Explicit solutions to Generalized Lyapunov Matrix Inequality

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Presentation Outline

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Lyapunov Equation

Consider the Linear Time Invariant system dynamics

$$\dot{x} = Ax, \quad x(0) = x_0 \in \mathbb{R}^n, \tag{1}$$

we study its stability properties via the Lyapunov Inequality

$$PA + A^\top P \prec 0, \quad P = P^\top \succ 0. \tag{2}$$

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The standard approach to compute such a P involves an additional matrix $Q = Q^\top \succ 0$ [1], so that we can define the so-called Lyapunov Equation

$$PA + A^\top P = -Q, \tag{3}$$

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Fact: Such a solution P to (3), exists and is unique if and only if A has all negative real parts, and such a P is given by

$$P = \int_0^\infty \exp(A^\top s) Q \exp(As) ds$$

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Parameters of exponential convergence

Definition (Global Exponential Convergence)

A dynamical system $\dot{x} = Ax$ with sate $x \in \mathbb{R}^n$ is said to have a Global Exponential Convergence if there exist $\kappa \in \mathbb{R}_{\geq 1}, \alpha \in \mathbb{R}_{>0}$ such that

$$|x(t)| \leq \kappa \exp(-\alpha t)|x_0|, \quad \forall x(0) = x_0 \in \mathbb{R}^n, \tag{4}$$

κ is called the *scaling factor* and α the *convergence rate*.

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The pair P, Q, with P solution of (3) can be exploited to determine such a κ and α, i.e., we have

$$|x(t)| \leq \frac{\sigma_{\max}(P)}{\sigma_{\min}(P)} \exp\left(-\frac{\sigma_{\min}(Q)}{2\sigma_{\max}(P)}\, t\right) |x_0| \quad \forall\, t \geq 0, \tag{5}$$

where $\kappa := \mu(P)$ and $\alpha = \frac{\sigma_{\min}(Q)}{2\sigma_{\max}(P)}$.

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Question: How do we link the eigenvalues of P to the algebraic properties of A and to the choice of Q?

Lyapunov Inequality

Another way to get the values of κ and α is to solve the Lyapunov Inequality, i.e.,

$$PA + A^\top P \preceq -2\alpha P. \tag{6}$$

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Problem: What happens when A has a non-trivial Jordan Block form?

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A recap on the Jordan Block form

Given a matrix A of dimensions $n \times n$, its eigenvalues are ordered $\Re\{\lambda_1\} \geq \Re\{\lambda_2\} \geq \ldots \geq \Re\{\lambda_n\}$. Let $m \leq n$ be the total number of linearly independent eigenvectors $T_i^1 \neq 0$ [2] relative to an eigenvalue $\bar{\lambda}_i \in \sigma(A)$, $i = 1, \ldots, m$, such that

$$AT_i^1 = \bar{\lambda}_i T_i^1 \quad \forall i = 1, \ldots, m.$$

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Definition (Jordan blocks dimension)

For each $i \in \{1, \ldots, m\}$, we define the values[a] $g_i \geq 1$ satisfying $\sum_{i=1}^{m} g_i = n$, such that there exist $g_i - 1$ linearly independent generalized eigenvectors $T_i^k \neq 0$, for $k = 2, \ldots, g_i$, associated to the corresponding eigenvalue $\bar{\lambda}_i$ and satisfying

$$(A - \bar{\lambda}_i I)T_i^k = T_i^{k-1} \quad \forall k = 2, \ldots, g_i.$$

[a] g_i is not a geometric multiplicity.

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The standard Jordan form is given by

$$J = TAT^{-1}, \quad J = \text{blckdiag}\left(\bar{\lambda}_1 I_{g_1} + N_{g_1}, \quad , \bar{\lambda}_m I_{g_m} + N_{g_m}\right)$$

where I_g is the identity matrix of dimension g and $N_g \in \mathbb{R}^{g \times g}$ is the ‘shifted’ identity matrix $g \in \mathbb{N}$.

