Toward nonlinear tracking and rejection using LPV control

Gérard Scorletti

with

V. Fromion, S. de Hillerin

Laboratoire Ampère (CNRS)  Ecole Centrale de Lyon  COMUE Université de Lyon

Sessions GT Identification et MOSAR
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1. A typical control problem

2. Extension of $H_\infty$ control approach to nonlinear systems

3. Nonlinear $\mathcal{L}_2$ gain control using LPV

4. Limitations of the $\mathcal{L}_2$ gain control

5. Typical specs are ensured using $\mathcal{L}_2$ incremental gain

6. LPV control for ensuring typical nonlinear specs
A typical control problem
Usual control problem involves both tracking and rejection specifications.

Let us focus on a simple problem: Given a nonlinear plant $G_{NL}$, find $K_{NL}$ such that

$$G_{NL} K_{NL}$$

Typical control specs:

- tracking of step reference with a null static error and a response time $\leq 0.1$ s
- rejection of step disturbance at the plant input
- limited control energy
Nonlinear plant under consideration

\[ y = G_{NL}(u) \text{ with } \]

\[
\begin{align*}
\dot{x}_1(t) &= -100 \varphi(x_1(t)) - 70x_2(t) + 300u(t) \\
\dot{x}_2(t) &= 70x_1(t) - 14x_2(t) \\
y(t) &= x_1(t)
\end{align*}
\]

with \( \varphi \) defined by

that is

\[ 0 \leq \varphi(x_1) \leq 2x_1 \]
Extension of $H_\infty$ control approach to nonlinear systems
Typical approach for LTI plant and controller

LTI case: $H_\infty$ control approach

- Integral control
- With weighting functions $W_1, W_2, W_3$ suitable for the specs
- Compute $K_{LTI}$ such that $H_\infty$ norm of the closed loop system less than 1
$H_\infty$ control approach (recall)

Given an (augmented) LTI plant $P_{LTI}$

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B_w w(t) + B_u u(t) \\
z(t) &= C_z x(t) + D_{zw} w(t) + D_{zu} u(t) \\
y(t) &= C_y x(t) + D_{yw} w(t) + D_{yu} u(t)
\end{align*}
\]

Compute an LTI controller $K_{LTI}$

\[
\begin{align*}
\dot{x}(t) &= A_K \bar{x}(t) + B_K y(t) \\
u(t) &= C_K \bar{x}(t)
\end{align*}
\]

Such that

\[
\|T_{w\to z}\|_\infty \leq 1
\]

- Efficient solution (Riccati or LMI)
Typical approach for LTI plant and controller

Reference signal

Output signal
Typical approach for LTI plant and controller

Reference signal

Output signal
- Nice steady state behaviour
  - Stability of LTI system $\Rightarrow$ for constant input, output $\longrightarrow$ constant
  - Stability + integral control $\Rightarrow$ null static error

- Nice transient behavior
  - Inequality on the weighted $H_{\infty}$ norm of the closed loop system
First possible extension using the $L_2$ gain

$H_\infty$ Norm

$H_\infty$ Controller

LTI Plant

NL Plant
First possible extension using the $\mathcal{L}_2$ gain
First possible extension using the $\mathcal{L}_2$ gain

Given an (augmented) nonlinear plant $P_{NL}$

\[
\begin{align*}
\dot{x}(t) &= f(x(t), w(t), u(t)) \\
z(t) &= g(x(t), w(t), u(t)) \\
y(t) &= h(x(t), w(t))
\end{align*}
\]

Compute a nonlinear controller $K_{NL}$

\[
\begin{align*}
\dot{x}(t) &= f_K(x(t), y(t)) \\
u(t) &= g_K(x(t), y(t))
\end{align*}
\]

Such that the $\mathcal{L}_2$ gain of the closed loop system is less than 1: for all $w$

\[
\forall T > 0, \quad \int_0^T z(t)^T z(t) \, dt \leq \int_0^T w(t)^T w(t) \, dt
\]
For LTI system, \( \| T_{w \rightarrow z} \|_\infty \leq 1 \) is equivalent to the \( \mathcal{L}_2 \) gain is less than 1: for all \( w \)

\[
\forall \ T > 0, \quad \int_0^T z(t)^T z(t) \, dt \leq \int_0^T w(t)^T w(t) \, dt
\]

