A_i invertible, $i \in \mathbb{N}_q = \{1; \cdots, q\}$,
- $\sigma : \mathbb{N} \to \mathbb{N}_q$ is the switching law.

Several frameworks depending on σ :

σ is a controlled input with constraints.

$A_1 = \begin{pmatrix} 1.2 & 5 \\ 0 & 0.5 \end{pmatrix}$, and $A_2 = \begin{pmatrix} 0.5 & 0 \\ 0 & 1.4 \end{pmatrix}$ are not Schur, but $A_1 A_2 A_2 = \begin{pmatrix} 0.3 & 9.8 \\ 0 & 0.98 \end{pmatrix}$ is Schur, with the constraint : mode 2 appears twice when occuring.

- Large literature for specific classes of switching law.
- Language constrained switching law : CNS by geometrical approach [*Fiacchini, Jungers, Girard, ECC 2016*].
- Language constrained switching law : CS by Lyapunov–Metlzer and LMIs approach [*Jungers, Girard, Fiacchini, CDC 2016, ADHS 2018*].

Outline of the talk

What is a switched system ?

About stability

Sufficient conditions for stability

- The joint spectral radius
- The common Lyapunov function approach
- Multiple Lyapunov functions

Stabilization without constraints

Stabilization constrained by a language

Geometric approach : the joint spectral radius

The joint spectral radius of a set of matrices $\mathcal{A} = \{A_1, \cdots, A_N\}$, denoted $\rho(\mathcal{A})$ is an extension of the radius of a matrix A (i.e. $\rho(A)$) and gives a necessary and sufficient condition for the stability of the system (15) and solves P1. See [*Theys 2005*].

We define

$$\rho(\mathcal{A}) = \limsup_{p \to +\infty} \rho_p(\mathcal{A}),$$

where

$$\rho_p(\mathcal{A}) = \sup_{A_{i_1}, A_{i_2}, \cdots, A_{i_p} \in \mathcal{A}} \left\| A_{i_1} A_{i_2} \times \cdots \times A_{i_p} \right\|^{\frac{1}{p}}.$$

Theorem

The switched system (15) is GAS if and only if

$$\rho(\mathcal{A}) < 1. \quad (10)$$

Main difficulty : this is difficult in the generic case to practically compute the joint spectral radius. Several approximations are provided in the literature.

The common Lyapunov function approach

Theorem

If all the modes share a common Lyapunov function, then the switched system is GUAS.

Theorem

If the switched system is GUAS, then all the modes share a common Lyapunov function.

Remark : be careful, there is no assumption concerning the class of the Lyapunov function. Especially, this Lyapunov function is not necessary on the form $V(x) = x^T Px$ as it will be seen in the following. This existence result does not help roughly speaking about how to find this Lyapunov function. In addition, there exists a common Lyapunov function on the form $V(x) = x^T P(x)x$, where $P(\lambda x) = P(x), \forall \lambda \neq 0$ (homogeneous of degree zero).

The common Lyapunov function approach : sufficient conditions

The previous theorem suggests to look for a common quadratic Lyapunov function in the class $V(x) = x^T Px$.

Theorem

Consider the discrete-time linear switched system (15). If there exists a matrix $P \in \mathbb{R}^{n \times n}$ such that

$$P > 0_n \tag{11}$$

and

$$A_i^T PA_i - P < 0, \qquad \forall i \in \mathcal{I}, \tag{12}$$

then the system (15) admits the common quadratic Lyapunov function $V(x)$ and is GUAS.

Remark : the system (15) may be GUAS without feasible LMI (12).

The common Lyapunov function approach : unfeasibility test

To complete the previous remark, we have the following theorem.

Theorem

If there exist positive definite matrices $R_i \in \mathbb{R}^{n\times n}$, $R_i > 0_n$ such that

$$\sum_{i\in\mathcal{I}} A_i R_i A_i^T - R_i > 0_n, \tag{13}$$

then there does not exist $P > 0_n$ such that

$$A_i^T P A_i - P < 0, \qquad \forall i \in \mathcal{I}, \tag{14}$$

Proof : If there exist R_i $(\in \mathcal{I})$ such that Inequalities (13) hold, then for every $P > 0_0$,

$$0 < \mathrm{Tr}\left[P\left(\sum_{i\in\mathcal{I}} A_i R_i A_i^T - R_i\right)\right] = \mathrm{Tr}\left[R_i\left(A_i^T P A_i - P\right)\right],$$

then there exists $i_0 \in \mathcal{I}$ such that $A_{i_0}^T P A_{i_0} - P > 0$.

Multiple Lyapunov functions

Definition : We consider functions of the form

$$V(\sigma(k), x_k) = V_{\sigma(k)}(x_k) = x_k^T P(\sigma(k), x_k) x_k. \qquad (15)$$

Theorem [*Daafouz, Riedinger, Iung, TAC 2002*]

If there exist P_i, $i \in \mathcal{I}_N$ such that $P_i > 0$ and

$$A_i^T P_j A_i - P_i < 0, \qquad \forall (i,j) \in \mathcal{I}_N^2, \qquad (16)$$

then the discrete-time switched system (15) is GUAS.

Sketch of proof : By chosing $i = \sigma(k)$ and $j = \sigma(k+1)$, we have $V_{\sigma(k+1)}(x_{k+1}) - V_{\sigma(k)}(x_k) < 0$, $\forall x_k \neq 0$.

Outline of the talk

What is a switched system ?

About stability

Sufficient conditions for stability

Stabilization without constraints

- Lyapunov–Metzler inequalities approach
- Geometric approach
- LMI approach
- Periodic stabilizability

Stabilization constrained by a language

Stabilization of linear discrete-time switched systems

The problem P3 is to design a switching law that stabilizes the system (15).

Assumption : A_i ($\forall i \in \mathcal{I}$) are not Schur.

This assumption is to avoid a trivial solution : if there exists i_0 such that A_{i_0} is Schur, then $\sigma(k) = i_0$ globally asymptotically stabilizes the system.

Min-switching strategy : Lyapunov-Metzler inequalities approach

Idea : « min-switching approach » (*Wicks & DeCarlo ACC 1997*, *Liberzon 2003*).

Set of Metzler matrices in discrete-time domain (stochastic matrices) :

The matrix $\Pi \in \mathcal{M}_{\mathrm{d}}$, where $\mathcal{M}_{\mathrm{d}}$ is the particular Metzler matrices set :

$$\mathcal{M}_{\mathrm{d}} = \{\Pi \in \mathbb{R}^{N\times N},\ \pi_{ij} \geq 0,\ \textstyle\sum_{\ell\in\mathcal{I}_N}\pi_{\ell i} = 1, \forall (i,j) \in \mathcal{I}_N^2\}.$$

Theorem *[Geromel & Colaneri, IJC 2006]*

If the Lyapunov-Metzler inequalities

$$\sum_{j\in\mathcal{I}_N} \pi_{ji} A_i' P_j A_i - P_i < 0_n, \quad \forall i \in \mathcal{I}_N$$

with $P_i = P_i' > 0_n$, $\Pi \in \mathcal{M}_{\mathrm{d}}$ hold then the switched system (9) is stabilized by the switching law

$$\sigma(k) = g(x_k) \in \arg\min_{i\in\mathcal{I}_N} x_k' P_i x_k. \qquad (17)$$

$$V_{\min}(x) = \min_{i\in\mathcal{I}_N} x' P_i x; \quad \Rightarrow \quad V_{\min}(x_{k+1}) \leq \sum_{j\in\mathcal{I}_N} \pi_{ji} x_k' A_{\sigma(k)}' P_j A_{\sigma(k)} x_k < V_{\min}(x_k).$$

Sketch of proof

Lyapunov function considered

$$V_{\min} : \begin{cases} \mathbb{R}^n & \to & \mathbb{R}, \\ x_k & \mapsto & \min_{i \in \mathcal{I}} x_k^T P_i x_k, \end{cases} \tag{18}$$

Notation : $(P)_{p,i} = \sum_{\ell \in \mathcal{I}} \pi_{\ell i} P_\ell$.

Elements of proof

- By post-multiplying by $x_k \neq 0$ and pre-multiplying by x_k',

$$x_{k+1}'(P)_{p,i} x_{k+1} - x_k' P_i x_k < 0 \tag{19}$$

- the minimum scalar value of convex polytopes is reached on one of the vertices

$$V_{\min}(x_{k+1}) = \min_{j \in \mathcal{I}} x_{k+1}' P_j x_{k+1} = \min_{\substack{\sum_{j \in \mathcal{I}} \lambda_j = 1 \\ \lambda_j \in \mathbb{R}^+;}} \sum_{j \in \mathcal{I}} \lambda_j x_{k+1}' P_j x_{k+1}. \tag{20}$$

  Each column of the Metzler matrix $\Pi \in \mathcal{M}$ is in the unit simplex, then

$$V_{\min}(x_{k+1}) \leq x_{k+1}'(P)_{p,i} x_{k+1}. \tag{21}$$

$\Rightarrow$ global asymptotic stability holds with

$$V_{\min}(x_{k+1}) - V_{\min}(x_k) < 0, \quad \forall x_k \neq 0. \tag{22}$$

Example

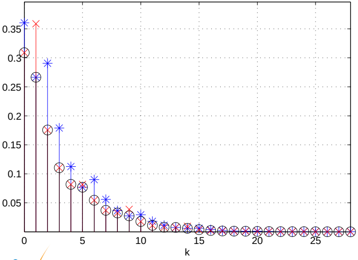
Consider

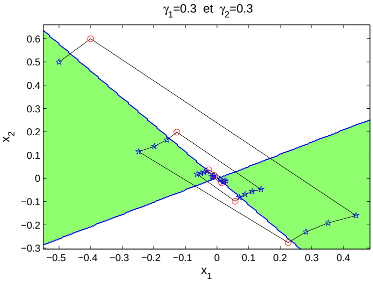
$$A_1 = \begin{bmatrix} -1.1 & 0 \\ 1 & 0.4 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 1.3 \end{bmatrix}, \; x_0 = \begin{pmatrix} -0.5 \\ 0.5 \end{pmatrix}$$

