Méthodes et outils issus de la Commande Robuste pour la conception efficace de systèmes

Application à la synthèse fréquentielle de systèmes homogènes interconnectés

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Congrès SAGIP 2024 - Session MOSAR

Collaborations and publication

Joint work with

- Gérard Scorletti (lab. Ampère)
- Anton Korniienko (lab. Ampère)
- Mykhailo Zarudniev (CEA-LETI)

Publication

- A. Perodou, A. Korniienko, M. Zarudniev and G. Scorletti. Frequency Synthesis of Interconnected Homogeneous LTI Systems. In *IEEE Transactions on Automatic Control (IEEE TAC) : Full Papers*, March 2024.

Outline

Contents

Interconnected homogeneous LTI systems

- Increase complexity in engineering problems
  → Common approach: interconnect simple local systems in a way that fulfills a global objective
- Interconnected systems framework → takes explicitly into account the structure of the interconnection
- Robust Control framework → tools for efficient design methods
- In this work
  - Homogeneous systems = systems with same model
  - LTI systems
  - Frequency filtering application

Interconnected homogeneous LTI systems

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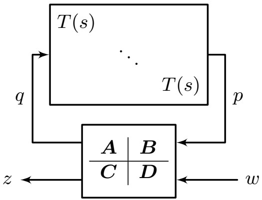
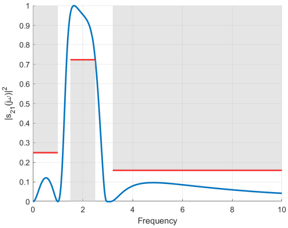
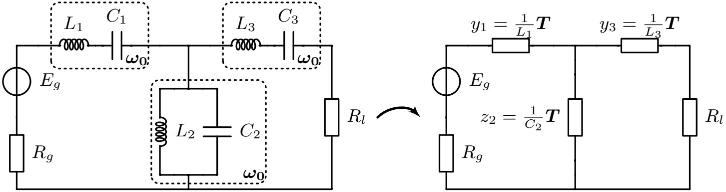




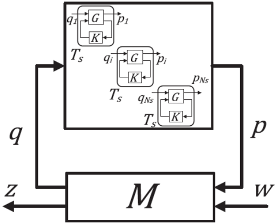
Motivation by applications

- Passive radiofrequency filter design

$$\boldsymbol{T}(s) = \frac{s}{s^2+\omega_0^2}$$



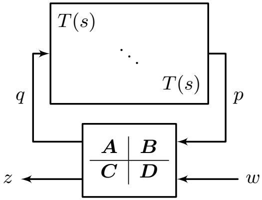
- Weights synthesis for decentralized $\mathscr{H}_\infty$-control



$$\left\| W_s(T_s)(T_s I_{N_S} \star M)\, W_e(T_s) \right\|_\infty < 1$$

Representation of the interconnection

- Interconnection of n homogeneous LTI systems



$$\begin{cases} p = (T(s) \cdot I_n) q \\ q = Ap + Bw \\ z = Cp + Dw \end{cases}$$

- Linear Fractionnal Transformation (LFT)

$$H(T(s)) = (T(s) \cdot I_n) \star \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = D + CT(s)(I - AT(s))^{-1} B$$

$$= \frac{b_0 + b_1 \cdot T(s) + \ldots + b_m \cdot T(s)^m}{a_0 + a_1 \cdot T(s) + \ldots + a_n \cdot T(s)^n}$$

→ Particular case $T(s) = \frac{1}{s}$

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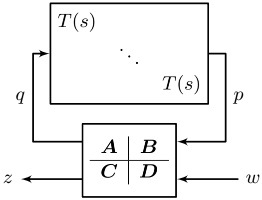

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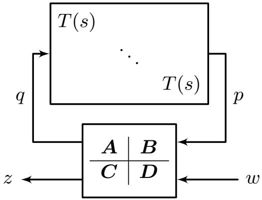

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Characterization of the sub-systems

Definition ($\{x, y, z\}$-dissipative system)