A natural idea is then to extend the \( H_\infty \) control to nonlinear systems by the \( \mathcal{L}_2 \) gain control: usually referred to as “nonlinear \( H_\infty \) control”
Two questions

1. How to compute a solution (nonlinear controller) to the $L_2$ gain control problem?
   
   No efficient direct approach $\Rightarrow$ indirect approach: Quasi LPV control

2. Does the $L_2$ gain controller ensures nice tracking and rejection properties?

   See application on the illustrative example
Two questions

1. How to compute a solution (nonlinear controller) to the $\mathcal{L}_2$ gain control problem?

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Two questions

1. How to compute a solution (nonlinear controller) to the $\mathcal{L}_2$ gain control problem?

   No efficient direct approach $\Rightarrow$ indirect approach: Quasi LPV control

2. Does the $\mathcal{L}_2$ gain controller ensures nice tracking and rejection properties?

   See application on the illustrative example
Nonlinear $\mathcal{L}_2$ gain control using LPV
Given a Linear Parameter Varying (LPV) plant $G_{LPV}$

\[
\begin{align*}
\dot{x}(t) &= A(\theta(t))x(t) + B_1(\theta(t))w(t) + B_2(\theta(t))u(t) \\
z(t) &= C_1(\theta(t))x(t) + D_{11}(\theta(t))w(t) + D_{12}(\theta(t))u(t) \\
y(t) &= C_2(\theta(t))x(t) + D_{21}(\theta(t))w(t) + D_{22}(\theta(t))u(t)
\end{align*}
\]

- $\theta(t) =$ vector of time varying parameters, measured in real-time, which belong to a given interval
- $A(\cdot), B_1(\cdot), \ldots$ rational function of $\theta_i(t)$

Compute an LPV controller $K_{LPV}$

\[
\begin{align*}
\dot{x}(t) &= A_K(\theta(t))\bar{x}(t) + B_K(\theta(t))y(t) \\
u(t) &= C_K(\theta(t))\bar{x}(t)
\end{align*}
\]

Such that the $\mathcal{L}_2$ gain of the closed loop system is less than 1: for all $w$

\[
\forall \ T > 0, \quad \int_0^T z(t)^T z(t) \ dt \leq \int_0^T w(t)^T w(t) \ dt
\]
Interest of the LPV control problem

- Solutions of the LPV control problem can be computed using LMI optimization.

- A strong motivation of the LPV control problem is to propose, in contrast with the gain scheduling control, a rigorous solution to the nonlinear $\mathcal{L}_2$ gain control problem.\(^1\)

---

 Connecting LPV control and nonlinear $L_2$ gain control via quasi LPV

To the (augmented) nonlinear plant $P_{NL}$

$$
\begin{align*}
\dot{x}(t) &= f(x(t), w(t), u(t)) \\
z(t) &= g(x(t), w(t), u(t)) \\
y(t) &= h(x(t), w(t))
\end{align*}
$$

(1)

is associated an LPV plant $P_{LPV}$

$$
\begin{align*}
\dot{x}(t) &= A(\theta(t))x(t) + B_1(\theta(t))w(t) + B_2(\theta(t))u(t) \\
z(t) &= C_1(\theta(t))x(t) + D_{11}(\theta(t))w(t) + D_{12}(\theta(t))u(t) \\
y(t) &= C_2(\theta(t))x(t) + D_{21}(\theta(t))w(t) + D_{22}(\theta(t))u(t)
\end{align*}
$$

(2)

such that with

$$
\Omega_{NL} = \left\{ (x, z, y, w, u) \mid (1) \text{ is satisfied} \right\}
$$

and

$$
\Omega_{LPV} = \left\{ (x, z, y, w, u) \mid (2) \text{ is satisfied} \right\}
$$

we have

$$
\Omega_{NL} \subset \Omega_{LPV}
$$
Extension of $H_\infty$ to nonlinear systems: $\mathcal{L}_2$ gain?

