Using model order reduction to compute fast frequency sweeps of vibro-acoustic systems described by indirect boundary element models

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1 Problem description

2 System assembly: an efficient interpolation approach
   - Frequency dependency of the system matrices
   - Frequency interpolation of the system matrices
   - Determining the frequency windows

3 System solving: computing Padé approximants

4 Proposed algorithm

5 Numerical examples
   - Exterior application
   - Interior/exterior application

6 Conclusion
Using model order reduction to compute...

What is model order reduction?

= mathematical technique to reduce the complexity of dynamical systems.

- First used in control (controller’s complexity same as that of the system to be controlled). Problems:
  - storage
  - accuracy
  - computational speed

- Later used for speeding up simulations and decrease the time-to-market of products when parallelization is not feasible

- MOR offers a trade-off between accuracy and complexity
  - For non-minimal systems, the reduction to minimal system is error-free
  - There are systems for which MOR is not suitable.
Using MOR to compute fast frequency sweeps...
Why frequency, why frequency sweeps and why fast sweeps?

Analyzing systems in the **frequency domain** allows one to infer properties regarding resonances (e.g., vibro-acoustic systems), filtering properties (e.g., electrical systems), etc.
Using MOR to compute fast frequency sweeps...

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A frequency sweep amounts to solving a linear system \( A(f)x(f) = b(f) \) (e.g., \( A(f) = j2\pi f I - A \)) at many frequencies.
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A **frequency sweep** amounts to solving a linear system $A(f)x(f) = b(f)$ (e.g., $A(f) = j2\pi f I - A$) at many frequencies.

A **fast frequency sweep** avoids solving the large linear system for each frequency by using extrapolation.
Using... of vibro-acoustic systems...

What are vibro-acoustic systems?

Vibro-acoustics or structural acoustics is the study of the acoustic waves in structures and how they interact with and radiate into adjacent media.
Using... of vibro-acoustic systems...

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The steady-state acoustic pressure generated by harmonic excitations at wavenumber $k$ is described by the wave (Helmholtz) equation:

$$\nabla^2 p + k^2 p = 0 \quad \text{in the domain } \Omega,$$

where $\nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}$ denotes the complex amplitude of the pressure representing a time harmonic variation given by $p' = \text{Re}(p e^{i\omega t})$ with $\omega$ the angular frequency, $c$ the speed of sound in the domain $\Omega$. 
Using... of vibro-acoustic systems...

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where

- $\nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}$
- $p$ denotes the complex amplitude of the pressure representing a time harmonic variation given by $p' = \text{Re}(pe^{i\omega t})$
- $k = \omega/c$, with $\omega$ the angular frequency, $c$ the speed of sound
- domain $\Omega$ containing an inviscid compressible fluid.
Using ... by **indirect boundary element models**

What is IBEM?

Indirect formulation is employed for interior and exterior problems.

In IBEM, the unknowns are $\sigma = \frac{\partial p^+}{\partial n^+} - \frac{\partial p^-}{\partial n^-}$ (single layer potential) and $\mu = p^+ - p^-$ (double layer potential). Acoustic pressure at field point $X$ is

$$p(X) = \int_S \left( G(X, Y)\sigma(Y) - \frac{\partial G(X, Y)}{\partial n(Y)} \mu(Y) \right) dS.$$ 

$G(X, Y) = \frac{\exp(-ikR)}{4\pi R}$, $R = |X - Y|$ is the 3D Green’s function.
What is IBEM?

The surface $S$ is discretized into boundary elements $S \approx \sum_{e} S^e$. The unknowns are expressed at the discretization points (nodes) as

$$\mu(X) = N_\mu \cdot \mu, \quad \sigma(X) = N_\sigma \cdot \sigma$$

with $\mu$ and $\sigma$, vectors of nodal double and single layer potentials, and $N_\mu$ and $N_\sigma$, shape functions. This yields the system of equations of size $N_{DOF}$:

$$\begin{bmatrix} A_{\sigma\sigma} & A_{\sigma\mu} \\ A_{\sigma\mu}^H & A_{\mu\mu} \end{bmatrix} \begin{bmatrix} \sigma \\ \mu \end{bmatrix} = \begin{bmatrix} 0 \\ b_{\mu} \end{bmatrix}$$

$$A(f) \begin{bmatrix} \sigma \\ \mu \end{bmatrix} = \begin{bmatrix} b(f) \end{bmatrix}$$

with the matrix $A$ being complex and symmetric.
Challenges and goals

Challenges for IBEM:

- system matrix $A(f)$ is dense
- assembling and solving are equally expensive
- complicated frequency dependency because of $G(X, Y) = \frac{\exp(-ikR)}{4\pi R}$. 
Challenges and goals

Challenges for IBEM:

- system matrix $A(f)$ is dense
- assembling and solving are equally expensive
- complicated frequency dependency because of $G(X, Y) = \exp\left(-i k R \right) \frac{4 \pi R}{4 \pi R}$.

