Robustness of an adaptive predictor-based output feedback for a wave PDE in presence of in-domain viscous damping

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Facilities/vibrations

Well length between 500m and 5km

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Vibrations types

- whirl oscillations

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Well length between 500m and 5km

Vibrations types
- whirl oscillations
- vertical vibrations (bit bounce)
Facilities/vibrations

- whirl oscillations
- vertical vibrations (bit bounce)
- torsional oscillations (stick-slip)

Well length between 500m and 5km
Stick-slip oscillations

Modeling of the angular displacement $u(x, t)$

Damped wave equation + friction

$$u_{tt}(x, t) = u_{xx}(x, t) + \lambda u_t(x, t)$$
Modeling of the angular displacement $u(x, t)$

Damped wave equation + friction

$u_{tt}(x, t) = u_{xx}(x, t) + \lambda u_t(x, t)$

$u_x(1, t) = U(t)$ (applied torque)
Modeling of the angular displacement $u(x, t)$

Damped wave equation + friction

\[ u_{tt}(x, t) = u_{xx}(x, t) + \lambda u_t(x, t) \]

\[ u_x(1, t) = U(t) \] (applied torque)

\[ u_{tt}(0, t) = a F(u_t(0, t)) + au_x(0, t) \]

friction
Modeling of the angular displacement $u(x, t)$

- **Modeling**

- **Experimental data**
Modeling of the angular displacement $u(x, t)$

Damped wave equation + friction

$$u_{tt}(x, t) = u_{xx}(x, t) + \lambda u_t(x, t)$$

$$u_x(1, t) = U(t) \text{ (applied torque)}$$

$$u_{tt}(0, t) = aF(u_t(0, t)) + au_x(0, t)$$

Very uncertain, depends on
- the nature of the rock;
- the weight on the bit...
1. Adaptive feedback without distributed damping
2. Robustness to distributed damping
Linearized model

Wave equation + friction

\[ u_{tt}(x, t) = u_{xx}(x, t) \]
\[ u_x(1, t) = U(t) \]
\[ u_{tt}(0, t) = aF(u_t(0, t)) + au_x(0, t) \]

Desired equilibrium: uniform velocity \( u_t' \)

\[
\begin{align*}
u_t'(x, t) &= u_t' t - F(u_t') x + u_0 \quad (u_0 \in \mathbb{R}) \\
U_t' &= -F(u_t') \\
u_x' &= U_t'
\end{align*}
\]
Linearized model

Wave equation + friction

\( u_{tt}(x, t) = u_{xx}(x, t) \)

\( u_x(1, t) = U(t) \)

\( u_{tt}(0, t) = aF(u_t(0, t)) + au_x(0, t) \)

\[ \Rightarrow \]

Linearized model for \( \tilde{u} = u - u^r \)

\( \tilde{u}_{tt}(x, t) = \tilde{u}_{xx}(x, t) \)

\( \tilde{u}_x(1, t) = U(t) - U^r \)

\( u_{tt}(0, t) = aq\tilde{u}_t(0, t) + a\tilde{u}_x(0, t) \)

Desired equilibrium: uniform velocity \( u_t^r \)

\[ \begin{cases} 
    u^r(x, t) = u_t^r t - F(u_t^r)x + u_0 \ (u_0 \in \mathbb{R}) \\
    U^r = -F(u_t^r) \\
    u_x^r = U^r 
\end{cases} \]

with \( q = \frac{dF}{du_t}(u_t^r) > 0 \) for high velocities
Problem at stake

PDE + ODE

\[
\begin{align*}
    u_{tt}(x, t) &= u_{xx}(x, t) \\
    u_x(1, t) &= U(t) \\
    u_{tt}(0, t) &= a q u_t(0, t) + a(u_x(0, t) - d)
\end{align*}
\]

Adaptive controller: \( q \rightarrow \hat{q}(t) \) and \( d \rightarrow \hat{d}(t) \)

Assumptions

- There exist known constants \( q, \bar{q}, d \) and \( \bar{d} \) such that \( q > \bar{q}, d < \bar{d} \) and \( q \in [q, \bar{q}], d \in [d, \bar{d}] \).
- We measure \( u_t(0, t) \) and \( u_t(1, t) \) for all \( t \geq 0 \).
Reformulation