An LPV model is a differential inclusion
Limitations of the $\mathcal{L}_2$ gain control
Application to the illustrative case of the quasi LPV method

To the nonlinear plant $G_{NL}$:

\[
\begin{align*}
\dot{x}_1(t) &= -100\varphi(x_1(t)) - 70x_2(t) + 300u(t) \\
\dot{x}_2(t) &= 70x_1(t) - 14x_2(t) \\
y(t) &= x_1(t)
\end{align*}
\]

we associate the LPV plant $G_{LPV}$:

\[
\begin{align*}
\dot{x}(t) &= A_G(\theta(t))x(t) + \begin{bmatrix} 300 \\ 0 \end{bmatrix} u(t), \quad \theta(t) \in [0, 2] \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)
\end{align*}
\]

with\(^2\)

\[
A_G(\theta(t)) = \begin{bmatrix} 0 & -70 \\
70 & -14 \end{bmatrix} + \theta(t) \begin{bmatrix} -100 & 0 \\
0 & 0 \end{bmatrix}
\]

\(^2\theta(t) = \frac{\varphi(y(t))}{y(t)}\) with $0 \leq \varphi(y) \leq 2y$
For step tracking and step rejection, an LPV controller is computed using the augmented plant defined as follows.

\[
\begin{align*}
K_{LPV} & = \int \begin{bmatrix} W_1 z_1(t) + W_2 z_2(t) \\
G_{LPV} y(t) - u(t) \end{bmatrix} \\
& = \begin{bmatrix} z_1(t) \\
z_2(t) \end{bmatrix} \\
& = \begin{bmatrix} \int & W_1 \\
W_2 & G_{LPV} \end{bmatrix} \begin{bmatrix} z_1(t) \\
z_2(t) \end{bmatrix} \\
& = \begin{bmatrix} z_1(t) \\
z_2(t) \end{bmatrix}
\end{align*}
\]

Thanks to the embedding process, this controller is a solution to the nonlinear $\mathcal{L}_2$ gain control problem.

Does the controller ensure satisfying tracking and rejection?
Behaviour of the LPV closed loop system with respect to initial conditions & zero inputs

For a given function $\theta(t) \in [0, 2]$

Output $y(t)$ for different initial conditions $x_0$, $r = 0$, $b = 0$
Output $y(t)$ for different initial conditions $x_0$, $r = 0$, $b = 0$

($\mathcal{L}_2$ gain) stability ensures convergence to 0 for different initial conditions
Behaviour of the LPV closed loop system with respect to step reference & disturbance

For a given function $\theta(t) \in [0, 2]$

Output $y(t)$ for $x_0 = 0$, a step disturbance at 4s and a square reference signal
Output $y(t)$ for $x_0 = 0$, a step disturbance at 4s and a square reference signal
Behaviour of the LPV closed loop system with respect to step reference & disturbance

Output $y(t)$ for $x_0 = 0$, a step disturbance at 4s and a square reference signal

($L_2$ gain) stability + integral control do not ensure step tracking/rejection for LTV system
Behaviour of the nonlinear closed loop system with respect to step reference & disturbance

Output $y(t)$ for $x_0 = 0$, a step disturbance at 4s and a square reference signal
Behaviour of the nonlinear closed loop system with respect to step reference & disturbance

Output $y(t)$ for $x_0 = 0$, a step disturbance at 4s and a square reference signal

($\mathcal{L}_2$ gain) stability + integral control do not ensure step tracking/rejection for nonlinear system

Except perhaps for inputs close to 0
Discussion on the illustrative example

- For inputs close to 0, the $\mathcal{L}_2$ gain control solution reduces to the $H_\infty$ one\(^3\)

- Null static errors for step reference & disturbance by integral control depend on the property that for constant inputs, the system signals tend to a constant

- Unfortunately, this property is not ensured by ($\mathcal{L}_2$ gain) stability

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\(^3\) A. J. van der Schaft, “$\mathcal{L}_2$-gain analysis of nonlinear systems and nonlinear state feedback $H_\infty$ control,” *IEEE Trans. Automatic Control*, vol. 37, no. 6, pp. 770–784, June 1992
(\mathcal{L}_2 \text{ gain}) \text{ stability does not ensure a good behavior}

- For periodic inputs: different steady states for input \( u(t) = \sin(2t) \)

\[
G(s) = \frac{909}{(s^2 + 0.1s + 1)(s + 100)}
\]

How to ensure a good behavior?
Typical specs are ensured using $\mathcal{L}_2$ incremental gain
How to ensure a good behaviour? Use the $\mathcal{L}_2$ incremental gain

Nonlinear plant $G_{NL}$

\[
\begin{align*}
\dot{x}(t) &= f(x(t), w(t)) \\
z(t) &= g(x(t), w(t))
\end{align*}
\]