Let x, y, z be real scalars such that $xz - y^2 < 0$. Denote $p_{\mathbb{C}^+}$ the set of poles of $T(s)$ in $\mathbb{C}^+$. Then $T(s)$ is said to be $\{x, y, z\}$-dissipative if

$$\forall s \in \mathbb{C}^+ \setminus p_{\mathbb{C}^+}, \quad \begin{bmatrix} T(s) \\ 1 \end{bmatrix}^* \begin{bmatrix} x & y \\ y & z \end{bmatrix} \begin{bmatrix} T(s) \\ 1 \end{bmatrix} \geq 0$$

→ $T(s)$ positive-real if $\{x, y, z\} = \{0, 1, 0\}$, bounded-real if $\{x, y, z\} = \{-1, 0, 1\}$

- How to find suitable $\{x, y, z\}$? Graphical test (Nyquist style) on $T(j\omega)$
  - Eg with $T(s)$ stable and $x < 0$

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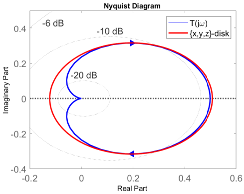

  - Particular cases: lossless dissipative

$$\begin{bmatrix} \frac{1}{j\omega} \\ 1 \end{bmatrix}^* \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{j\omega} \\ 1 \end{bmatrix} = 0 \qquad \begin{bmatrix} T(j\omega) \\ 1 \end{bmatrix}^* \begin{bmatrix} x & y \\ y & z \end{bmatrix} \begin{bmatrix} T(j\omega) \\ 1 \end{bmatrix} = 0$$

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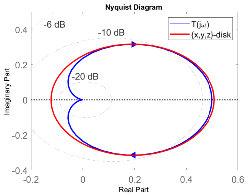

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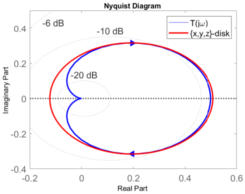

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Results - Stability analysis

Lemma

Consider n homogeneous systems $T(s)$ that are $\{x, y, z\}$-dissipative

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The interconnection $H(T(s)) = (T(s) \cdot I_n) \star \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *is internally stable if*

$$\exists \boldsymbol{P} = \boldsymbol{P}^{\boldsymbol{T}} > 0, \qquad \begin{bmatrix} I \\ A \end{bmatrix}^T \begin{bmatrix} x\boldsymbol{P} & y\boldsymbol{P} \\ y\boldsymbol{P} & z\boldsymbol{P} \end{bmatrix} \begin{bmatrix} I \\ A \end{bmatrix} < 0$$

- Integrator $\frac{1}{s}$ such as $\begin{bmatrix} \frac{1}{s} \\ 1 \end{bmatrix}^* \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s} \\ 1 \end{bmatrix} \geq 0 \Rightarrow \exists \boldsymbol{P} = \boldsymbol{P}^{\boldsymbol{T}} > 0,\ A^T \boldsymbol{P} + \boldsymbol{P} A < 0$

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Results - Performance analysis

Lemma (KYP Lemma)

Consider a stable interconnection $H(T(s)) = (T(s)\cdot I_n) \star \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *of n homogeneous systems* $T(s)$ *that are* $\{x, y, z\}$*-dissipative:*

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Then $(i) \Rightarrow (ii)$

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(ii) $\forall \omega \in \mathbb{R},$

$$\begin{bmatrix} H(T(j\omega)) \\ I \end{bmatrix}^* \Psi \begin{bmatrix} H(T(j\omega)) \\ I \end{bmatrix} \geq 0$$

→ Eg - $\forall \omega \in \mathbb{R}, \quad |H(T(j\omega))|^2 \leq U^2 \Leftrightarrow \Psi = \begin{bmatrix} -I & 0 \\ 0 & U^2 \end{bmatrix}$

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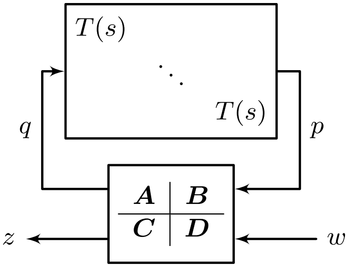
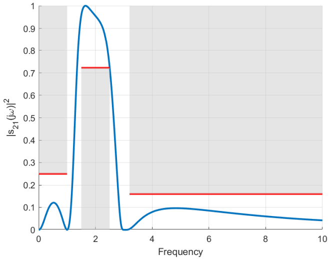
→ Synthesis: $\boldsymbol{\Psi}$ as a decision variable