[2] Associated to different Jordan Blocks

A modified Jordan form

For any $\lambda \in \mathcal{C}$ and $g \in \mathbb{N}$, we introduce the matrix $D_g(\lambda) \in \mathbb{R}^{g \times g}$ as

$$D_g(\lambda) := \begin{cases} \mathrm{diag}(1, \lambda, \ldots, \lambda^{g-1}), & \text{if } \lambda \neq 0, \\ I_g, & \text{if } \lambda = 0, \end{cases} \tag{7}$$

and

$$\mathbb{J}_g := I_g + N_g. \tag{8}$$

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We can use the matrix $D_g(\bar{\lambda})$ defined in (7) to obtain

$$D_g^{-1}(\bar{\lambda}) J_{\bar{\lambda}} D_g(\bar{\lambda}) = D_g^{-1}(\bar{\lambda}) \left(\bar{\lambda} I_g + N_g\right) D_g(\bar{\lambda}) = \bar{\lambda} I_g + \bar{\lambda} N_g = \bar{\lambda} \mathbb{J}_g$$

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because $D_g^{-1}(\lambda) N_g D_g(\lambda) = \lambda N_g$. The extension of $D_g(\lambda)$ will be the matrix

$$\mathbb{D} := \operatorname{blckdiag}\left(D_{g_1}(\bar{\lambda}_1), \ldots, D_{g_m}(\bar{\lambda}_m)\right) \tag{9}$$

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Modified Jordan form: It is given by

$$\mathbb{J} := \mathbb{T}^{-1} A \mathbb{T}, \qquad \mathbb{T} := T\mathbb{D}, \tag{10}$$

where

$$\mathbb{J} = \mathrm{blckdiag}\left(\bar{\lambda}_1 \mathbb{J}_{g_1}, \ \ldots \ , \bar{\lambda}_m \mathbb{J}_{g_m}\right) = \Lambda(I + N), \tag{11}$$

with $\Lambda = \mathrm{diag}\left(\lambda_1, \ldots, \lambda_n\right)$ and $N = \mathrm{blckdiag}\left(N_{g_1}, \ldots, N_{g_m}\right)$.

Explicit solutions of Lyapunov Inequality

Given a Hurwitz matrix A, we can explicitly write the norm of the solution to the system dynamics

$$\dot{x} = Ax, \quad x(0) = x_0 \in \mathbb{R}^n, \tag{12}$$

via the Jordan blocks normal form, with $g_{\max} = \max_{i \in \{1,\dots,m\}} \{g_i\}$, as

$$|x(t)| = |T \exp(\Lambda t) \sum_{k=0}^{g_{\max}} \frac{N^k t^k}{k!} T^{-1} x(0)|$$

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Theorem

Consider an asymptotically stable system (12)*, and the transformation matrix* $\mathbb{T}$ *that puts* A *into its Modified Jordan form* $\mathbb{J}$*. Then the scaling factor* κ *and the convergence rate* α *are given by*

$$\kappa = \mu(\mathbb{T}), \ \alpha = -\Re\{\lambda_{\max}(\mathrm{sym}(\mathbb{J}))\}. \tag{13}$$

Where $\lambda_{\max}(\mathrm{sym}(\mathbb{J}))$ can be written as an explicit function of $\bar{\lambda}_i$ and g_i, with $i = 1, \dots, m$.
$P = \mathbb{T}^{-\top}\mathbb{T}^{-1}$ is solution of the Lyapunov Inequality (6).

Spirito & Astolfi "Explicit convergence rate parameters for linear autonomous systems", submitted to MICNON 2024

An example

Take $A = -1(I + N) \in \mathbb{R}^{10\times 10}$, i.e., there is an eigenvalue $\lambda = -1$, with geometric multiplicity $g = n = 10$.