- ($\mathcal{L}_2$ gain) stability if $\exists \gamma \geq 0$, $\forall w$,

\[
\forall T > 0, \quad \int_0^T z(t)^T z(t) \, dt \leq \gamma^2 \int_0^T w(t)^T w(t) \, dt
\]

$\mathcal{L}_2$ gain of $G_{NL}$ ($\|G_{NL}\|_{\mathcal{L}_2}$) = the smallest value of such $\gamma$

- ($\mathcal{L}_2$) incremental (gain) stability if stability and $\exists \eta \geq 0$, $\forall T > 0$, $\forall w_1$, $\forall w_2$,

\[
\int_0^T (z_1(t) - z_2(t))^T (z_1(t) - z_2(t)) \, dt \leq \eta^2 \int_0^T (w_1(t) - w_2(t))^T (w_1(t) - w_2(t)) \, dt
\]

Incremental $\mathcal{L}_2$ gain of $G_{NL}$ ($\|G_{NL}\|_\Delta$) = the smallest value of such $\eta$

- For an LTI system, $H_\infty$ norm = $\mathcal{L}_2$ gain = $\mathcal{L}_2$ incremental gain
Why incremental (\(L_2\)) gain is nice for control performance?

<table>
<thead>
<tr>
<th>Specs \ Norm</th>
<th>LTI</th>
<th>NL</th>
<th>NL</th>
</tr>
</thead>
<tbody>
<tr>
<td>(H_\infty)</td>
<td>(L_2) gain</td>
<td>incremental gain</td>
<td></td>
</tr>
</tbody>
</table>

- Unique steady state: YES | NO | YES
- Convergence of the unperturbed motions: YES | NO | YES
- Constant input → constant output: YES | NO | YES
- T periodic input → T periodic output: YES | NO | YES
- Quantitative perf.: YES | NO | YES
- Robustness | YES | YES | YES

V. Fromion and S. Monaco and D. Normand-Cyrot, The weighted incremental norm approach: from linear to nonlinear \(H_\infty\) control, Automatica 2001

V. Fromion and G. Scorletti. The behavior of incrementally stable discrete time systems, System and Control Letters 2002

V. Fromion, Some results on the behavior of Lipschitz continuous systems, ECC 97
Qualitative specifications

Unique steady state

\[ u_r(t) \]

\[ \tilde{u}_r(t) \]

\[ t_0 \rightarrow t \]

\[ \bar{x}(t) = \phi(t, t_0, x_0, \tilde{u}_r) \]

\[ x(t) = \phi(t, t_0, x_0, u_r) \]

Convergence of the unperturbed motions

\[ \tilde{x}_0 \]

\[ x_0 \]

\[ t_0 \rightarrow t \]
Qualitative specs (II)

Constant (periodic) input → Constant (periodic) output

Diagram:
- Two graphs showing the relationship between time in seconds and acceleration output for constant (periodic) input and constant (periodic) output.
Quantitative specs: Performance

Disturbance attenuation of a set of perturbation $d$, for any initial condition

for $d$ such that $\|W_p^{-1}(d)\|_{2,T} \leq \|d\|_{2,T} \Rightarrow \|y\|_{2,T} \leq \alpha$
Extension of $H_\infty$ to nonlinear systems

- Norme $H_\infty$
- $\mathcal{L}_2$ gain
- Incremental Norm

Controller $H_\infty$

$\mathcal{L}_2$ gain controller

LTI Plant

NL Plant

QUASI-LPV
The incremental norm as a rigorous extension of the $H_\infty$ norm

Given an (augmented) nonlinear plant $P_{NL}$

\[
\begin{align*}
\dot{x}(t) &= f(x(t), w(t), u(t)) \\
z(t) &= g(x(t), w(t), u(t)) \\
y(t) &= h(x(t), w(t))
\end{align*}
\]

Compute a nonlinear controller $K_{NL}$

\[
\begin{align*}
\dot{x}(t) &= f_K(x(t), y(t)) \\
u(t) &= g_K(x(t), y(t))
\end{align*}
\]

Such that the $L_2$ incremental gain of the closed loop system is less than 1: for all $w_1, w_2$

\[
\forall T > 0, \quad \int_0^T (z_1(t) - z_2(t))^T (z_1(t) - z_2(t)) \, dt < \int_0^T (w_1(t) - w_2(t))^T (w_1(t) - w_2(t)) \, dt
\]