Contents

Problem overview

- Frequency synthesis of the interconnection of n homogeneous systems





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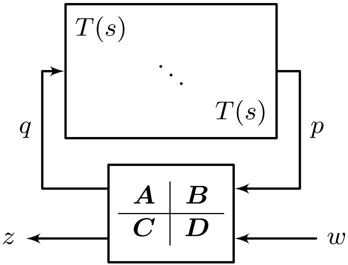

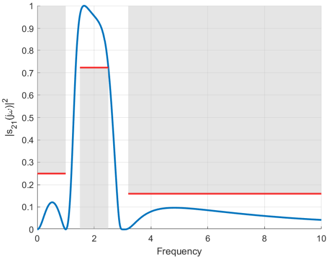

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Usual convex approach: revisited, extended and generalized

Problem (Classical frequency filter synthesis - revisited)

Find a stable $H(s) = \left(\frac{1}{s}\cdot I_n\right) \star \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *such that* $\begin{cases} \forall\omega \in \Omega_u^U, \ |H(j\omega)|^2 \leq U_u^2 \\ \forall\omega \in \Omega_l^L, \ |H(j\omega)|^2 \geq L_l^2 \end{cases}$

Usual Convex Approach

(*i*) Magnitude Design $|H(j\omega)|^2$

- Finite parametrization $|H(j\omega)|^2 = \frac{|H_N(j\omega)|^2}{|H_D(j\omega)|^2}$
- KYP Lemmas: LMI feasibility problem

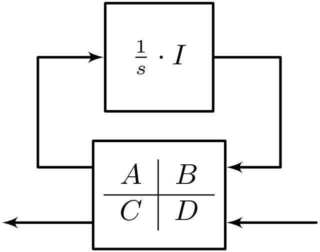
(*ii*) Spectral Factorization

$$|H(j\omega)|^2 \to H(s)$$

- Algebraic Riccati Equation (ARE)

Dissipative property

$$\begin{bmatrix} \frac{1}{j\omega} \\ 1 \end{bmatrix}^* \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{j\omega} \\ 1 \end{bmatrix} = 0$$



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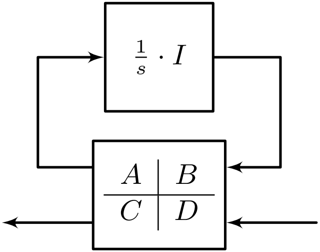

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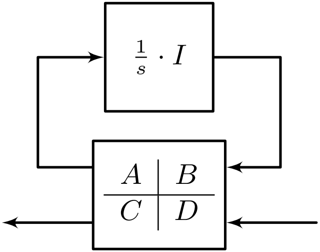

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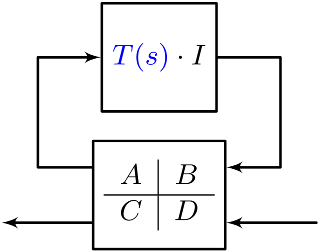

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(ii) Spectral Factorization ✗

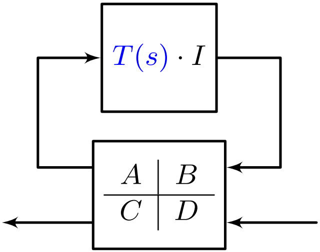
$$|H(T(j\omega))|^2 \to H(T(s)) + E(T(s))$$

Generalized Convex Approach ✔

- Error management

Dissipative property

$$\begin{bmatrix} T(j\omega) \\ 1 \end{bmatrix}^* \begin{bmatrix} x & y \\ y & z \end{bmatrix} \begin{bmatrix} T(j\omega) \\ 1 \end{bmatrix} \geq 0$$