(Modified) Riemman variables

\[
\begin{align*}
\zeta(x, t) &= u_t(x, t) + u_x(x, t) - \hat{d}(t) \\
\omega(x, t) &= u_t(x, t) - u_x(x, t) - \hat{d}(t) \\
W(t) &= u_t(1, t) + U(t) - \hat{d}(t)
\end{align*}
\]
Reformulation

(Modified) Riemann variables

\[
\begin{align*}
\zeta(x,t) &= u_t(x,t) + u_x(x,t) - \hat{d}(t) \\
\omega(x,t) &= u_t(x,t) - u_x(x,t) - \hat{d}(t) \\
W(t) &= u_t(1,t) + U(t) - \hat{d}(t)
\end{align*}
\]

PDEs + ODE Cascade (with \(d\) known)

\[
\begin{align*}
\frac{d^2 u}{dt^2}(0,t) &= a q u_t(0,t) + a(u_x(0,t) - d) \\
\frac{d^2 u}{dt^2}(x,t) &= u_{xx}(x,t) \\
\frac{d u}{dt}(1,t) &= U(t)
\end{align*}
\]

\[
\begin{align*}
\frac{d u}{dt}(0,t) &= a(q-1)u_t(0,t) + a\zeta(0,t) \\
\zeta_t(x,t) &= \zeta_x(x,t) \\
\zeta(1,t) &= W(t) \\
\omega_t(x,t) &= -\omega_x(x,t) \\
\omega(0,t) &= 2u_t(0,t) - \zeta(0,t)
\end{align*}
\]

\(\hat{d}(t) = d\) \Rightarrow \text{input-delay system}

\text{transport PDE (stable)}
Control law

**ODE**

\[ u_{tt}(0, t) = a(q - 1)u_{t}(0, t) + a[W(t) - \tilde{d}(t)] \]

Adaptive PI controller for the error \( e(t) = u_{t}(0, t) - u'_{t} \)

\[ U(t) = -u_{t}(1, t) + \hat{d}(t) + W(t) \]

\[ W(t) = -(c_{0} + \hat{q}(t) - 1)\Omega(t) \]

\[ \Omega(t) = e(t) \]
Control law

**ODE**

\[ u_{tt}(0, t) = a(q - 1)u_t(0, t) + a[W(t - 1) - \tilde{d}(t)] \]

Adaptive PI controller for the **predicted** error \( e(t + 1) = u_t(0, t + 1) - u_t' \)

\[
U(t) = -u_t(1, t) + \hat{d}(t) + W(t)
\]

\[
W(t) = - (c_0 + \hat{q}(t) - 1)\Omega(t)
\]

\[
\Omega(t) = e^{a(\hat{q}(t) - 1)}e(t) + a \int_{t-1}^{t} e^{a(\hat{q}(t) - 1)(t-\tau)} [U(\tau) + u_t(1, \tau) - \hat{d}(\tau)] d\tau
\]

= adaptive predictor of \( e(t+1) \)

Only needs

- the speed of the bit \( u_t(0, t) \) and the top velocity \( u_t(1, t) \)
- the top input \( U(t) \) (torque applied to the rotatory table)

(storage in memory over a length equal to the propagation time)
Update laws and stability result

\[ \dot{\hat{q}}(t) = \text{Proj}_{[\hat{q}, q]} \left\{ \frac{\gamma_q}{1 + N(t)} e(t) \gamma(t), \hat{q}(t) \right\} \]

\[ \dot{\hat{d}}(t) = - \text{Proj}_{[\hat{d}, d]} \left\{ \frac{\gamma_d}{1 + N(t)} \gamma(t), \hat{d}(t) \right\} \]

**Theorem [ACC14]**

For all \( c_0 > 0 \), there exist \( \gamma_q^*, \gamma_d^* > 0 \) such that, if \( \gamma_q < \gamma_q^* \) and \( \gamma_d < \gamma_d^* \), there exist \( R > 0 \) and \( \rho > 0 \) such that

\[ \Gamma(t) \leq R (e^{\rho t} \Gamma(0) - 1) \]

\[ \Gamma(t) = u_t(0, t)^2 + \int_0^1 (u_x(x, t) - \bar{d})^2 dx + \int_0^1 u_t(x, t)^2 dx + \tilde{q}(t)^2 + \tilde{d}(t)^2 \]