- As for $L_2$ gain control, no efficient direct method for solving this problem
LPV control for ensuring typical nonlinear specs
Equivalence between local properties and global properties

**NL Plant**

\[ y = G_{NL}(u) : \begin{cases} \dot{x}(t) = f(x(t), u(t)) \\
y(t) = g(x(t), u(t)) \end{cases} \tag{1} \]

\[ \downarrow \]

**Gâteaux Derivative** TV Linearizations of \( G_{NL} \) at \( u_r \in \mathcal{L}_2 \)

\[ \bar{y} = DG_{NL}[u_r](\bar{u}) : \begin{cases} \dot{x}(t) = \bar{A}(t)\bar{x}(t) + \bar{B}(t)\bar{u}(t) \\
y(t) = \bar{C}(t)\bar{x}(t) + \bar{D}(t)\bar{u}(t) \end{cases} \]

with

\[
\left[ \begin{array}{cc} \bar{A}(t) & \bar{B}(t) \\ \bar{C}(t) & \bar{D}(t) \end{array} \right] = \left[ \begin{array}{cc} \frac{\partial f}{\partial x} (x_r(t), u_r(t)) & \frac{\partial f}{\partial u} (x_r(t), u_r(t)) \\ \frac{\partial g}{\partial x} (x_r(t), u_r(t)) & \frac{\partial g}{\partial u} (x_r(t), u_r(t)) \end{array} \right]
\]

where \( x_r(t) \) is the solution of (1) for the input \( u(t) \equiv u_r(t) \)
Equivalence between local properties and global properties

**NL Plant**

\[
y = G_{NL}(u) : \begin{cases} 
\dot{x}(t) = f(x(t), u(t)) \\
y(t) = g(x(t), u(t)) 
\end{cases} \quad (1)
\]

\[
\downarrow
\]

**(Gâteaux Derivative) TV Linearizations** of \( G_{NL} \) at \( u_r \in L_2 

\[
\bar{y} = DG_{NL}[u_r](\bar{u}) : \begin{cases} 
\dot{x}(t) = \bar{A}(t)\bar{x}(t) + \bar{B}(t)\bar{u}(t) \\
\bar{y}(t) = \bar{C}(t)\bar{x}(t) + \bar{D}(t)\bar{u}(t) 
\end{cases}
\]

**Mean Value Theorem in Norm**

\[
\|G_{NL}\|_\Delta \leq \gamma \iff \|DG_{NL}[u_r]\|_{i,2} \leq \gamma, \quad \forall \ u_r \in L_2
\]
Equivalence between local properties and global ones

**Global Properties**

$L_2$ incremental gain prop. of the NL system

**Local Properties**

$L_2$ gain prop. of TV linearisations
An LPV approach for incremental synthesis

$K_{NL}$ such that $||F_i(G_{NL}, K_{NL})||_\Delta < \gamma$

$G_{NL}$

NL systems

Incremental Norm

$K_{NL}$

NL controllers
An LPV approach for incremental synthesis

\[ K_{NL} \text{ such that } \| F_i(G_{NL}, K_{NL}) \|_\Delta < \gamma \]

\[ DK_{NL} \text{ such that } \| F_i(DG_{NL}, DK_{NL}) \|_{i,2} < \gamma \]

\[ \| F(l(DG_{NL}, DK_{NL})) \|_{i,2} < \gamma \]
Two questions

1. How to compute for any \( u_r \in \mathcal{L}_2, DK_{NL}[u_r] \)?

   \[ \rightarrow \text{Use an LPV method with } G_{LPV} \text{ which embeds } DG_{NL}[u_r] \text{ for any } u_r \]

2. From \( DK_{NL}[u_r] \), defined for any \( u_r \in \mathcal{L}_2 \), how to compute \( K_{NL} \)?

   \[ \rightarrow \text{focus on a special class of nonlinear control problem with the appropriated LPV control method} \]
Two questions

1. How to compute for any \( u_r \in L_2, DK_{NL}[u_r] \)?

   \( \rightarrow \) Use an LPV method with \( G_{LPV} \) which embeds \( DG_{NL}[u_r] \) for any \( u_r \)