A solution to the Lyapunov–Metzler inequalities is given by

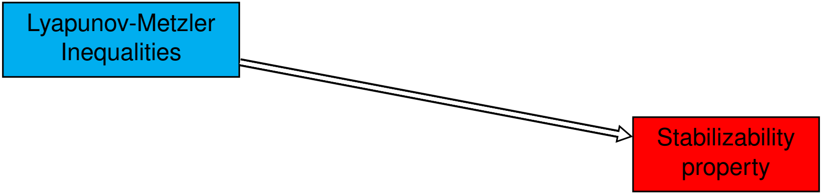
$$\Pi = \begin{bmatrix} 0.3 & 0.7 \\ 0.7 & 0.3 \end{bmatrix}, P_1 = \begin{bmatrix} 1.7097 & 0.3734 \\ 0.3734 & 0.4786 \end{bmatrix}, \; P_2 = \begin{bmatrix} 1.1978 & 0.6398 \\ 0.6398 & 1.3173 \end{bmatrix}.$$

$$A_1'(\pi_{11}P_1 + \pi_{21}P_2)A_1 - P_1 < 0_2$$
$$A_2'(\pi_{12}P_1 + \pi_{22}P_2)A_2 - P_2 < 0_2.$$





Sum up


Lyapunov-Metzler Inequalities
Stabilizability property

Geometric tools

Definition

A C-set is a compact, convex set containing the origin in its interior.

A set $\Omega \subseteq \mathbb{R}^n$ is a C^*-set if it is compact, star-convex with respect to the origin and $0 \in \mathrm{int}(\Omega)$.

Notice a set is

- convex if $\forall x_0 \in \Omega$ and $\forall x \in \Omega$, then $\alpha x_0 + (1-\alpha)x \in \Omega,\ \forall \alpha \in [0,1]$.
- star-convex if $\exists x_0 \in \Omega$, such that $\forall x \in \Omega$, then $\alpha x_0 + (1-\alpha)x \in \Omega,\ \forall \alpha \in [0,1]$.

Definition

Minkowski function of a C^*-set Ω : $\Psi_\Omega(x) = \min_\alpha \{\alpha \in \mathbb{R} : x \in \alpha\Omega\}$.

- Any C-set is a C^*-set.
- Given a C^*-set Ω, we have that $\alpha\Omega$ is a C^*-set and $\alpha\Omega \subseteq \Omega$ for all $\alpha \in [0,1]$.
- $\Psi_\Omega(\cdot)$ is : defined on $\mathbb{R}^n$; homogenous of degree one ; positive definite and radially unbounded. But nonconvex in general !

Geometric approach

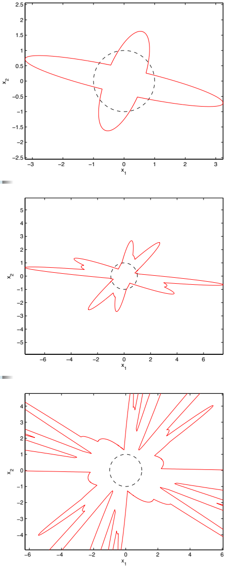
Algorithm

Control λ-contractive C-set for the switched system (15).

- **Initialization** : given the C^*-set $\Omega \subseteq \mathbb{R}^n$, define $\Omega_0 = \Omega$ and $k = 0$;
- **Iteration** for $k \geq 0$:

$$\begin{aligned} \Omega^i_{k+1} &= A_i^{-1}\Omega_k, \quad \forall i \in \mathcal{I}_N, \\ \Omega_{k+1} &= \bigcup_{i \in \mathcal{I}_N} \Omega^i_{k+1}; \end{aligned}$$

- **Stop** if $\Omega \subseteq \operatorname{int}\left(\bigcup_{j \in \mathbb{N}_{k+1}} \Omega_j\right)$; denote $\check{N} = k + 1$ and $\check{\Omega} = \bigcup_{j \in \{1;\cdots;\check{N}\}} \Omega_j$.



Geometric approach

Geometrical interpretation :

- the set Ω_k^i is the set of x that can be stirred in Ω in k steps by a switching sequence beginning with $i \in \mathcal{I}_N$;
- then Ω_k is the set of points that can be driven in Ω in k steps ;
- and hence $\check{\Omega}$ the set of those which can reach Ω in $\check{N}$ or less steps, by an adequate switching law.

Necessary and sufficient condition for stabilizability.

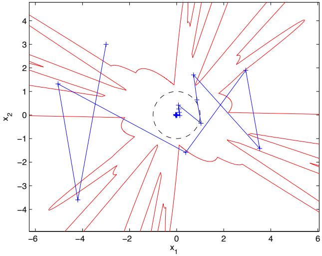
Theorem [*Fiacchini, Jungers, Automatica 2014*]

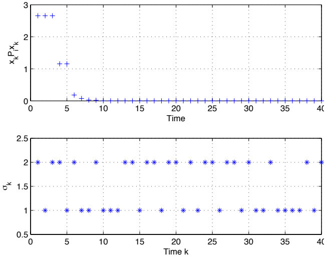
There exists a control Lyapunov function for the switched system if and only if the Algorithm 1 ends with finite $\check{N}$.

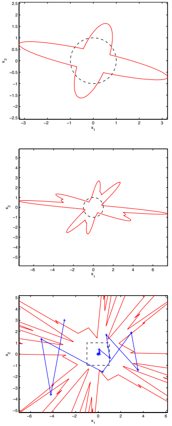
Example 1

Non-Schur switched system with $N = n = 2$.

$$A_1 = \begin{bmatrix} 1.2 & 0 \\ -1 & 0.8 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.6 & -2 \\ 0 & -1.2 \end{bmatrix},$$





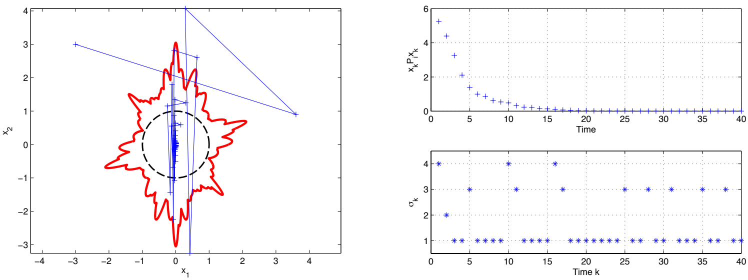


Example 2

System with $N = 4$, $n = 2$ and

$$A_1 = \begin{bmatrix} 1.5 & 0 \\ 0 & -0.8 \end{bmatrix}, \qquad A_2 = 1.1\, R(\tfrac{2\pi}{5})$$
$$A_3 = 1.05\, R(\tfrac{2\pi}{5} - 1), \qquad A_4 = \begin{bmatrix} -1.2 & 0 \\ 1 & 1.3 \end{bmatrix}.$$

The matrices A_i, with $i \in \mathbb{N}_4$, are not Schur. Notice : only one stable eigenvalue !



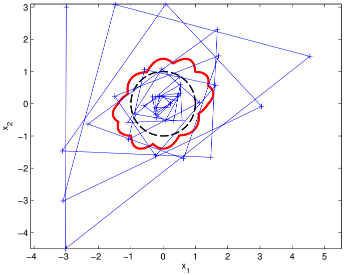
Example 3

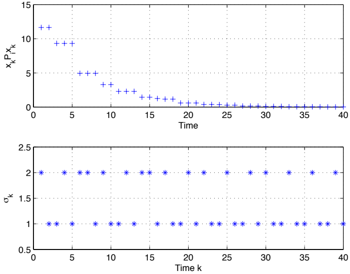
Switched system with

$$A_1 = \begin{bmatrix} 0 & -1.01 \\ 1 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & -1.01 \\ 1 & -0.5 \end{bmatrix}.$$

The technique based on Lyapunov-Metzler inequalities has been numerically checked (gridding) and it results not feasible.

Nevertheless...



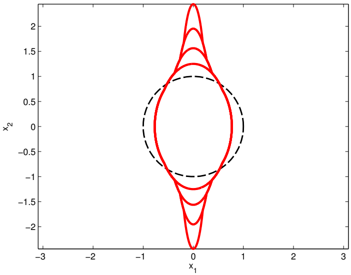


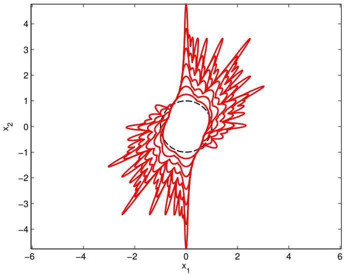
Example 4

Switched system with

$$A_1 = \begin{bmatrix} 1.3 & 0 \\ 0 & 0.9 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1.4 & 0 \\ 0 & 0.8 \end{bmatrix},$$

for $\theta = 0$ (left) and $\theta = \frac{\pi}{5}$ (right).





Sum up

Lyapunov-Metzler Inequalities
Geometric condition
Stabilizability property

CS for stabilization : Idea of LMI approach

Idea : based on particular sufficient conditions to ensure stabilizability.

- $P_i = I$;
- Consider a fictitious switched system with augmented modes as follows.

Notation :

- $\mathcal{I} = \mathcal{I}_N$: finite set of switching modes.
- $\mathcal{I}^{[M:K]} = \bigcup_{k=M}^{K} \mathcal{I}^k$: all the possible sequences of modes of length from M to K.
- $\bar{K} = \sum_{k=1}^{K} q^k$: given $K \in \mathbb{N}$, number of elements $i \in \mathcal{I}^{[1:K]}$.
- Given $i = (i_1, \ldots, i_k) \in \mathcal{I}^{[1:K]}$, $\mathbb{A}_i = \prod_{j=1}^{k} A_{i_j} = A_{i_k} \cdots A_{i_1}$.