Goals for FFS:

- avoid assembling the system matrices at each frequency: perform polynomial interpolation on appropriate frequency scaled matrices
- avoid solving a linear system at each frequency: employ Padé approximations
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Frequency dependency of the system matrices

Motivation: understand this to be able to design well-suited strategies to approximate $A(f)$ by interpolating appropriate frequency-scaled quantities.

Figure: Frequency behavior of various entries
Scaling the system matrices

The scaled entries are \( \hat{A}_{m,n} = \begin{cases} e^{ikR_{m,n}} A_{m,n} , & m, n = 1, \ldots, N_{\text{DOF}}. \\ k A_{m,n} & \end{cases} \)

**Figure:** The effect of applying the scaling factor on two matrix entries
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The scaled matrices are interpolated by Lagrange polynomials:

$$\hat{A}_{m,n}(k) = \sum_{j=1}^{N+1} P_j(k) \hat{A}_{m,n}(k_j), \quad P_j(k) = \prod_{i=1}^{N+1} \frac{k - k_i}{k_j - k_i}$$

with $P_j(k) = 1$, for $k = k_j$, and $P_j(k) = 0$, for $k \neq k_j$.

We call the Lagrange nodes $k_j$ the master wavenumbers (frequencies).

The interpolation order $N$ can be 1 (linear interpolation as in [Benthien, 1989]), 2, or higher. This approach requires assembling and storing $N + 1$ system matrices, so one needs to find a trade-off.
Inverse scaling of the system matrices

The approximated system matrix entries are obtained by multiplying $\tilde{A}[m,n](k)$ with the inverse of the scaling factor:

$$
\tilde{A}[m,n](k) = \left\{ \begin{array}{ll}
e^{-ikR[m,n]}\tilde{A}[m,n](k) = \sum_{j=1}^{N+1} P_j(k)e^{i(k_j-k)R[m,n]}A[m,n](k_j) \\
\frac{1}{k}\tilde{A}[m,n](k) = \sum_{j=1}^{N+1} P_j(k)\frac{k_j}{k}A[m,n](k_j). 
\end{array} \right.
$$

Remark The approximated matrix is equal to the original at $k_j$: $\tilde{A}(k_j) = A(k_j)$. 
Inverse scaling of the system matrices

The approximated system matrix entries are obtained by multiplying \( \tilde{A}_{[m,n]}(k) \) with the inverse of the scaling factor:

\[
\tilde{A}_{[m,n]}(k) = \begin{cases} 
  e^{-ikR_{[m,n]}} \tilde{A}_{[m,n]}(k) = \sum_{j=1}^{N+1} P_j(k) e^{i(k_j-k)R_{[m,n]}} A_{[m,n]}(k_j) \\
  \frac{1}{k} \tilde{A}_{[m,n]}(k) = \sum_{j=1}^{N+1} P_j(k) \frac{k_j}{k} A_{[m,n]}(k_j).
\end{cases}
\]

Remark The approximated matrix is equal to the original at \( k_j: \tilde{A}(k_j) = A(k_j) \).

Recap To avoid assembling the system matrix at each \( f \) [Benthien, 1989]:
- assemble & store matrices @ master frequencies
- perform the interpolation described above @ slave frequencies.
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Determining the frequency windows

**Motivation:** A large polynomial order $N$ required when performing interpolation over entire frequency band $\Rightarrow$ smaller intervals.

A few **representative matrix entries** are carefully chosen and assembled at all frequencies. These entries are interpolated simultaneously by an order $N$ polynomial with an a-priori or user-defined accuracy.

**Windows** determined as intervals which contain highest possible number of frequencies in ascending order such that the fitting error for the representative entries inside the interval is below the tolerance.
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The Taylor series for $x(f)$ around $f_0$, the expansion frequency:

$$x(f) = x(f_0) + x'(f_0)(f - f_0) + \ldots + x^{(q)}(f_0)\frac{(f - f_0)^q}{q!} + \ldots $$
Taylor series for vector functions

Recall that we wish to solve $A(f)x(f) = b(f)$ for many $f$.