Sketch of proof - Main ideas - Technical details

(i) Magnitude design

- Parametrization $|H(j\omega)|^2 = \frac{|H_N(j\omega)|^2}{|H_D(j\omega)|^2} = \frac{\mathscr{B}(j\omega)^* X_N \mathscr{B}(j\omega)}{\mathscr{B}(j\omega)^* X_D \mathscr{B}(j\omega)}$
- Find $\boldsymbol{X_N}$, $\boldsymbol{X_D}$ such that

$$\frac{\mathscr{B}(j\omega)^* \boldsymbol{X_N} \mathscr{B}(j\omega)}{\mathscr{B}(j\omega)^* \boldsymbol{X_D} \mathscr{B}(j\omega)} \leq U^2 \quad \Leftrightarrow \quad \mathscr{B}(j\omega)^* \left(U^2 \boldsymbol{X_D} - \boldsymbol{X_N}\right) \mathscr{B}(j\omega) \geq 0$$

- KYP Lemmas - LMI feasibility problem

$$\exists \boldsymbol{P_N} = \boldsymbol{P_N^T}, \quad -\begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^T \begin{bmatrix} 0 & \boldsymbol{P_N} \\ \boldsymbol{P_N} & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \left(U^2 \boldsymbol{X_D} - \boldsymbol{X_N}\right) \begin{bmatrix} C & D \end{bmatrix} \geq 0$$

(ii) Spectral Factorization

- Solve $ARE(\boldsymbol{P_{N_R}}) = 0 \Rightarrow$ stable $H_N(s)$ such that

$$F(j\omega)^* \underbrace{\begin{bmatrix} \frac{1}{j\omega} \\ 1 \end{bmatrix}^* \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{j\omega} \\ 1 \end{bmatrix}}_{=0} (-\boldsymbol{P_{N_R}}) F(j\omega) + \mathscr{B}(j\omega)^* \boldsymbol{X_N} \mathscr{B}(j\omega) = |H_N(j\omega)|^2$$

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Sketch of proof - Main ideas - Technical details

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Sketch of proof - Main ideas - Technical details

(i) Magnitude design

- Parametrization $|H(T(j\omega))|^2 = \frac{|H_N(T(j\omega))|^2}{|H_D(T(j\omega))|^2} = \frac{\mathscr{B}(T(j\omega))^* X_N \mathscr{B}(T(j\omega))}{\mathscr{B}(T(j\omega))^* X_D \mathscr{B}(T(j\omega))}$
- Find $\boldsymbol{X_N}$, $\boldsymbol{X_D}$ such that

$$\frac{\mathscr{B}(T(j\omega))^* \boldsymbol{X_N} \mathscr{B}(T(j\omega))}{\mathscr{B}(T(j\omega))^* \boldsymbol{X_D} \mathscr{B}(T(j\omega))} \leq U^2 \quad \Leftrightarrow \quad \mathscr{B}(T(j\omega))^* \left(U^2 \boldsymbol{X_D} - \boldsymbol{X_N}\right) \mathscr{B}(T(j\omega)) \geq 0$$

- KYP Lemmas - LMI feasibility problem

$$\exists \boldsymbol{P_N} = \boldsymbol{P_N^T} > 0, \qquad - \begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^T \begin{bmatrix} x\boldsymbol{P_N} & y\boldsymbol{P_N} \\ y\boldsymbol{P_N} & z\boldsymbol{P_N} \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \left(U^2 \boldsymbol{X_D} - \boldsymbol{X_N}\right) \begin{bmatrix} C & D \end{bmatrix} \geq 0$$

(ii) Spectral Factorization

- Solve $ARE(\boldsymbol{P_{N_R}}) = 0 \Rightarrow$ stable $H_N(T(s))$ such that

$$\underbrace{F(T(j\omega))^* \begin{bmatrix} T(j\omega) \\ 1 \end{bmatrix}^* \begin{bmatrix} x & y \\ y & z \end{bmatrix} \begin{bmatrix} T(j\omega) \\ 1 \end{bmatrix} (-\boldsymbol{P_{N_R}}) F(T(j\omega))}_{E_N(T(j\omega), \boldsymbol{P_{N_R}})} + \mathscr{B}(T(j\omega))^* \boldsymbol{X_N} \mathscr{B}(T(j\omega)) = |H_N(T(j\omega))|^2$$