Property

The stabilizability of the system $x_{k+1} = A_{\sigma(k)} x_k$ with $\{A_i\}_{i \in \mathcal{I}}$ is equivalent to the one of the system

$$z_{k+1} = \mathbb{A}_{\gamma(k)} z_k \text{ with } \gamma : \mathbb{N} \to \mathcal{I}^{[1:K]}.$$

CS for stabilization : LMI approach

Theorem *[Fiacchini, Girard, Jungers, TAC 2016]*

The switched system (9) is stabilizable if there exist $K \in \mathbb{N}$ and $\eta \in \mathbb{R}^{\bar{K}}$ such that $\eta \geq 0$, $\sum_{i \in \mathcal{I}^{[1:K]}} \eta_i = 1$ and

$$\sum_{i \in \mathcal{I}^{[1:K]}} \eta_i \mathbb{A}_i^T \mathbb{A}_i < I. \tag{23}$$

Comments :

- The condition (23) ensures the exponential stabilization of a switched system (9) ;
- Nevertheless neither the Euclidean norm $x^T x$ nor $\min_{i \in \mathcal{I}^{[1:K]}} x^T \mathbb{A}_i^T \mathbb{A}_i x$ are Control Lyapunov Functions ;
- The condition (23) is equivalent to the periodically stabilizability.
- The condition is just sufficient (except for particular cases), is it also necessary ? No !

CS for stabilization : LMI approach

Idea : modify the later inequality to provide a control Lyapunov function :

Theorem *[Fiacchini, Girard, Jungers, TAC 2016]*

If there exists $\mu \in [0,1)$ and $\eta \in \mathbb{R}^{\bar{K}}$ such that $\eta \geq 0$, $\sum_{i \in \mathcal{I}^{[1:K]}} \eta_i = 1$ such that

$$\sum_{i \in \mathcal{I}^{[1:K]}} \eta_i \mathbb{A}_i^T \mathbb{A}_i \leq \mu^2 I. \quad (24)$$

The switching law given by

$$\sigma_{k_p+j-1} = i_{p,j}, \quad \forall j \in \{1, \ldots, l(i_p)\}. \quad (25)$$

where $\{k_p\}_{p \in \mathbb{N}}$ with $k_0 = 0$, and $k_p < k_{p+1} \leq k_p + K$, for all $p \in \mathbb{N}$ and

$$i_p = \arg \min_{i \in \mathcal{I}^{[1:K]}} (x_{k_p}^T \lambda^{-l(i)} \mathbb{A}_i^T \mathbb{A}_i x_{k_p}), \qquad k_{p+1} = k_p + l(i_p), \qquad \lambda = \mu^{2/K}, \quad (26)$$

with $l(i_p)$ length of i_p, globally exponentially stabilizes the system and $V(x) = min_{i \in \mathcal{I}^{[1:K]}} (x^T \lambda^{-l(i)} \mathbb{A}_i^T \mathbb{A}_i x)$ is a Control Lyapunov function.

Periodic stabilizability

A periodic switching law is given by $\sigma(k+K) = \sigma(k)$.

The stabilizability through periodic switching law, i.e. periodic stabilizability, is formalized below.

Definition

The switched system is periodic stabilizable if there exist a periodic switching law $\sigma : \mathbb{N} \to \mathcal{I}$, such that the system is stabilizable for all $x \in \mathbb{R}^n$.

Notice that for stabilizability the switching function might be state-dependent, hence a state feedback, whereas for having periodic stabilizability the switching law must be independent on the state.

Is there an equivalence relation between periodic stabilizability and the LMI condition ? The answer is below.

Theorem

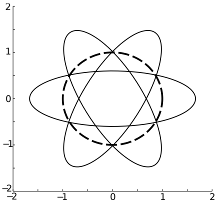
A stabilizing periodic switching law for the switched system exists if and only if the LMI condition holds.

Counterexample

Consider the three modes given by the matrices

$$A_1 = AR(0), \quad A_2 = AR\left(\frac{2\pi}{3}\right), \quad A_3 = AR\left(\frac{-2\pi}{3}\right), \quad \text{with} \quad A = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$$

and $a = 0.6$. The geometric condition holds with $N = 1$.

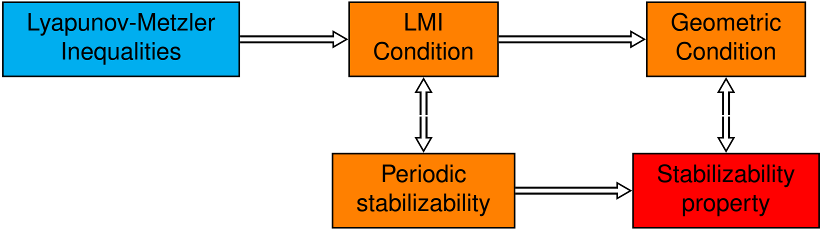


For every N and every $\mathbb{B}_i$ with $i \in \mathcal{I}$, the related $\mathbb{A}_i$ is such that $\det(\mathbb{A}_i^T \mathbb{A}_i) = 1$ and $\mathrm{Tr}(\mathbb{A}_i^T \mathbb{A}_i) \geq 2$.

Notice that, for all the matrices $Q > 0$ in $\mathbb{R}^{2\times 2}$ such that $\det(Q) = 1$, then $\mathrm{Tr}(Q) \geq 2$ and $\mathrm{Tr}(Q) = 2$ if and only if $Q = I$.

Thus, for every subset $K \subseteq \mathcal{I}$, we have that $\sum_{i \in K} \eta_i \mathbb{A}_i^T \mathbb{A}_i < I$, cannot hold, since either $\mathrm{Tr}(\mathbb{A}_i^T \mathbb{A}_i) > 2$ or $\mathbb{A}_i^T \mathbb{A}_i = I$.

Sum up


Lyapunov-Metzler Inequalities
LMI Condition
Geometric Condition
Periodic stabilizability
Stabilizability property

Outline of the talk

What is a switched system ?

About stability

Sufficient conditions for stability

Stabilization without constraints

Stabilization constrained by a language

- Language constrained switching law
- Geometric Necessary and Sufficient Condition for Recursive ECLF
- Lyapunov-Metzler inequalities approach

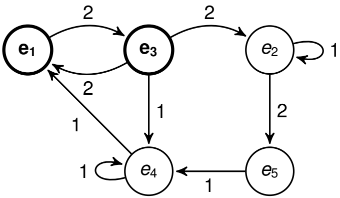
Language constrained switching law

Constraints : dwell time, modal constraints, ... σ belongs to a language specified by a nondeterministic finite automaton.

Definition

A nondeterministic finite automaton is a tuple $\mathcal{A} = (\mathcal{S}, \Sigma, \delta, \mathcal{S}_0)$ where :

- $\mathcal{S}$ is a finite set of p states : $\mathcal{S} = \{e_i\}_{i \in \mathbb{N}_p}$;
- $\Sigma = \mathbb{N}_N$ is a finite alphabet (active mode) ;
- $\delta : \mathcal{S} \times \Sigma \to 2^{\mathcal{S}}$ is a set-valued transition map ;
- $\mathcal{S}_0 \subseteq \mathcal{S}$ is a subset of initial states.

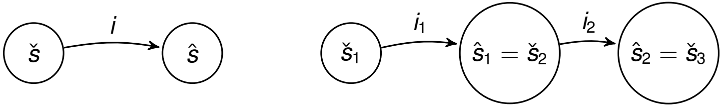


Notation : $\sigma : \mathbb{N} \to \Sigma$ belongs to the language of $\mathcal{A}$, i.e. $\sigma \in L(\mathcal{A})$, if there exists $s^\sigma : \mathbb{N} \to \mathcal{S}$ such that $s^\sigma_0 \in \mathcal{S}_0$ and $s^\sigma_{k+1} \in \delta(s^\sigma_k, \sigma_k)$ for all $k \in \mathbb{N}$.

Additional tools

Definition

A transition is a triplet $\tau = (\check{s}, i, \hat{s}) \in \mathcal{S} \times \Sigma \times \mathcal{S}$ such that $\hat{s} \in \delta(\check{s}, i)$. A path of $\mathcal{A}$ is a sequence of transitions $p = \{(\check{s}_1, i_1, \hat{s}_1), (\check{s}_2, i_2, \hat{s}_2), \cdots\}$ such that $\check{s}_{k+1} = \hat{s}_k$. $\mathcal{P}_m(\mathcal{A})$ and $\mathcal{P}_\infty(\mathcal{A})$ are the sets of m-length and infinite paths.



Definition

For a path $p = \{(\check{s}_1, i_1, \hat{s}_1), \cdots, (\check{s}_m, i_m, \hat{s}_m)\} \in \mathcal{P}_m(\mathcal{A})$,

- $w(p) = (i_1, i_2, \cdots, i_m) \in \Sigma^m$ is a word, admissible of the language $L(\mathcal{A})$, and we denote $w_j(p) = i_j$ for $j \in \mathbb{N}_m$ $(\mathbb{A}_{w(p)} = A_{i_m} A_{i_{m-1}} \cdots A_{i_1})$;
- $\pi(p) = (\check{s}_1, \cdots, \check{s}_m, \hat{s}_m) \in \mathcal{S}^{m+1}$ is the projection of the path over the set of automaton states, and we denote $\pi_j(p) = \check{s}_j$ with $j \in \mathbb{N}_m$ and $\pi_{m+1}(p) = \hat{s}_m$.
- For two paths $p_1 \in \mathcal{P}_{m_1}(\mathcal{A})$ and $p_2 \in \mathcal{P}_{m_2}(\mathcal{A})$, if these paths are compatible, $(\pi_{l(p_1)+1}(p_1) = \pi_1(p_2))$, then we can define the concatenation of these paths denoted $p_1 \circ p_2 \in \mathcal{P}_{m_1+m_2}(\mathcal{A})$.