Notation: $w_{q+1} = \frac{x^{(q)}(f_0)}{q!}$, $A_q = \frac{A^{(q)}(f_0)}{q!}$, $b_q = \frac{b^{(q)}(f_0)}{q!}$.

\[
x(f) = A^{-1}b_0 = w_1,
\]

\[
x'(f) = A^{-1}(b_1 - A_1w_1) = w_2,
\]

\[
x^{(q)}(f_0) = A^{-1}\left(b_q - \sum_{i=1}^{q} A_i w_{q-i+1}\right) = w_{q+1}.
\]

This moments-computation process is ill-conditioned.
A Padé approximant of order \([q_1/q_2]\) of a scalar \(g(f)\) is a rational function

\[
\frac{a_0 + a_1(f - f_0) + \ldots + a_{q_1}(f - f_0)^{q_1}}{1 + b_1(f - f_0) + \ldots + b_{q_2}(f - f_0)^{q_2}},
\]

whose Taylor expansion around \(f_0\) matches the first \(q = q_1 + q_2 + 1\) terms in the Taylor series of \(g(f)\).
Padé approximants

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\]

whose Taylor expansion around \(f_0\) matches the first \(q = q_1 + q_2 + 1\) terms in the Taylor series of \(g(f)\).

Asymptotic Waveform Evaluation (AWE)
Given derivatives of \(g(f)\) up to order \(q\), a linear system with a Hankel matrix is solved for the coefficients \(a_0, \ldots, a_{q_1}\) and \(b_1, \ldots, b_{q_2}\). For vector functions, such an approximant must be computed for each component of the solution vector \(x(f)\).

Very ill-conditioned and time consuming!
Galerkin Asymptotic Waveform Evaluation

Galerkin AWE amounts to forming the moment-matching subspace

\[ \mathbf{W}_q = [\mathbf{w}_1 \mathbf{w}_2 \ldots \mathbf{w}_q] \in \mathbb{C}^{N_{DOF} \times q} \]

and imposing that the residual is perpendicular to \( \mathbf{W}_q \), yielding the following solution vector

\[
\mathbf{x}_q(f) = \mathbf{W}_q \left( \mathbf{W}_q^H \mathbf{A}(f) \mathbf{W}_q \right)^{-1} \left( \mathbf{W}_q^H \mathbf{b}(f) \right).
\]

It can be proven that the approximated vector \( \mathbf{x}_q(f) \) matches the solution, as well as the value of \( q - 1 \) derivatives around \( f_0 \).
Galerkin Asymptotic Waveform Evaluation

Galerkin AWE amounts to forming the moment-matching subspace $W_q = \begin{bmatrix} w_1 & w_2 & \ldots & w_q \end{bmatrix} \in \mathbb{C}^{N_{DOF} \times q}$ and imposing that the residual is perpendicular to $W_q$, yielding the following solution vector

$$x_q(f) = W_q \left( W_q^H A(f) W_q \right)^{-1} \left( W_q^H b(f) \right).$$

It can be proven that the approximated vector $x_q(f)$ matches the solution, as well as the value of $q - 1$ derivatives around $f_0$.

Advantages of GAWE:

- a much smaller linear system needs to be solved, namely
  $$(W_q^H A(f) W_q)^{-1} (W_q^H b(f))$$
  where $W_q^H A(f) W_q$ is of size $q \times q$
  yields the Padé approximant of the entire vector $x(f)$. 

WCAWE [Slone et al., 2003]

Uses GAWE with the moments computed in a well conditioned manner.
**WCAWE [Slone et al., 2003]**

Uses GAWE with the moments computed in a well conditioned manner.

Before: \( w_{q+1} = A_0^{-1} \left( b_q - \sum_{i=1}^{q} A_i w_{q-i+1} \right) \).

\[ \text{WCAWE: } \tilde{w}_{q+1} = A_0^{-1} \left( \sum_{i=1}^{q} b_i c_i - A_1 w_q - \sum_{i=2}^{q} A_j W_{q-i+1} d_i \right), \]

where \( c_i, d_i \) are correction factors.

Moreover, they are orthonormalized via a modified Gram-Schmidt process:

\[ \text{for } i = 1, \ldots, q - 1 \]
\[ U_{[i,q]} = w_i^H \tilde{w}_q \]
\[ \tilde{w}_q = \tilde{w}_q - U_{[i,q]} w_j \]
\[ U_{[q,q]} = \| \tilde{w}_q \|, \ w_q = \tilde{w}_q U_{[q,q]}^{-1}. \]
Derivatives of the system matrix