2. From \( DK_{NL}[u_r] \), defined for any \( u_r \in L_2 \), how to compute \( K_{NL} \)?

   \( \rightarrow \) focus on a special class of nonlinear control problem with the appropriated LPV control method
Two questions

1. How to compute for any $u_r \in \mathcal{L}_2$, $DK_{NL}[u_r]$?
   \[ \mapsto \text{Use an LPV method with } G_{LPV} \text{ which embeds } DG_{NL}[u_r] \text{ for any } u_r \]

2. From $DK_{NL}[u_r]$, defined for any $u_r \in \mathcal{L}_2$, how to compute $K_{NL}$?
   \[ \mapsto \text{focus on a special class of nonlinear control problem with the appropriated LPV control method} \]
LPV model embeds time varying linearizations

To the time varying linearizations $DP_{NL}[w_r, u_r]$

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B_w(t)w(t) + B_u(t)u(t) \\
\tilde{z}(t) &= C_z(t)x(t) + D_{zw}(t)w(t) + D_{zu}(t)u(t) \\
\tilde{y}(t) &= C_y(t)x(t) + D_{yw}(t)w(t) + D_{yu}(t)u(t)
\end{align*}
\]

is associated an LPV plant

\[
\begin{align*}
\dot{x}(t) &= A(\theta(t))x(t) + B_1(\theta(t))w(t) + B_2(\theta(t))u(t) \\
z(t) &= C_1(\theta(t))x(t) + D_{11}(\theta(t))w(t) + D_{12}(\theta(t))u(t) \\
y(t) &= C_2(\theta(t))x(t) + D_{21}(\theta(t))w(t) + D_{22}(\theta(t))u(t)
\end{align*}
\]

such that with

\[
\Omega_{DNL} = \left\{ (\bar{x}, \bar{z}, \bar{y}, \bar{w}, \bar{u}) \mid \exists u_r, w_r, (3) \text{ is satisfied} \right\}
\]

and

\[
\Omega_{LPV} = \left\{ (x, z, y, w, u) \mid (4) \text{ is satisfied} \right\}
\]

we have

\[
\Omega_{DNL} \subset \Omega_{LPV}
\]
Roughly speaking, nonlinear system of the form

\[
\begin{aligned}
\dot{x}(t) &= Ax(t) + B_2 u(t) + \tilde{f}(x(t)) \\
\quad x(0) &= x_0
\end{aligned}
\]

with \(\tilde{f}(x(t)) = B_0 p(t)\)

where \(p(t)\) is measured on-line or where the components of \(x(t), w(t)\) and \(u(t)\) necessary for the computation of \(p(t)\) are measured, that is, there exists a function \(\alpha\) such that

\[p(t) = \alpha(x(t), w(t), u(t))\]

An LPV approach for incremental synthesis

\( \| F_i(G_{NL}, K_{NL}) \|_\Delta < \gamma \)

\( K_{NL} \) such that \( \| F_i(G_{NL}, K_{NL}) \|_\Delta < \gamma \)

\( \| F_i(DG_{NL}, DK_{NL}) \|_{i,2} < \gamma \)

\( DK_{NL} \) such that \( \| F_i(DG_{NL}, DK_{NL}) \|_{i,2} < \gamma \)
An LPV approach for incremental synthesis

Let $\dot{x} = Ax + B_f(x) + B_u u$ be the nonlinear plant $G_{NL}$.

Linearization:

- For $G_{NL}$, find $K_{NL}$ such that $||\mathcal{F}(G_{NL}, K_{NL})||_\Delta < \gamma$.
- For $DG_{NL}$, find $\bar{K}$ such that $||\mathcal{F}(DG_{NL}, \bar{K})||_{l_2} < \gamma$.

$DG_{NL}$ is obtained by linearization at $x_r$.
An LPV approach for incremental synthesis

\[
\dot{x} = Ax + B_f(x) + B_u u
\]

NL Plant

\[
\dot{x} = A\bar{x} + B_f(x) \partial f(x_r)\bar{x} + B_u \bar{u}
\]

Linéarisation

\[
\dot{x}_K = A_K \bar{x}_K + B_{Kp} \partial f(x_r)\bar{x}
+ B_{Ky} \bar{y}
\]

\[
K_{NL} \text{ such that } ||\mathcal{F}_i(G_{NL}, K_{NL})||_\Delta < \gamma
\]

\[
\mathcal{K} \text{ such that } ||\mathcal{F}_i(DG_{NL}, \mathcal{K})||_{i, 2} < \gamma
\]
An LPV approach for incremental synthesis

\[ \dot{x} = Ax + B_f(x) + B_u u \]