Examples of language constraints (i).

Automaton 1 : $p = 1$, $\mathcal{A} = (\{e_1\}, \mathbb{N}_N, \delta, \{e_1\})$, where $e_1 \in \delta(e_1, i)$, $\forall i \in \mathbb{N}_N$.

Automaton 2 : $p = N$, $\mathcal{A} = (\{e_1, \cdots, e_N\}, \mathbb{N}_N, \delta, \{e_1, \cdots, e_N\})$, where $\{e_i\} = \delta(e_j, i)$, $\forall (i, j) \in \mathbb{N}_N^2$.

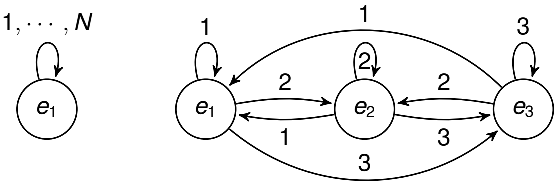


FIGURE 1 – Automatons 1 with $p = 1$ (left) and 2 with $p = N = 3$ (right).

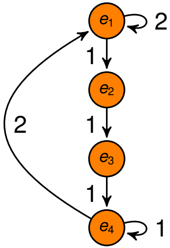
Examples of language constraints (ii).

Definition

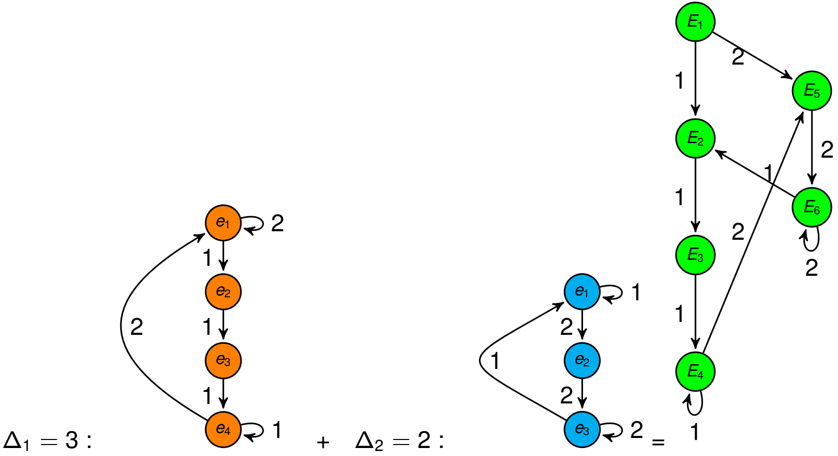
For an integer $\Delta \in \mathbb{N}^*$, the set of the switching laws satisfying a dwell time at least equal to Δ is defined by

$$\mathcal{D}_\Delta = \{\sigma : \mathbb{N} \to \mathcal{I};\ \exists \{\ell_q\}_{q\in\mathbb{N}}, \ell_{q+1} - \ell_q \geq \Delta; \\ \sigma(k) = \sigma(\ell_q), \forall \ell_q \leq k < \ell_{q+1}; \sigma(\ell_q) \neq \sigma(\ell_{q+1})\}.$$

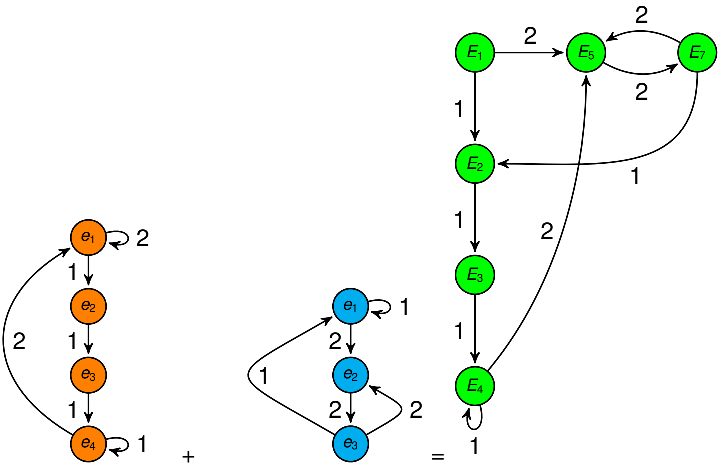
$\Delta_1 = 3; \Delta_2 = 1$



Language constraints (iii) : intersections.



Language constraints (iv) : intersections.



Problem formulation

Definition

The system is globally exponentially stabilizable relatively to language $L(\mathcal{A})$ if there are $c \geq 0$ and $\lambda \in [0, 1)$ and, for all $x \in \mathbb{R}^n$, there exists $\sigma \in L(\mathcal{A})$, such that : $\|x_k^\sigma(x)\| \leq c\lambda^k\|x\|$.

Problem :

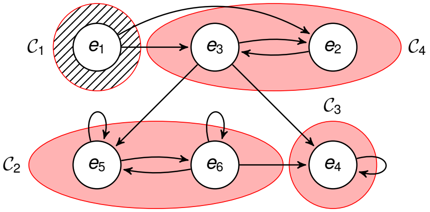
Let us consider the system (9) and the automaton $\mathcal{A}$ defining the language constraints for the switching laws. Determine a path $p \in \mathcal{P}_\infty(\mathcal{A})$ that generates switching laws $w(p) = \sigma \in L(\mathcal{A})$ verifying the language constraints and that globally exponentially stabilizes the closed-loop system (9). p is assumed to depend on the states of the system and automaton.

Directed graph properties

Based on $\mathcal{A}$: directed graph (digraph) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$: with vertices $\mathcal{V} = \mathcal{S}$ and edges $\mathcal{E} = \{(s, r) \in \mathcal{S}^2,\ \exists \ell \in \Sigma,\ r \in \delta(s, \ell)\}$.

Definition

Let $(s, r) \in \mathcal{V}^2$. s and r are strongly connected if $s = r$ or if there exist a directed path from s to r and a directed path from r to s. $\to$ equivalence relation on the nodes.



Definition

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a finite digraph and $\mathcal{C} \subset \mathcal{V}$. $\mathcal{C}$ is strongly connected if for every pair of vertices $(s, r) \in \mathcal{C}^2$, s and r are strongly connected. A strongly connected component (SCC) of the digraph $\mathcal{G}$ is a maximally strongly connected set of vertices. This is an equivalence class for the relation of strongly connectivity. A SCC $\mathcal{C}$ is called trivial if $\mathcal{C} = \{s\}$ and $(s, s) \notin \mathcal{E}$.

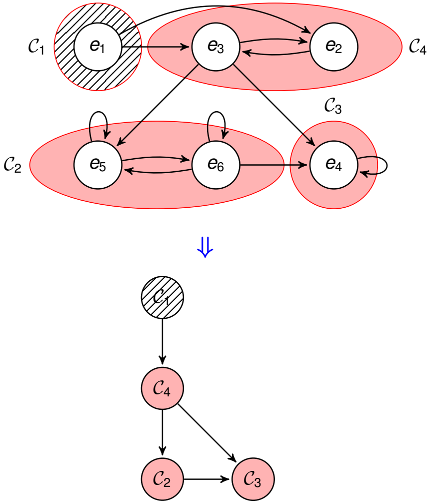
Condensation of the digraph $\mathcal{G}$

Definition

The condensation of $\mathcal{G}$ is a digraph $\mathcal{G}^{\text{SCC}} = (\mathcal{V}^{\text{SCC}}, \mathcal{E}^{\text{SCC}})$. $\mathcal{V}^{\text{SCC}}$ one vertex per SCC. $\mathcal{E}^{\text{SCC}}$ links between the SCCs that is a partial order relation between the SCCs as $\mathcal{C}_i \succeq \mathcal{C}_j$ if there exists a path between one vertex in $\mathcal{C}_i$ and a vertex in $\mathcal{C}_j$.

Proposition

Let $\mathcal{G}$ be the digraph associated with the automaton $\mathcal{A}$ and its condensation $\mathcal{G}^{SCC}$. Every trajectory of the constrained switched system has a projection on the automaton state that ultimately enters and does not exit a nontrivial SCC.



$\mathcal{C}_1 \succeq \mathcal{C}_4 \succeq \mathcal{C}_2 \succeq \mathcal{C}_3.$

Reformulation on a strongly connected component

Definition

Let $\mathcal{C}$ be a nontrivial SCC induced by the automaton $\mathcal{A}$. We define the set of finite paths of m transitions restricted to $\mathcal{C}$, $\mathcal{P}_m(\mathcal{A},\mathcal{C})$ such that

$$\mathcal{P}_m(\mathcal{A},\mathcal{C}) = \{p \in \mathcal{P}_m(\mathcal{A}),\ \pi_j(p) \in \mathcal{C},\ \forall j \in \mathbb{N}_{m+1}\}. \tag{27}$$

Definition

Let us consider a nontrivial SCC $\mathcal{C}$ of the digraph $\mathcal{G}$ induced by the automaton $\mathcal{A}$. A nonnegative continuous function $V : \mathbb{R}^n \times \mathcal{C} \to \mathbb{R}^+$ is an exponentially stabilizing control Lyapunov function (ECLF) of the system (9) in $\mathcal{C}$ if for any $(x,r) \in \mathbb{R}^n \times \mathcal{C}$, we have

1. $\kappa_1\|x\|^2 \leq V(x,r) \leq \kappa_2\|x\|^2$ for some finite positive constants κ_1 and κ_2 ;
2. There exists $p_\nu : \mathbb{R}^n \times \mathcal{C} \to \mathcal{P}_1(\mathcal{A},\mathcal{C})$, such that $\pi_1(p_\nu(x,r)) = r$, and $V(A_{w(p_\nu(x,r))}x, \pi_2(p_\nu(x,r))) - V(x,r) \leq -\kappa_3\|x\|^2$, for a constant $\kappa_3 > 0$.