The $q^{th}$ derivative at the expansion wave number $k_0$: 

\[
\frac{\partial^q \tilde{A}_{[m,n]}(k)}{\partial k^q} \bigg|_{k=k_0} = \sum_{j=1}^{N+1} \frac{\partial^q}{\partial k^q} \left( \frac{P_j(k)}{k} \right) \bigg|_{k=k_0} k_j A_{[m,n]}(k_j),
\]

for entries scaled by $k$, and 

\[
\frac{\partial^q \tilde{A}_{[m,n]}(k)}{\partial k^q} \bigg|_{k=k_0} = \sum_{j=1}^{N+1} \frac{\partial^q}{\partial k^q} \left( P_j(k) e^{-ikR_{[m,n]}} \right) \bigg|_{k=k_0} e^{ik_j R_{[m,n]} A_{[m,n]}(k_j)}
\]

otherwise.
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Proposed algorithm

1. Choose a few representative matrix entries, assemble at all frequencies
2. Apply polynomial interpolation of order $N$ to scaled entries with deviation $d_{\text{tol}} = 10^{-4}$ ⇒ frequency windows
3. Each frequency window contains $N + 1$ master frequencies ⇒ set the middle one as the expansion frequency
4. Apply WCAWE inside each window by matching moments at the expansion frequency
   1. Start with a small moment subspace
   2. Add new vectors to the moments subspace as long as residual
      $$r(f) = \frac{\|\tilde{A}(f)x_q(f) - b(f)\|_\infty}{\|b(f)\|_\infty}$$ is larger than $\varepsilon_{\text{tol}} = 10^{-3}$
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**Sphere with rigid cap**

The pressure outside the sphere verifies:

\[
p(r, \theta) = \frac{-i \rho c v_0(f)}{2} \sum_{n=0}^{\infty} \left[ \tilde{P}_{n-1} (\cos \alpha) - \tilde{P}_{n+1} (\cos \alpha) \right] \frac{h_n(kr)}{h'_n(ka)} \tilde{P}_n(\cos \theta),
\]

where:
- \( r \), distance to evaluation point,
- \( h_n \), spherical Hankel functions of first kind,
- \( \tilde{P}_n \), Legendre polynomials,
- \( v_0(f) \), uniform normal velocity of spherical cap,
- and \( a \), radius of sphere.

The infinite summation truncated at \( 2k \).
Parameters for the problem

- sphere radius $a = 0.6$ m, angle defining the vibrating cap $\alpha = \pi/3$ rad
- sound speed is $c = 340$ m/s, fluid density is $\rho = 1.225$ kg/m$^3$
- cap normal velocity $v_0(f)$ is taken as the response of a classical 3 DOF mass-spring-damper: $M_1 = 60$, $M_2 = 40$, $M_3 = 20$ (kg); $K_{1,2,3} = 2.7 \times 10^5$ (N/m); $C_{1,2,3} = 20$ (Ns/m) $\Rightarrow$ 3 resonances
- mesh with 8653 nodes, 17302 triangular elements $\Rightarrow N_{\text{DOF}} = 15136$
- $F = [200, 1000]$ Hz with 1 Hz increment (1101 individual frequencies)
- $N = 2$ interpolation for 8 representative matrix entries $\Rightarrow$ 10 frequency windows (4 min, 95% on assembly)
Results

2 h 07 min vs 58 h ⇒ speed up factor of 27.4
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Car engine compartment

Motivation:

- Vehicles should comply with noise emission regulations
- Engine is a major contributor to vehicle pass by noise
- Acoustic treatments in various locations of engine compartment (e.g., under-bonnet, dash, firewall, floor, etc) are employed
- Interior/exterior acoustics problem: cavity with interior resonances and acoustic radiation in free field.

![Diagram of car engine compartment with dimensions and views: top view, 3D isometric view, frontal view, lateral view.](image)
Parameters for the problem

- Mesh with 9,326 nodes, 18,408 elements $\Rightarrow N_{DOF} = 10,151$
- Discontinuous impedance is applied on the internal sides of the bonnet (light grey elements) and the firewall (dark grey elements)
- Remaining elements in white are considered acoustically rigid
- 6 field points measured by microphones
- A spherical point source is located at $(x = 3 \text{ m}, y = 7 \text{ m}, z = 0 \text{ m})$
- $F = [100, 1000] \text{ Hz}$ with 1 Hz frequency increment (901 frequencies)
- 29 frequency windows
Results

(b) Residual and interpolation frequency windows (vertical lines).

5 h 07 min vs 55 h ⇒ speed up factor of 10
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MOR for computing FFS for IBEM:

- avoids **assembling and storing** the system matrix at each frequency
- avoids **solving** the linear system at each frequency

**Current work**: compute moments at several expansion frequencies per window and combine these subspaces (multi-point or rational approach).
Thank you for your attention!