Potential + kinetic energy

Besides, the regulation in \( L_2 \)-norm is achieved.
Contributions

- Pointwise velocity regulation
- Adaptive control wrt the friction with the rock
- Only measurement of $u_t(1, t)$ and $u_t(0, t)$
  - *in-domain damping neglected*, $\lambda = 0$
1. Adaptive feedback without distributed damping
2. Robustness to distributed damping
Distributed damping introduces coupling

\[ u_{tt}(x, t) = u_{xx}(x, t) - 2\lambda u_t(x, t) \]

\[ x = 1 \quad U(t) \]

\[ u_x(1, t) = U(t) \]

\[ x = 0 \]

\[ u_{tt}(0, t) = au_x(0, t) + a[qu_t(0, t) - d] \]
Distributed damping introduces coupling

\[ W(t) \]

\[ \zeta_t(x, t) = \zeta_x(x, t) - \dot{\hat{d}}(t) \]

\[ \omega_t(x, t) = -\omega_x(x, t) + \dot{\hat{d}}(t) \]

\[ \text{ODE} \quad \dot{y}(t) = a(q - 1)y(t) + a[\zeta(0, t) - \tilde{d}(t)] \]

\[ \rightarrow \text{For } \lambda = 0, \text{ no coupling} \]
Distributed damping introduces coupling

\[ W(t) \]

\[ \zeta_t(x, t) = \zeta_x(x, t) - \dot{\tilde{d}}(t) - \lambda(\zeta(x, t) + \omega(x, t)) \]

\[ \omega_t(x, t) = -\omega_x(x, t) + \dot{\tilde{d}}(t) - \lambda(\omega(x, t) + \zeta(x, t)) \]

ODE \[ \dot{y}(t) = a(q - 1)y(t) + a[\zeta(0, t) - \tilde{d}(t)] \]
Elements of analysis: Reformulation

\[
\begin{align*}
\dot{\xi}_t(x, t) &= \dot{\xi}_x(x, t) - \dot{d}(t) \\
\dot{\zeta}_t &= \dot{\zeta}_x - \lambda (\dot{\omega} + \dot{\zeta} + \ddot{\omega} + \ddot{\xi}) \\
\dot{\omega}_t &= -\ddot{\omega}_x - \lambda (\dot{\omega} + \dot{\zeta} + \ddot{\omega} + \ddot{\xi}) \\
\dot{\omega}_t(x, t) &= -\ddot{\omega}_x(x, t) + \dot{d}(t)
\end{align*}
\]

\[
\zeta(x, t) = \tilde{\zeta}(x, t) + \tilde{\zeta}(x, t), \quad \omega(x, t) = \tilde{\omega}(x, t) + \tilde{\omega}(x, t)
\]
Robustness of [BPK2014]

**Theorem [ACC16]**

Consider the closed-loop system with in-domain damping and the previous control law developed in [ACC14], define the functional

\[
\Gamma(t) = u_t(0, t)^2 + \int_0^1 u_t(x, t)^2 \, dx + \int_0^1 (u_x(x, t) - d)^2 \, dx \\
+ (q - \hat{q})^2 + (d - \hat{d})^2
\]

There exists \( \lambda^* > 0 \) such that, if \( \lambda < \lambda^* \), there are \( R > 0 \) and \( \rho > 0 \) such that

\[
\Gamma(t) \leq R \left( e^{\rho \max_{s \in [0,2]} \Gamma(-s) - 1} \right), \quad \forall t \geq 0
\]

and the regulation in \( L_2 \)-norm follows

\[
\lim_{t \to \infty} u_t(0, t) = \lim_{t \to \infty} \|u_t(t)\| = \lim_{t \to \infty} \|u_x(t) - d\| = \lim_{t \to \infty} (\hat{d} - d) = 0
\]
Simulations on the nonlinear model for $\lambda \neq 0$

![Graph showing velocity and control response](image-url)
Directions of future works

Contributions
- Pointwise velocity regulation
- Adaptive control w.r.t. the friction with the rock
- Only measurement of $u_t(1, t)$ and $u_t(0, t)$
- **robustness to in-domain damping, $\lambda \neq 0$**

On-going and future works
- for larger values of $\lambda$, extend the prediction approach for coupled transport equations (first step: tailored observer-based backstepping controller, CDC16)
- extend this work to the measurement of $u_t(1, t)$ only
- develop a general observation strategy for 1D hyperbolic equations cascaded with an ODE