Main results

Solution in two steps :

- Check if there exists at least one nontrivial SCC for which a ECLF exists ;
- Check if at least one initial condition can reach a SCC that admits a ECLF.
  If d is the number of SCCs, let be $\mathcal{Q} = \{i \in \mathbb{N}_d,\ \mathcal{C}_i \text{ admits an ECLF}\}$, then

$$\tilde{\mathcal{S}}^0 = \mathcal{S}^0 \bigcap \Big(\bigcup_{j \in \mathcal{Q}} \bigcup_{\substack{i \in \mathbb{N}_d, \\ \mathcal{C}_i \succeq \mathcal{C}_j}} \mathcal{C}_i \Big) \tag{28}$$

  is the set of initial automaton states that can be chosen to reach a SCC that admits an ECLF. If $\tilde{\mathcal{S}}^0 \neq \emptyset$, then Problem 1 admits a solution.

Extensions :

- Geometric approach : [*Fiacchini, Jungers, Girard, Automatica 2018*] ;
- Lyapunov–Metzler inequalities : [*Jungers, Girard, Fiacchini, CDC 2016*] ;
- LMI approach : [*Jungers, Girard, Fiacchini, ADHS 2018*] ;

Geometric Necessary and Sufficient Condition for Recursive ECLF

Definition

The automaton trajectory $r^\sigma : \mathbb{N} \to S$ is recurrent in $s \in S$ under the switching sequence $\sigma \in L(\mathcal{A})$ if there exist $N \in \mathbb{N}$ and a sequence $l_k : \mathbb{N} \to \mathbb{N}$ such that $l_1 = 0$ and $r^\sigma_{l_k} = s, \;\text{and}\; 1 \leq l_{k+1} - l_k \leq N, \qquad \forall k \in \mathbb{N}$.

Definition

The function $V : \mathbb{R}^n \times R \to \mathbb{R}^+$ is a recurrent ECLF in $R \subseteq S$ if it is an ECLF in R under a control policy ν such that $\nu \in L(\mathcal{A})$ and it generates trajectories recurrent in a state $s \in R$.

Objective : we will be searching for contractive C*-sets such that the related gauge functions are recurrent ECLF for the switched systems subject to the language constraints induced by $\mathcal{A}$.

Sets and algorithm

For every set $\Omega \subseteq \mathbb{R}^n$ and state $s \in S$, define :

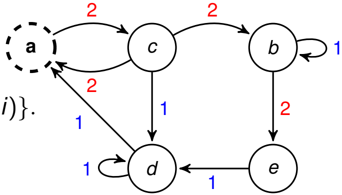
- the one-step operator for every mode $i \in \mathcal{I}$ as

$$Q_i^s(\Omega) = \{(x,\, r) \in \mathbb{R}^n \times S : \; A_i x \in \Omega, \;\; s \in \delta(r, i)\}.$$

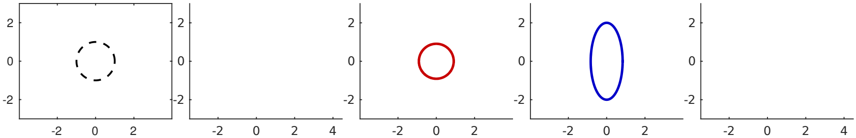
- the one-step operator as :

$$Q^s(\Omega) = \bigcup_{i \in \mathcal{I}} Q_i^s(\Omega) = \bigcup_{i \in \mathcal{I}} \bigcup_{r \in \gamma(s,i)} (A_i^{-1}\Omega \times r).$$

where $\gamma(s, i) = \{r \in S : \; s \in \delta(r, i)\}$;



$$A_1 = \begin{bmatrix} 1.2 & 0 \\ 0 & 0.5 \end{bmatrix}, \; A_2 = 1.1\, R(\pi/3)$$



Sets and algorithm

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- the one-step operator as :

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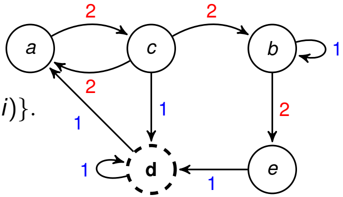

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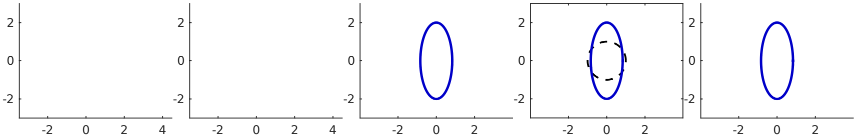

Sets and algorithm

For every set $\Omega \subseteq \mathbb{R}^n$ and state $s \in S$, define :

- the one-step operator for every mode $i \in \mathcal{I}$ as

$$Q_i^s(\Omega) = \{(x, r) \in \mathbb{R}^n \times S : \; A_i x \in \Omega, \;\; s \in \delta(r, i)\}.$$

- the one-step operator as :

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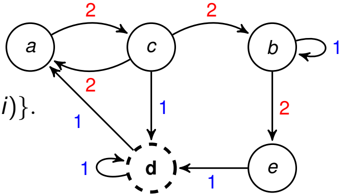

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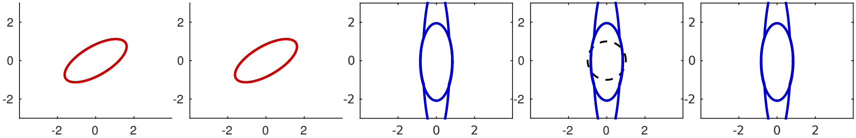

Sets and algorithm

Computation of a contractive C*-set for the switched system recurrent in s.

Algorithm

- **Initialization** : given the C*-set $\Omega_0 \subseteq \mathbb{R}^n$ and a state $s \in S$, define $\Lambda_0^s = \Omega_0 \times s$ and $k = 0$;
- **Iteration** for $k \geq 0$:

$$\Lambda_{k+1}^s = \bigcup_{(\Omega \times r) \subseteq \Lambda_k^s} Q^r(\Omega),$$
$$\Omega_{k+1}^{s,s} = \{x \in \mathbb{R}^n : \ (x,\ s) \in \Lambda_{k+1}^s\},$$

- **Stop** if $\Omega_0 \subseteq \operatorname{int}\left(\bigcup_{j \in \mathbb{N}_{k+1}} \Omega_j^{s,s}\right)$; denote $N^s = k+1$ and $\Omega^s = \bigcup_{j \in \mathbb{N}_{N^s}} \Omega_j^{s,s}$.

- For every $\Omega \subseteq \mathbb{R}^n$ and $s \in S$:

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- Every set Λ_j^s have the form

$$\Lambda_j^s = \bigcup_{r \in S} \left(\Omega_j^{r,s} \times r\right), \quad \forall j \in \mathbb{N},$$

with $\Omega^{r,s}$ possibly empty, with

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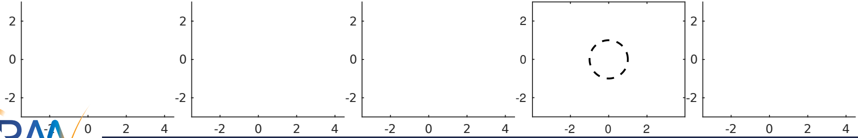

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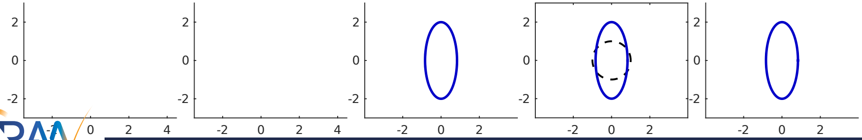

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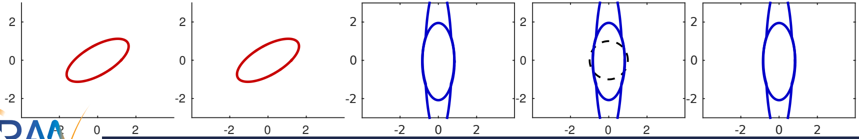

(Counter)-Example

We take the Example 17 in the paper *[F., Girard, Jungers, TAC16]* to consider the relation with periodic stabilizability.

- The matrices are

$$A_1 = AR(0), \quad A_2 = AR\left(\frac{2\pi}{3}\right), \quad A_3 = AR\left(\frac{-2\pi}{3}\right),$$

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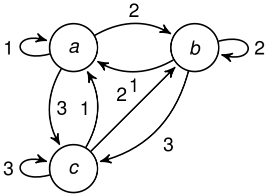

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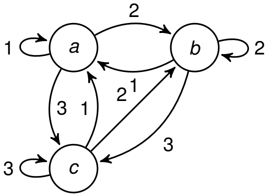

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Results :

- The system is stabilizable through a recurrent Laypunov function : then recurrent ECLF are strictly less conservative than periodic (state-independent) once.

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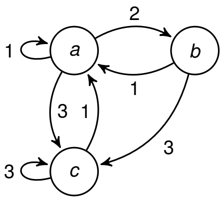

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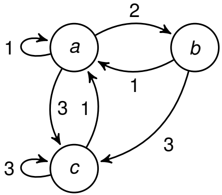

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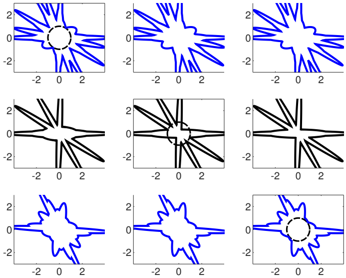
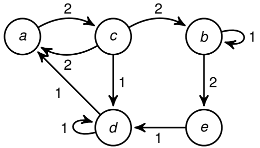
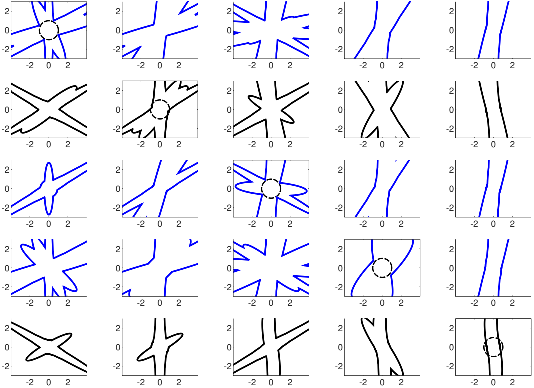


Example 3

System with $A_1 = \begin{bmatrix} 1.2 & 0 \\ 0 & 0.5 \end{bmatrix}$, $A_2 = 1.1\, R(\pi/3)$, and stop for $N^s = 6$.