Generalized approach - Error management from scratch

(i) Magnitude design

- Parametrization

$$|H(T(j\omega))|^2 = \frac{|H_N(T(j\omega))|^2}{|H_D(T(j\omega))|^2} = \frac{E_N(T(j\omega), \boldsymbol{P_{N_R}}) + \mathscr{B}(T(j\omega))^* X_N \mathscr{B}(T(j\omega))}{E_D(T(j\omega), \boldsymbol{P_{D_R}}) + \mathscr{B}(T(j\omega))^* X_D \mathscr{B}(T(j\omega))}$$

- Find $\boldsymbol{X_N}$, $\boldsymbol{X_D}$ such that

$$\frac{E_N(T(j\omega), \boldsymbol{P_{N_R}}) + \mathscr{B}(T(j\omega))^* \boldsymbol{X_N} \mathscr{B}(T(j\omega))}{E_D(T(j\omega), \boldsymbol{P_{D_R}}) + \mathscr{B}(T(j\omega))^* \boldsymbol{X_D} \mathscr{B}(T(j\omega))} \leq U^2$$

- KYP Lemmas - LMI feasibility problem

$+$ $ARE(\boldsymbol{P_{N_R}}) = 0$

(ii) Spectral Factorization

- Solve $ARE(\boldsymbol{P_{N_R}}) = 0 \Rightarrow$ stable $H_N(T(s))$ such that

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Generalized approach - Error management from scratch

(i) Magnitude design

- Parametrization

$$|H(T(j\omega))|^2 = \frac{|H_N(T(j\omega))|^2}{|H_D(T(j\omega))|^2} = \frac{E_N(T(j\omega), \boldsymbol{P_{N_R}}) + \mathscr{B}(T(j\omega))^* X_N \mathscr{B}(T(j\omega))}{E_D(T(j\omega), \boldsymbol{P_{D_R}}) + \mathscr{B}(T(j\omega))^* X_D \mathscr{B}(T(j\omega))}$$

- Find $\boldsymbol{X_N}$, $\boldsymbol{X_D}$ such that

$$\frac{E_N(T(j\omega), \boldsymbol{P_{N_R}}) + \mathscr{B}(T(j\omega))^* \boldsymbol{X_N} \mathscr{B}(T(j\omega))}{E_D(T(j\omega), \boldsymbol{P_{D_R}}) + \mathscr{B}(T(j\omega))^* \boldsymbol{X_D} \mathscr{B}(T(j\omega))} \leq U^2$$

- KYP Lemmas - LMI feasibility problem

$+$ $ARE(\boldsymbol{P_{N_R}}) = 0$

(ii) Spectral Factorization

- Solve $ARE(\boldsymbol{P_{N_R}}) = 0 \Rightarrow$ stable $H_N(T(s))$ such that

$$E_N(T(j\omega), \boldsymbol{P_{N_R}}) + \mathscr{B}(T(j\omega))^* \boldsymbol{X_N} \mathscr{B}(T(j\omega)) = |H_N(T(j\omega))|^2$$

Link ARE and LMI

- $ARE(\boldsymbol{P_R}) = 0$

$$A^T(-\boldsymbol{P_R}) + (-\boldsymbol{P_R})A + C^TXC - (-\boldsymbol{P_R}B + C^TXD)(D^TXD)^{-1}(-\boldsymbol{P_R}B + C^TXD)^T = 0$$

- Minimal solution P_R^{min} computable by LMI resolution:

$$\min_{\boldsymbol{P}} \quad \text{trace}\,\boldsymbol{P}$$

$$-\begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^T \begin{bmatrix} 0 & \boldsymbol{P} \\ \boldsymbol{P} & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} X \begin{bmatrix} C & D \end{bmatrix} \geq 0$$