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We depict the evolution of the state norm $|z(t)|$, from initial conditions

$$z_0 = [0.01001979,\ 0.02185996,\ -0.01413963,\ 0.08315555,\ -0.14693675,\ 0.31947075 \\ -0.4683713,\ 0.59627956,\ -0.4827674,\ 0.24631203]^\top. \tag{14}$$

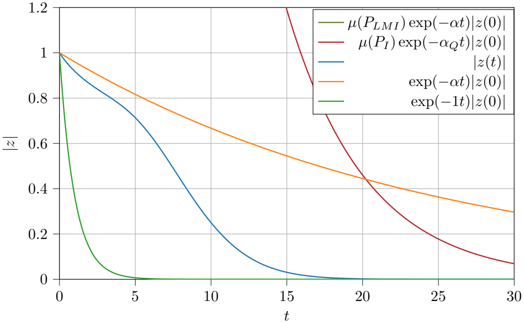

Figure: Evolution of the norm of $z(t)$ and its upper bounds.

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Generalized Lyapunov Inequality

Definition (Matrix inertia)

The inertia of a matrix P with respect to the imaginary axis (c-inertia) is defined by the triplet of integers

$$\mathrm{In}_c(P) = \{\pi_-(P), \pi_0(P), \pi_+(P)\}$$

where $\pi_-(P)$, $\pi_0(P)$ and $\pi_+(P)$, denote the numbers of eigenvalues of P, counted with their algebraic multiplicities, with negative, zero, and positive real part, respectively.

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For a real matrix $A \in \mathbb{R}^{n\times n}$, we take its $m \leq n$ Jordan-distinct eigenvalues in order withh decreasing real part, i.e.,

$$\Re\{\bar{\lambda}_1\} \geq \ldots \geq \Re\{\bar{\lambda}_q\} > 0 > \Re\{\bar{\lambda}_{q+1}\} \geq \ldots \geq \Re\{\bar{\lambda}_m\}, \tag{15}$$

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Problem (Generalized Lyapunov Inequality)

Given a matrix A of c-inertia $\{n-p, 0, p\}$, Find a solution P to the generalized Lyapunov matrix inequality

$$PA + A^\top P \preceq 2\alpha P, \qquad \alpha \in \mathbb{R}, \quad P = P^\top, \tag{16}$$

for some $\alpha \in \mathbb{R}$ such that P is a symmetric matrix of c-inertia $\{p, 0, n-p\}$.

Some explicit solution to Generalized Lyapunov Inequality

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Given $\delta \in (0,1)$, we let ε_δ be defined as $\varepsilon_\delta = \min\{\varepsilon_+^*, \varepsilon_-^*\}$,

$$\begin{aligned} \varepsilon_+^* &= \min_{i\in\{1,\ldots,q\}} \left\{ \cos\left(\pi \frac{g_i}{g_i+1}\right)^{-1} \left[\frac{|\Re\{\bar{\lambda}_q\}|}{|\Re\{\bar{\lambda}_i\}|}(1-\delta) - 1 \right] \right\}, \\ \varepsilon_-^* &= \min_{i\in\{q+1,\ldots,m\}} \left\{ \cos\left(\pi \frac{g_i}{g_i+1}\right)^{-1} \left[\frac{|\Re\{\bar{\lambda}_{q+1}\}|}{|\Re\{\bar{\lambda}_i\}|}(1-\delta) - 1 \right] \right\}. \end{aligned} \tag{18}$$

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Theorem

Let A be a matrix with c-inertia $\{n-p, 0, p\}$. For any $\delta \in (0,1)$ there exists $\varepsilon_\delta > 0$ defined in (18), such that, for any $\varepsilon \in (0, \varepsilon_\delta]$, the following P with c-inertia $\{p, 0, n-p\}$

$$\begin{aligned} P := \mathbb{T}^{-\top}\mathbb{I}\mathbb{D}_\varepsilon^{-2}\mathbb{T}^{-1}, \qquad \mathbb{T} := T\mathbb{D}, \\ \mathbb{I} := \mathrm{blckdiag}\left(-I_p, I_{n-p}\right), \end{aligned} \tag{19}$$

is a solution to the generalized Lyapunov matrix inequality (16) with $\alpha \in \mathbb{R}$ satisfying

$$(1-\delta)\Re\{\bar\lambda_q\} \geq \alpha \geq (1-\delta)\Re\{\bar\lambda_{q+1}\}. \tag{20}$$

Spirito & Astolfi "Some explicit solutions and bounds to the generalized Lyapunov matrix inequality", submitted to S&CL

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Thank you for your attention.

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Any Questions?