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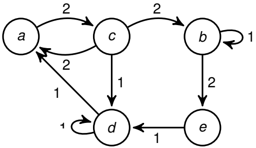
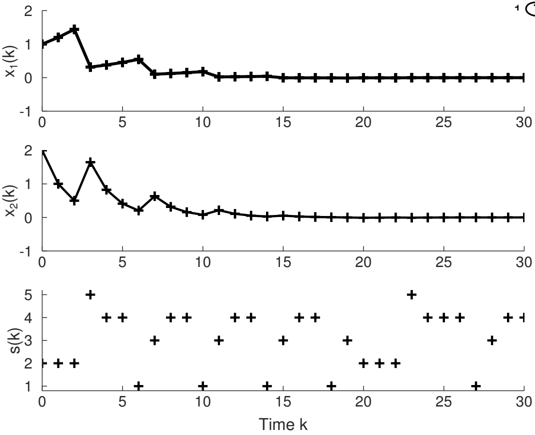




Lyapunov-Metzler inequalities

Theorem

Let be $\mathcal{C} = \{c_1, \cdots, c_h\}$ *a nontrivial SCC induced by the automaton* $\mathcal{A}$*. If there exist a stochastic matrix, admissible with the SCC* $\mathcal{C}$*, that is* $\Pi \in \mathbb{R}^{Nh \times h}$ *with* $\Pi' \in \mathcal{M}_{h \times Nh}$ *and h symmetric positive definite matrices* M_i*,* $i \in \mathbb{N}_h$*, such that the h bilinear matrix inequalities*

$$\forall j \in \mathbb{N}_h, \quad M_j > \sum_{\substack{(i,\ell) \in \mathbb{N}_h \times \Sigma \\ c_i \in \delta(c_j, \ell) \cap \mathcal{C}}} \pi_{i+(\ell-1)h,j} A'_\ell M_i A_\ell,$$

are satisfied, then $\mathcal{C}$ *admits an ECLF on the form*

$$V_{min} : \left\{ \begin{array}{ccc} \mathcal{C} \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^+, \\ (c_j, x) & \longmapsto & \min\limits_{\substack{(i,\ell) \in \mathbb{N}_h \times \Sigma \\ c_i \in \delta(c_j, \ell) \cap \mathcal{C}}} V(c_i, A_\ell x), \end{array} \right. \qquad V : \left\{ \begin{array}{ccc} \mathcal{C} \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^+, \\ (c_i, x) & \longmapsto & x' M_i x. \end{array} \right.$$

Moreover, after an (arbitrary) prefix allowing to reach in finite time an automaton state $c_{i_0} \in \mathcal{C}$ *(*$i_0 \in \mathbb{N}_h$*) from* $s_0 \in \tilde{\mathcal{S}}^0$*, apply*

$$(s_{k+1}, \sigma(k)) = \nu^{\mathcal{C}}(x_k, s_k) \in \arg \min_{\substack{(i,\ell) \in \mathbb{N}_h \times \Sigma \\ c_i \in \delta(s_k, \ell)}} x'_k A'_\ell M_i A_\ell x_k. \qquad (29)$$

Links with unconstrained case

Automaton 1 : $p = 1$, $\mathcal{A} = (\{e_1\}, \mathbb{N}_N, \delta, \{e_1\})$, where $e_1 \in \delta(e_1, i)$, $\forall i \in \mathbb{N}_N$. We have the inequality from [Lemma 1, *Geromel & Colaneri, IJC 2006*]

$$M_1 > \sum_{\ell \in \mathbb{N}_N} \pi_{\ell,1} A'_\ell M_1 A_\ell. \quad (30)$$

Automaton 2 : $p = N$, $\mathcal{A} = (\{e_1, \cdots, e_N\}, \mathbb{N}_N, \delta, \{e_1, \cdots, e_N\})$, where $\{e_i\} = \delta(e_j, i)$, $\forall (i,j) \in \mathbb{N}_N^2$.

$$\forall j \in \mathbb{N}_N,\ M_j > \sum_{\ell \in \mathbb{N}_h} \pi_{\ell+(\ell-1)q,j} A'_\ell M_\ell A_\ell. \quad (31)$$

By introducing $P_i = A'_i M_i A_i$, $\forall i \in \mathbb{N}_N$, it yields the Lyapunov-Metzler inequalities

$$P_j = A'_j M_j A_j > A'_j (\sum_{\ell \in \mathbb{N}_q} \tilde{\pi}_{\ell,j} P_\ell) A_j,\ \forall j \in \mathbb{N}_N. \quad (32)$$

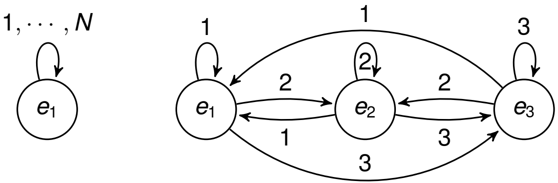
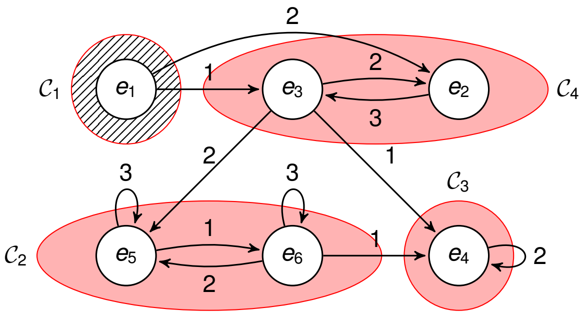


FIGURE 2 – Automatons 1 with $p = 1$ (left) and 2 with $p = N = 3$ (right).

Illustration : $n = 2$; $N = 3$ modes ; $p = 6$

$$\begin{bmatrix} A_1 & A_2 & A_3 \end{bmatrix} = \left[\begin{array}{cc|cc|cc} 0.9 & 0 & 0.6 & 0 & 1.2 & 1 \\ 0 & 0.7 & 0 & 1/0.6 & 0 & 0.8 \end{array}\right], \quad x_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$



- $\mathcal{S}^0 = \{e_1, e_2, e_4\}$;
- $p = 6$ states ;
- $d = 4$ SCCs ;

$\mathcal{C}_1 = \{e_1\}$ is a trivial SCC.

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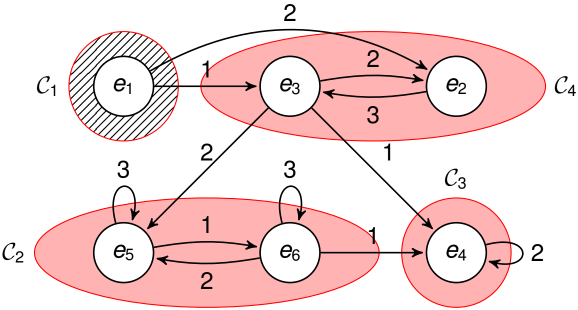

- $\mathcal{S}^0 = \{e_1, e_2, e_4\}$;
- $p = 6$ states ;
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$\mathcal{C}_2 = \{e_5, e_6\}$ is a nontrivial SCC. Inequalities (18) admit (at least) a solution.

$$M_5 = \begin{bmatrix} 0.0407 & 0.1665 \\ 0.1665 & 2.4735 \end{bmatrix}; \; M_6 = \begin{bmatrix} 0.0167 & 0.1668 \\ 0.1668 & 6.5058 \end{bmatrix}; \; \begin{array}{l} M_5 > 0.4A_1' M_6 A_1 + 0.6 A_3' M_5 A_3, \\ M_6 > 0.8 A_2' M_5 A_2 + 0.2 A_3' M_6 A_3. \end{array}$$

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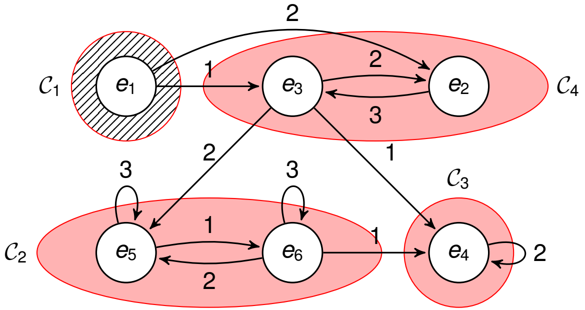

- $\mathcal{S}^0 = \{e_1, e_2, e_4\}$;
- $p = 6$ states ;
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$\mathcal{C}_3 = \{e_4\}$ is a nontrivial SCC. A_2 being unstable, Inequality (18) associated with $\mathcal{C}_3$ cannot admit a solution.