- Heuristic : replace AREs by LMIs

$$\min_{\boldsymbol{P_N},\boldsymbol{P_D}} \quad \text{trace}\,\boldsymbol{P_N} + \text{trace}\,\boldsymbol{P_D}$$

$$-\begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^T \begin{bmatrix} 0 & \boldsymbol{P_N} \\ \boldsymbol{P_N} & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \boldsymbol{X_N} \begin{bmatrix} C & D \end{bmatrix} \geq 0$$

$$-\begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^T \begin{bmatrix} 0 & \boldsymbol{P_D} \\ \boldsymbol{P_D} & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \boldsymbol{X_D} \begin{bmatrix} C & D \end{bmatrix} \geq 0$$

$$\mathscr{L}\mathscr{M}(\boldsymbol{P_N}, \boldsymbol{P_D}, \boldsymbol{X_N}, \boldsymbol{X_D}, \boldsymbol{P}_i) \geq 0$$

Link ARE and LMI

- $ARE(\boldsymbol{P_R}) = 0$

$$A^T(-\boldsymbol{P_R}) + (-\boldsymbol{P_R})A + C^T XC - (-\boldsymbol{P_R}B + C^T XD)(D^T XD)^{-1}(-\boldsymbol{P_R}B + C^T XD)^T = 0$$

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- Heuristic : replace AREs by LMIs

$$\min_{\boldsymbol{P_N}, \boldsymbol{P_D}} \quad \operatorname{trace} \boldsymbol{P_N} + \operatorname{trace} \boldsymbol{P_D}$$

$$-\begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^T \begin{bmatrix} 0 & \boldsymbol{P_N} \\ \boldsymbol{P_N} & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \boldsymbol{X_N} \begin{bmatrix} C & D \end{bmatrix} \geq 0$$

$$-\begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^T \begin{bmatrix} 0 & \boldsymbol{P_D} \\ \boldsymbol{P_D} & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \boldsymbol{X_D} \begin{bmatrix} C & D \end{bmatrix} \geq 0$$

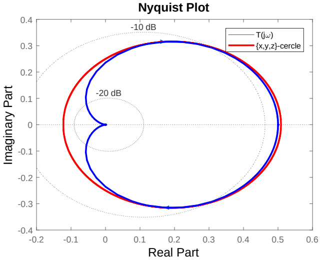
$$\mathscr{L}\mathscr{M}(\boldsymbol{P_N}, \boldsymbol{P_D}, \boldsymbol{X_N}, \boldsymbol{X_D}, \boldsymbol{P}_i) \geq 0$$

Illustration

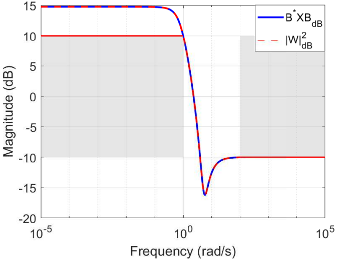
- Example with lossy dissipative $T(s)$

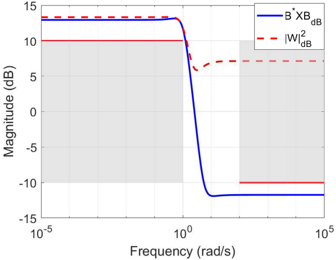
$$T(s) = \frac{1}{(s+1)(s+2)}$$

$$\begin{bmatrix} T(j\omega) \\ 1 \end{bmatrix}^* \underbrace{\begin{bmatrix} x & y \\ y & z \end{bmatrix}}_{\begin{bmatrix} -1 & 0.193 \\ 0.193 & 0.0624 \end{bmatrix}} \begin{bmatrix} T(j\omega) \\ 1 \end{bmatrix} \geq 0$$



- With and without factorization error management

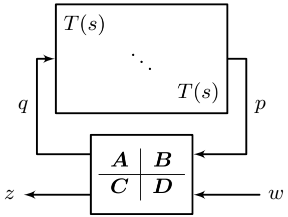


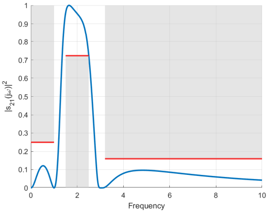


Contents

To sum up

- Synthesis of the interconnection of homogeneous LTI systems for frequency filtering





- Contributions based on Robust Control methods and tools
  - → Original problem formulation
  - → Revisit, extend and generalize the usual convex approach for frequency filter synthesis

- Main remaining issue: solve coupled AREs $\Leftarrow$ solve convex problem?