Linearization

\[ \dot{x} = Ax + B_{f_{\partial x}}(x_r) + B_u \bar{u} \]

NL Plant

Incremental Norm

\[ \dot{x}_K = A_K x_K + B_{Kp} f(x) + B_{Ky} y \]

NL Controller

Integration

\[ \dot{x}_K = A_K x_K + B_{Kp} \frac{\partial f}{\partial x}(x_r) + B_{Ky} \bar{y} \]

\[ K_{NL} \text{ such that } \| \mathcal{F}(G_{NL}, K_{NL}) \|_\Delta < \gamma \]

\[ \bar{K} \text{ such that } \| \mathcal{F}(DG_{NL}, \bar{K}) \|_{l,2} < \gamma \]
Application to the illustrative example

\[ r(t) \quad \epsilon(t) \quad \int \quad K_{LPV} \quad u(t) \quad G \quad y(t) \]

\[ z_1(t) \quad z_2(t) \quad W_1 \quad W_2 \quad W_3 \quad \theta \]

\[ W_3 \]

Tracking and rejection specs are satisfied.
Application to the illustrative example

\[ z_1(t) \rightarrow W_1 \rightarrow z_2(t) \rightarrow W_2 \rightarrow W_3 \rightarrow \theta \rightarrow G \rightarrow y(t) \]

\[ r(t) \rightarrow \epsilon(t) \rightarrow \int \rightarrow u(t) \rightarrow p(t) \]

\[ z_2(t) = W_2z_1(t) \]

\[ y(t) = Gp(t) \]

\[ p(t) = \theta u(t) \]

\[ u(t) = K_{LPV} \epsilon(t) \]

Tracking and rejection specs are satisfied.
Application to the illustrative example

\[ z_1(t) \rightarrow W_1 \quad z_2(t) \rightarrow W_2 \rightarrow W_3 \rightarrow \theta \rightarrow G \rightarrow y(t) \]

\[ r(t) \quad \epsilon(t) \rightarrow f \rightarrow \int \rightarrow K_{LPV} \rightarrow u(t) \rightarrow p(t) \rightarrow \] Tracking and rejection specs are satisfied.
Application to the illustrative example

\[
\begin{align*}
W_1 & \quad z_1(t) \\
W_2 & \quad z_2(t) \\
G & \quad y(t)
\end{align*}
\]

\[
\begin{align*}
r(t) & \quad \epsilon(t) \\
K_{LPV} & \quad u(t) \\
p(t) & \quad \theta
\end{align*}
\]

Tracking and rejection specs are satisfied.
Toward a new approach of nonlinear control using LPV

- Two existing approaches of nonlinear control using LPV
  - Gain scheduling
    - Main idea: LPV model embeds **time invariant linearizations** of nonlinear plant
    - Interest: improve a widespread engineering practise
    - Drawback: few guarantees on the closed loop behavior
  - Quasi LPV
    - Main idea: LPV model embeds **nonlinear plant**
    - Interest: stability guarantees
    - Drawback: typical specs are not ensured

- Proposition of a third LPV approach
  - LPV for incremental control
    - Main idea: LPV model embeds **time variant linearizations** of nonlinear plant
    - Interest: stability and typical specs are ensured
    - Drawback: more works for the integration
Toward a new approach of nonlinear control using LPV

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Conclusions

- Pave the way to a common LTI/NL framework for performance control, ensuring typical specifications.

- A key result of robust control is the translation of performance specs in a well-posed optimisation problem ($H_\infty$ norm).

- Its extension for typical specs is not the $L_2$ gain / stability approach but the $L_2$ incremental gain / incremental stability one.

- Combined with LPV methods, pave the way to the practical design of nonlinear controllers ensuring typical specifications.

- Objective: propose a rigorous alternative to the widespread gain-scheduling control used by the engineers.
On-going / Forthcoming projects involving this nonlinear performance approach

- Nonlinear robust performance analysis: extension of the $\mu$ analysis with less conservative approach than IQC (S. Waitman, P. Massioni, L. Bako)

- Identification for control: extension to nonlinear systems (X. Bombois)

- Nonlinear control design using LPV

- Design of systems with nonlinearities