Illustration : $n = 2$; $N = 3$ modes ; $p = 6$

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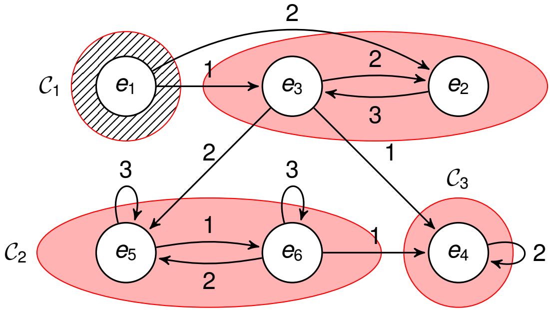


- $\mathcal{S}^0 = \{e_1, e_2, e_4\}$;
- $p = 6$ states ;
- $d = 4$ SCCs ;

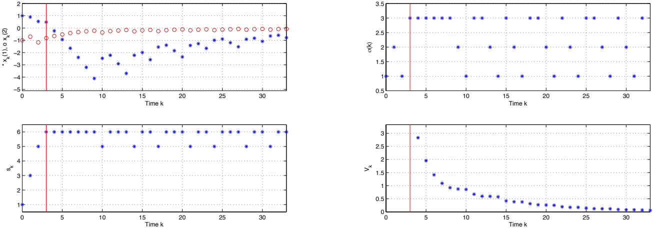
$\mathcal{C}_4 = \{e_2, e_3\}$ is a nontrivial SCC. The only possible cycle is a periodic one and A_2A_3 is not stable. Inequalities (18) associated with $\mathcal{C}_4$ cannot admit a solution.

Illustration

Prefix of the stabilizing policy : $\mathcal{S}^0 \bigcap (\mathcal{C}_2 \cup \mathcal{C}_4 \cup \mathcal{C}_1) = \{e_1, e_2\}$, which is not empty. We select $e_1 \in \tilde{\mathcal{S}}_0$ and $e_6 \in \mathcal{C}_2$. The Dijkstra path leads to $K = 3$ and

$$e_1 \xrightarrow{\sigma(0)=1} e_3 \xrightarrow{\sigma(1)=2} e_5 \xrightarrow{\sigma(2)=1} e_6.$$

Min-switching stabilizing policy : For $k \geq K$, apply the min-switching strategy (29).



The vertical red line split in time the Dijkstra path and the application of min-switching strategy. (left) : Trajectories $k \mapsto x_k$ and $k \mapsto s_k$. (right) : Functions $k \mapsto \sigma(k)$, for $k \in \mathbb{N}$ and $k \mapsto V_k$, for $k \geq K + 1$.

Language constrained sufficient condition for stabilizability with LMI conditions

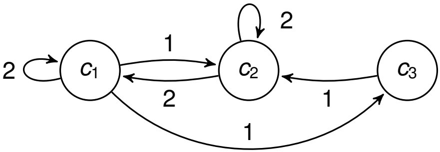
Definition

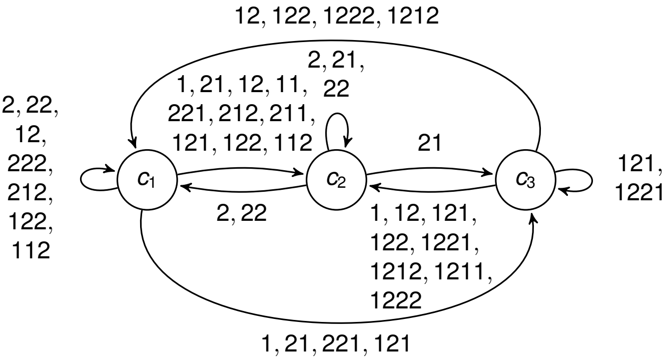
Let be $\mathcal{C}$ a nontrivial SCC of the automaton $\mathcal{A}$, containing $h = |\mathcal{C}|$ automaton states, denoted $\mathcal{C} = \{c_1, \cdots, c_h\}$. For given h positive integers $N_i \in \mathbb{N}$, $(i \in \mathbb{N}_h)$; let us define $\overline{\mathcal{P}}_{i,N_i}(\mathcal{A}, \mathcal{C})$ as the set of paths starting from the state automaton c_i, admissible to the language $\mathcal{A}$ remaining in the SCC $\mathcal{C}$ of length less than or equal to N_i, that is

$$\overline{\mathcal{P}}_{i,N_i}(\mathcal{A}, \mathcal{C}) = \bigcup_{j \in \mathbb{N}_{N_i}} \{p \in \mathcal{P}_j(\mathcal{A}, \mathcal{C}),\ \pi_1(p) = c_i\} \tag{33}$$

and $\overline{N}_i$ the number of paths in $\overline{\mathcal{P}}_{i,N_i}(\mathcal{A}, \mathcal{C})$.

Modified automaton





Language constrained sufficient condition for stabilizability

Theorem [*Jungers, Girard, Fiacchini, ADHS 2018*]

If there exist h vectors $\eta_i \in \mathbb{R}^{\overline{N}_i}$ ($i \in \mathbb{N}_h$), such that $\eta_i \geq 0$ and $\sum_{p \in \overline{\mathcal{P}}_{i,N_i}(\mathcal{A},\mathcal{C})} \eta_{i,p} = 1$ and finally such that the h Linear Matrix Inequalities (LMI) are satisfied.

$$\sum_{p \in \overline{\mathcal{P}}_{i,N_i}(\mathcal{A},\mathcal{C})} \eta_{i,p} \mathbb{A}_{w(p)}^T \mathbb{A}_{w(p)} < I_n, \quad i \in \mathbb{N}_h, \tag{34}$$

then the system (9) is exponentially stabilizable with a switching law admissible to the language $L(\mathcal{C})$. □

Language constrained sufficient condition for stabilizability

Algorithm designing p^*

Let us define iteratively as follows a sequence of indexes of automaton states $\{i_j\}_{j\in\mathbb{N}} \in (\mathbb{N}_h)^{\mathbb{N}}$ and a sequence of instants $\{k_j\}_{j\in\mathbb{N}} \in \mathbb{N}$, with $k_j < k_{j+1}$, $j \in \mathbb{N}$. The algorithm to build p^* is as follows :

Initialization : Choose $k_0 = 0$, arbitrarily $i_0 \in \mathbb{N}_h$, $x_0 \in \mathbb{R}^n$ and finally $p^* = \emptyset$.

Iteration j : Select $\tilde{p}$ such that

$$\tilde{p} \in \arg \min_{p\in\overline{\mathcal{P}}_{i_j,N_{i_j}}(\mathcal{A},\mathcal{C})} x_{k_j}^T \mathbb{A}_{w(p)}^T \mathbb{A}_{w(p)} x_{k_j}, \tag{35}$$

and define i_{j+1} as the unique value in $\mathbb{N}_h$ such that $c_{i_{j+1}}$ is the last automaton state of $\tilde{p}$:

$$\pi_{l(\tilde{p})+1}(\tilde{p}) = c_{i_{j+1}}, \tag{36}$$

and $k_{j+1} = k_j + l(\tilde{p})$. Build p^*_{k+1} by $p^*_k \circ \tilde{p}$, because the definition of i_{j+1} by equation (36) ensures the compatibility of the paths and allows the concatenation. The state is then given by

$$x_{k_j+z} = A_{w_z(\tilde{p})} x_{k_j+z-1}, \quad z \in \{1, \cdots, k_{j+1} - k_j\}, \tag{37}$$

and in closed form

$$x_{k_{j+1}} = \mathbb{A}_{w(\tilde{p})} x_{k_j}. \tag{38}$$

Links with unconstrained case

Automaton : $p = 1$, $\mathcal{A} = (\{e_1\}, \mathbb{N}_N, \delta, \{e_1\})$, where $e_1 \in \delta(e_1, i)$, $\forall i \in \mathbb{N}_N$.
We recover only one LMI : $\sum_{i \in \mathcal{I}^{[1:K]}} \eta_i \mathbb{A}_i^T \mathbb{A}_i < I$.

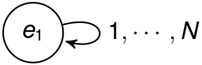


FIGURE 3 – Automaton 1 with $p = 1$

Links with the periodically stabilization

Definition

The system (9) is *periodic stabilizable with the language constraint* $L(\mathcal{A})$ if there exist an automaton state $s \in \mathcal{S}^0$ (or *ultimately periodic stabilizable* if there exists a reachable automaton state $s \in \mathcal{S}$ from $\mathcal{S}^0$) and a cyclic path $p_{\mathrm{per}} \in \mathcal{P}_m(\mathcal{A})$ with $\pi_1(p_{\mathrm{per}}) = s$ such that $\mathbb{A}_{w(p_{\mathrm{per}})}$ is Schur.

A cyclic path belongs to a SCC. Only a SCC $\mathcal{C}$ of the automaton $\mathcal{A}$ and a periodic stabilizability restricted to this SCC $\mathcal{C}$ are considered.

Theorem

The system (9) is periodic stabilizable on a SCC $\mathcal{C}$ of $\mathcal{A}$ if and only if there exist $h = |\mathcal{C}|$ natural integers N_i, ($i \in \mathbb{N}_h$) and vectors η_i in the simplex of dimensions $\overline{N}_i$, such that LMIs (34) are satisfied.

Illustration

Let us consider the switched system (9) with $N = 3$ modes, $x_0 = \begin{pmatrix} 2 & -1 \end{pmatrix}^T$ and

$$\left[\begin{array}{c|c|c} A_1 & A_2 & A_3 \end{array} \right] = \left[\begin{array}{cc|cc|cc} 1.3 & 0.8 & 0.6 & 0 & 1.1 & 0 \\ 0 & 0.5 & -0.4 & -1/0.6 & 0 & -0.9 \end{array} \right].$$

The automaton $\mathcal{A}$ defining the constrained language is the one in Figure 4, with all the automaton states as initial ones : $\mathcal{S}^0 = \{e_1, e_2\}$. We select the automaton state e_1 as the initial one in the simulation.
The inequalities (34) are feasible with $N_1 = 3$, and $N_2 = 4$, leading to the following number of terms in the LMIs : $\overline{N}_1 = 48$ and $\overline{N}_2 = 192$.

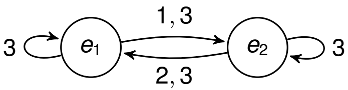
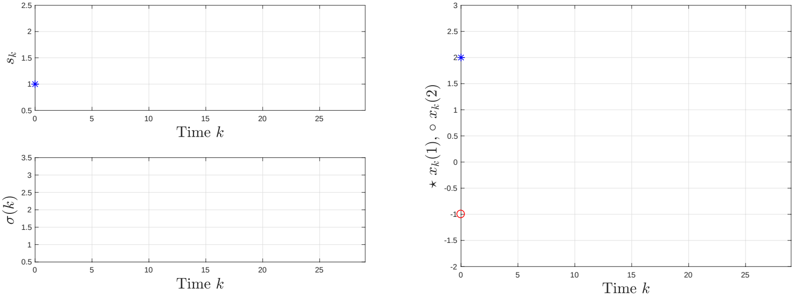


FIGURE 4 – Automaton related to the numerical example.

The constant mode 3 is admissible, roughly speaking there exists (at least) a path $p \in \mathcal{P}_\infty(\mathcal{A})$ such that $w_k(p) = 3$, but the matrix A_3 is not Schur.

Illustration

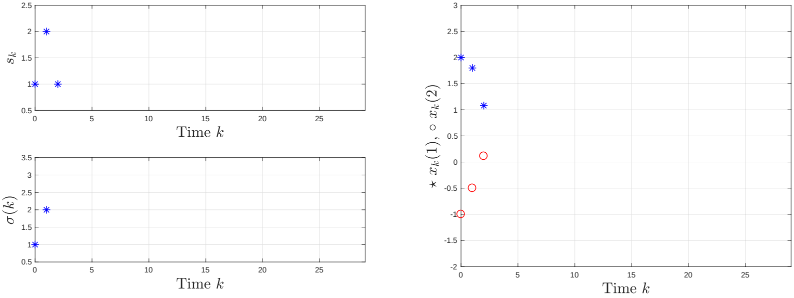


Solution :

$$\tilde{p} \in \arg \min_{p \in \overline{\mathcal{P}}_{1,N_1}(\mathcal{A},\mathcal{C})} x^T \mathbb{A}_{w(p)}^T \mathbb{A}_{w(p)} x,$$

Choix : $\tilde{p} = \{(e_1, 1, e_2), (e_2, 2, e_1)\}$

Illustration

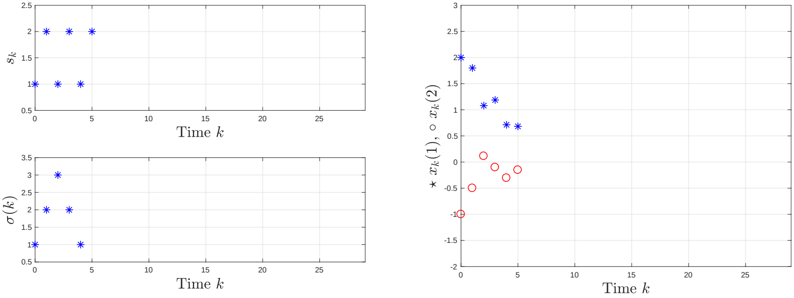


Solution :

$$\tilde{p} \in \arg \min_{p \in \overline{\mathcal{P}}_{1,N_1}(\mathcal{A},\mathcal{C})} x^T \mathbb{A}_{w(p)}^T \mathbb{A}_{w(p)} x,$$

Choix : $\tilde{p} = \{(e_1, 3, e_2), (e_2, 2, e_1), (e_1, 1, e_2)\}$

Illustration



Solution :

$$\tilde{p} \in \arg \min_{p \in \overline{\mathcal{P}}_{2,N_2}(\mathcal{A},\mathcal{C})} x^T \mathbb{A}_{w(p)}^T \mathbb{A}_{w(p)} x,$$

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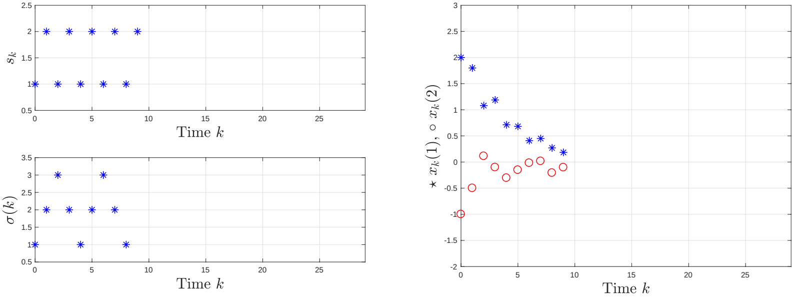
Illustration



Solution :

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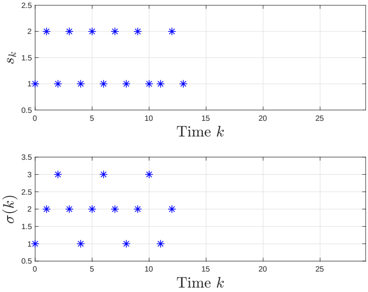
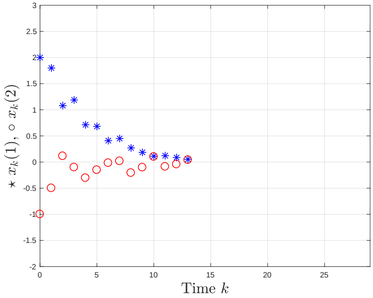
Illustration





Solution :

$$\tilde{p} \in \arg \min_{p \in \overline{\mathcal{P}}_{1,N_1}(\mathcal{A},\mathcal{C})} x^T \mathbb{A}_{w(p)}^T \mathbb{A}_{w(p)} x,$$

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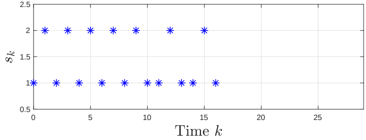
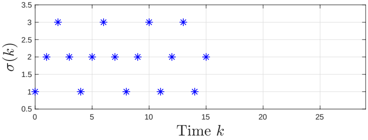
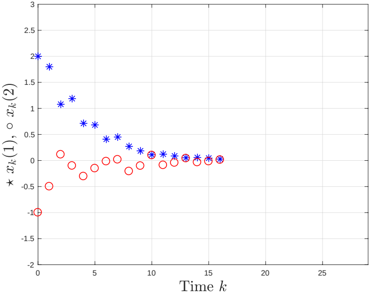
Illustration







Solution :

$$\tilde{p} \in \arg \min_{p \in \overline{\mathcal{P}}_{1,N_1}(\mathcal{A},\mathcal{C})} x^T \mathbb{A}_{w(p)}^T \mathbb{A}_{w(p)} x,$$

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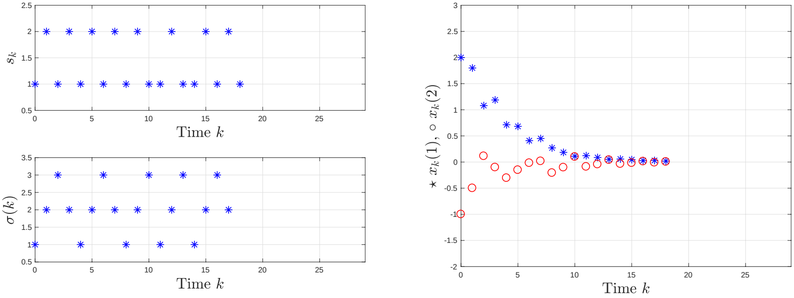

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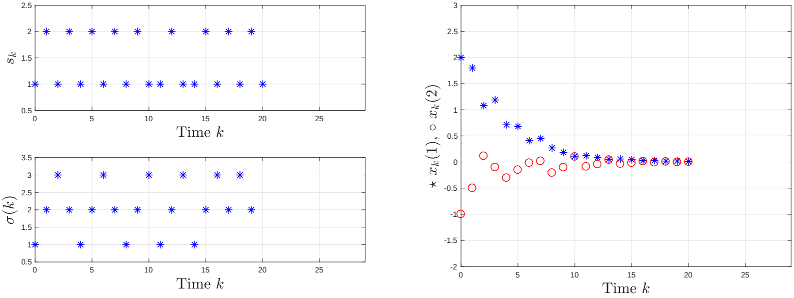

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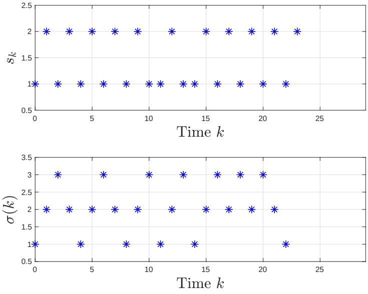

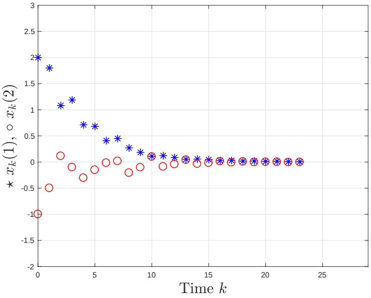

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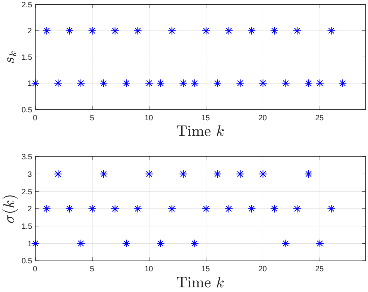
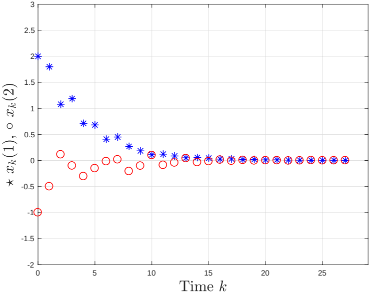
Illustration





Solution :

Choix : $\tilde{p} = \{(e_1, 3, e_2), (e_2, 2, e_1), (e_1, 1, e_2)\}$

Conclusion

Questions ?

Thank you very much for your attention !

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